# $F$-ISOCRYSTALS AS ISOSHTUKAS WITH NO LEGS

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ABSTRACT. We rederive some results about $F$-isocrystals on perfect schemes by interpreting these objects in terms of vector bundles on relative Fargues–Fontaine curves. This amounts to the notion of "isoshtukas" introduced by Gleason–Ivanov, specifically isoshtukas for $\mathrm{GL}_n$ with no legs.

## 1. INTRODUCTION

Throughout, fix a prime number $p$. We associate to every perfect $\mathbb{F}_p$-scheme $X$ the category of *convergent $F$-isocrystals* on $X$; when $X = \operatorname{Spec} R$ is affine this is simply the category of finite projective modules over $W(R)[p^{-1}]$, where $W$ denotes the functor of $p$-typical Witt vectors, equipped with isomorphisms with their pullbacks by the Witt vector Frobenius which must be respected by morphisms.

In connection with the study of crystalline cohomology for schemes of finite type over a perfect field of characteristic $p$, the category of convergent $F$-isocrystals has been studied classically by numerous authors. In recent years, a new link has emerged with the modern perspective on $p$-adic Hodge theory provided by the development of perfectoid spaces and relative Fargues–Fontaine curves. In particular, the convergent $F$-isocrystals on $X$ of a given rank $n$ are given a new interpretation in the work of Gleason–Ivanov [6], as *isoshtukas* for the group $\mathrm{GL}_n$ with no legs.

The purpose of this paper is to rederive some of the basic properties of $F$-isocrystals from the point of view of relative Fargues–Fontaine curves. As previously observed by Ivanov [8], this process is greatly simplified by the use of descent for the *arc-topology* [1]; in particular, most of the work is done in the case where $X$ is the spectrum of a valuation ring which is *absolutely integrally closed* (i.e., it has algebraically closed fraction field). The arguments in that setting are naturally expressed in the language of vector bundles on Fargues–Fontaine curves [4].

It would be natural to transfer further results from the theory of $F$-isocrystals on *smooth* schemes over a perfect field, such as Drinfeld's lemma [17, 20], to perfect schemes and then further to isoshtukas (for other groups and adding legs). However, some caution is required: for example, in the smooth setting the restriction functor induced by an open immersion with dense image is fully faithful [15, Theorem 5.3], but this fails for perfect schemes (Example 8.4). See Theorem 10.17 for a sample positive result, an adaptation of the relative Dieudonné–Manin decomposition from smooth schemes [17, Theorem 7.3] to perfect schemes.

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## 2. $F$-isocrystals over perfect rings

**Definition 2.1.** For $R$ a perfect $\mathbb{F}_p$-algebra, let $W(R)$ be the ring of $p$-typical Witt vectors over $R$; let $\varphi\colon W(R) \to W(R)$ be the functorial lift of absolute Frobenius. By an *$F$-isocrystal* on $R$, we mean a finite projective $W(R)[p^{-1}]$-module $M$ equipped with an isomorphism $\varphi^* M \cong M$; we interpret the latter as a semilinear action of $\varphi$ on $M$. These form a tensor category $\mathbf{F}\text{-}\mathbf{Isoc}(R)$ in which the morphisms are $\varphi$-equivariant $W(R)[p^{-1}]$-module morphisms.

The categories $\mathbf{F}\text{-}\mathbf{Isoc}(R)$ form a stack for the Zariski topology, the étale topology, and even the *arc-topology* of Bhatt–Mathew [1]; the latter statement is a consequence of arc-descent for finite projective $W(R)[p^{-1}]$-modules [8, Proposition 5.9]. In particular, we immediately also obtain a category $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ for every perfect $\mathbb{F}_p$-scheme $X$ in such a way that $\mathbf{F}\text{-}\mathbf{Isoc}(\operatorname{Spec} R)$ is canonically identified with $\mathbf{F}\text{-}\mathbf{Isoc}(R)$.

**Definition 2.2.** For $M \in \mathbf{F}\text{-}\mathbf{Isoc}(R)$, define

$$H^0_\varphi(M) = \ker(\varphi - 1, M), \qquad H^1_\varphi(M) = \operatorname{coker}(\varphi - 1, M);$$

we then have canonical isomorphisms

$$\operatorname{Hom}_{\mathbf{F}\text{-}\mathbf{Isoc}}(M_1, M_2) \cong H^0_\varphi(M_1^\vee \otimes M_2), \qquad \operatorname{Ext}^1_{\mathbf{F}\text{-}\mathbf{Isoc}}(M_1, M_2) \cong H^1_\varphi(M_1^\vee \otimes M_2).$$

**Definition 2.3.** For $s \in \mathbb{Q}$ with denominator $d$, let $\mathcal{O}(s)$ denote the isocrystal (over any base) which is free on the generators $\mathbf{e}_1, \dots, \mathbf{e}_d$ and satisfies

$$\varphi(\mathbf{e}_1) = \mathbf{e}_2, \dots, \varphi(\mathbf{e}_{d-1}) = \mathbf{e}_d, \varphi(\mathbf{e}_d) = p^{-ds}\mathbf{e}_1.$$

Note the sign convention, which is consistent with the literature on Fargues–Fontaine curves but not with the literature on isocrystals.

**Lemma 2.4** (Dieudonné–Manin decomposition)**.** *For $k$ an algebraically closed field, every object of $\mathbf{F}\text{-}\mathbf{Isoc}(k)$ decomposes as a direct sum in which each term is isomorphic to $\mathcal{O}(s)$ for some $s \in \mathbb{Q}$. This decomposition is not unique, but the associated* isotypical decomposition *is unique.*

*Proof.* References to follow. This can also be deduced via arc-descent from the classification of vector bundles on Fargues–Fontaine curves (Lemma 4.5); see the proof of [11, Theorem 4.5.7(a)]. □

**Definition 2.5.** For $R$ an algebraically closed field, we define the *Newton polygon* of $M \in \mathbf{F}\text{-}\mathbf{Isoc}(R)$ to be the graph of the convex piecewise linear function on $[0, \operatorname{rank}(M)]$ with the property that for each $s \in \mathbb{Q}$, the total width (or *multiplicity*) of the segment of the graph with slope $s$ equals the rank of the isotypical summand of $M$ corresponding to $\mathcal{O}(s)$. By the uniqueness aspect of Lemma 2.4, this definition is invariant under base change.

For $X$ a perfect $\mathbb{F}_p$-scheme, using the previous paragraph we may associate a *Newton polygon* to an object of $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ and a point of $X$. We say that the object is *isoclinic* if all of the slopes of its Newton polygon equal a single value, and *étale* or *unit-root* if moreover that common value is 0.

**Remark 2.6.** The notion of a convergent $F$-isocrystal is more usually associated to a smooth scheme over a perfect field of characteristic $p$, where it manifests as a vector bundle with connection on a certain rigid analytic space. In particular, the presence of the connection ensures that the resulting category is abelian. This is not true over perfect schemes; see Example 8.1.

## 3. Geometry of Witt vectors

We recall some key points about the geometry of Fargues–Fontaine curves and some closely related spaces.

**Definition 3.1.** For the remainder of the paper, let $K$ be an algebraically closed nonarchimedean field of characteristic $p$. Let $\mathfrak{o}_K$ be the valuation ring of $K$. Let $k$ be the residue field of $\mathfrak{o}_K$. Fix a pseudouniformizer $\varpi$ in $\mathfrak{o}_K$. Let $\eta$ and $z$ denote the generic point and closed point of $\operatorname{Spec} \mathfrak{o}_K$, respectively.

**Definition 3.2.** The ring $W(\mathfrak{o}_K)$ is complete with respect to the adic topology defined by the ideal $(p, [\varpi])$ (which does not depend on the choice of $\varpi$). Following Huber, we may construct its associated *adic spectrum* $\operatorname{Spa}(W(\mathfrak{o}_K), W(\mathfrak{o}_K))$, whose points are equivalence classes of continuous valuations on $W(\mathfrak{o}_K)$. By construction, elements of $W(\mathfrak{o}_K)$ define global sections on the adic spectrum, so there is a natural morphism of locally ringed spaces $\operatorname{Spa}(W(\mathfrak{o}_K), W(\mathfrak{o}_K)) \to \operatorname{Spec} W(\mathfrak{o}_K)$.

The space $\operatorname{Spa}(W(\mathfrak{o}_K), W(\mathfrak{o}_K))$ contains a single point whose residue field is trivially valued, namely the point corresponding to the trivial valuation on $k$. Removing this point leaves the *analytic locus* of $\operatorname{Spa}(W(\mathfrak{o}_K), W(\mathfrak{o}_K))$, denoted $Y_K$. Let $x_p, x_\varpi$ be the points of $Y_K$ cut out by $p$ and by $[\varpi]$, respectively.

The geometry of $Y_K$ is developed carefully in [4]; we summarize the key points here.

- The space $Y_K$ is quasicompact: it can be covered by two affinoid subspaces.
- The space $Y_K \setminus \{x_\varpi\}$ is covered by the adic spectra of principal ideal domains. Moreover, the latter are *strongly noetherian* in the sense that every Tate algebra over one of them is noetherian [13]; hence $Y_K \setminus \{x_\varpi\}$ is locally noetherian.
- The space $Y_K$ is a genuine adic space (that is, the structure presheaf is a sheaf). Away from $x_\varpi$ this follows from the locally noetherian property (reference to follow); at $x_\varpi$ we may use the fact that $Y_K$ is *sousperfectoid* in the sense of [7].
- The natural map $W(\mathfrak{o}_K) \to \Gamma(Y_K, \mathcal{O}_{Y_K})$ is an isomorphism. See Definition 3.3.

**Definition 3.3.** Define the ring of *overconvergent Witt vectors* over $K$ as the stalk $W^\dagger(K) := \mathcal{O}_{Y_K, x_p}$. It is $p$-adically separated, so we may view it naturally as a dense subring of $W(K)$.

Although we will not need this, we note that the image of $W^\dagger(K)$ in $W(K)$ can be described concretely: it consists of those $f = \sum_{n=0}^\infty p^n [\overline{f}_n] \in W(K)$ for which for some positive integer $a$, $\overline{f}_n \varpi^{an} \in \mathfrak{o}_K$ for all $n > 0$. See [12] for an algebraic development of the properties of $W^\dagger(K)$ from this point of view.

**Lemma 3.4.** *Let $U$ be an affinoid subspace of $Y_K \setminus \{x_p\}$. For each affinoid subspace $V$ of $U \setminus \{x_\varpi\}$, equip $\mathcal{O}(V)$ with the absolute value normalized with $|\varpi| = p^{-1}$. Then the restriction map $\mathcal{O}(U) \to \mathcal{O}(U \setminus \{x_\varpi\})$ identifies $\mathcal{O}(U)$ with the ring of* bounded *analytic functions on $U \setminus \{x_\varpi\}$.*

*Proof.* This can be deduced from either the log-convexity of Gauss norms [19, Lemma 4.2.3] or, using the fact that base extension of $Y_K \setminus \{x_p\}$ from $\mathbb{Q}_p$ to a perfectoid field gives a perfectoid space, the perfectoid Riemann extension theorem [21, Proposition 2.3.2]. □

**Definition 3.5.** Let $U$ be an open subspace of $Y_K$. By a *meromorphic vector bundle on* $U$, we will mean an object of the isogeny category of vector bundles on $U$, in which the morphisms are allowed to have poles at $x_p$ (but nowhere else). If $x_p \notin U$ then this is just the same as the category of vector bundles on $U$, so we will usually omit the word "meromorphic".

**Lemma 3.6.** *The pullback functor from the category of finite free $W(\mathfrak{o}_K)[p^{-1}]$-modules to the category of meromorphic vector bundles on $Y_K$ is an equivalence.*

*Proof.* This follows from the corresponding statement for ordinary vector bundles, for which see [16, Theorem 2.7]. □

**Remark 3.7.** An important subtlety in [16, Theorem 2.7] is that the pullback functor is not an *exact* equivalence, but Lemma 3.6 *is* an exact equivalence because we are inverting $p$. On the other hand, see Remark 4.11 for a similar but distinct issue.

**Remark 3.8.** The ring $W(\mathfrak{o}_K)$ is commonly denoted $\mathbf{A}_{\inf}(K)$, but we will not use this notation here.

## 4. Equivariant vector bundles and Fargues–Fontaine curves

We next consider categories of $\varphi$-equivariant (meromorphic) vector bundles on various subspaces of $Y_K$.

**Definition 4.1.** Let $\hat{x}_p$ denote the completion of $Y_K$ along $x_p$. The action of $\varphi$ on $W(\mathfrak{o}_K)$ induces an action on $Y_K$ fixing only $x_p$ and $x_\varpi$; more precisely, $x_\varpi$ is an attracting fixed point of $\varphi$ whereas $x_p$ is an attracting fixed point of $\varphi^{-1}$. Even more precisely, in the following diagram, every space on or above the diagonal inherits an action of $\varphi$ from $Y_K$, while every space on or below the diagonal inherits an action of $\varphi^{-1}$ from $Y_K$.

(4.2)
$$\begin{array}{ccccc} x_\varpi & & & & \\ \downarrow & & & & \\ Y_K \setminus \{x_p\} & \longleftarrow & Y_K \setminus \{x_p, x_\varpi\} & & \\ \downarrow & & \downarrow & & \\ Y_K & \longleftarrow & Y_K \setminus \{x_\varpi\} & \longleftarrow & \hat{x}_p \end{array}$$

The quotient $X_K := (Y_K \setminus \{x_p, x_\varpi\})/\varphi$ is the *(adic) Fargues–Fontaine curve* associated to $K$ (with coefficients in $\mathbb{Q}_p$).

**Definition 4.3.** For each space $U$ in (4.2), we may consider the category of *$\varphi$-equivariant* meromorphic vector bundles on $U$, which we denote by $\mathbf{F}\text{-}\mathbf{Vec}(U)$. In particular, we can interpret the bundles $\mathcal{O}(s)$ as objects of $\mathbf{F}\text{-}\mathbf{Vec}(U)$.

We may interpret the category $\mathbf{F}\text{-}\mathbf{Vec}(U)$ as follows.

- For $U = Y_K$, we obtain the category $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ thanks to Lemma 3.6.
- For $U = Y_K \setminus \{x_\varpi\}$, we obtain the category $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ of finite projective $W^\dagger(K)[p^{-1}]$-modules equipped with isomorphisms with their $\varphi$-pullbacks (i.e., *overconvergent $F$-isocrystals* on $K$).
- For $U = Y_K \setminus \{x_p, x_\varpi\}$, we obtain the category $\mathbf{Vec}(X_K)$ of vector bundles on $X_K$, which are classified by Lemma 4.5.

- For $U = Y_K \setminus \{x_p\}$, we obtain an as yet unidentified category admitting a restriction functor to vector bundles on $X_K$. We will show below that this is in fact an equivalence (Proposition 4.8).
- For $U = x_\varpi$, we recover the category $\mathbf{F}\text{-}\mathbf{Isoc}(k)$.
- For $U = \hat{x}_p$, we recover the category $\mathbf{F}\text{-}\mathbf{Isoc}(K)$, to which Lemma 2.4 applies.

In particular, once Proposition 4.8 is available, we obtain geometric restriction functors corresponding to the arrows in the following 2-commutative diagram:

$$\begin{array}{ccccc} \mathbf{F}\text{-}\mathbf{Isoc}(k) & & & & \\ \uparrow & & & & \\ \mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p\}) & \longrightarrow & \mathbf{Vec}(X_K) & & \\ \uparrow & & \uparrow & & \\ \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K) & \longrightarrow & \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K) & \longrightarrow & \mathbf{F}\text{-}\mathbf{Isoc}(K). \end{array} \tag{4.4}$$

**Lemma 4.5.** *Every object of* $\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p, x_\varpi\})$ *splits as a direct sum in which each term is isomorphic to* $\mathcal{O}(s)$ *for some* $s \in \mathbb{Q}$. *The associated isotypical decomposition is not necessarily unique, but the associated* slope filtration *is unique and coincides with the* Harder–Narasimhan (HN) filtration. *In particular, the first step of the filtration is the maximal subbundle of maximal slope.*

*Proof.* This is the classification of vector bundles on Fargues–Fontaine curves, which in this form first appears in [4, §8]. For further historical background, see [14, Theorem 3.6.13] and associated remarks. ∎

**Remark 4.6.** The functor $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K) \to \mathbf{F}\text{-}\mathbf{Isoc}(K)$ is essentially surjective by Lemma 2.4, but *not* fully faithful (see however Corollary 5.3 below). For instance, take $M$ to be free of rank 2 with the action of Frobenius on some basis given by $\begin{pmatrix} p & x \\ 0 & 1 \end{pmatrix}$ for some $x \in W^\dagger(K)[p^{-1}]$. By construction, there is an exact sequence

$$0 \to M_1^\dagger \to M^\dagger \to M_2^\dagger \to 0$$

in which $M_1^\dagger$ and $M_2^\dagger$ are objects of rank 1, and this sequence splits in $\mathbf{F}\text{-}\mathbf{Isoc}(K)$ by Lemma 2.4, but need not split in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$. Namely, this sequence represents a class in $H^1_\varphi(M_1^\dagger)$, which explicitly is the cokernel of $p\varphi - 1$ on $W^\dagger(K)[p^{-1}]$, and this cokernel can be shown to be nonzero in general.

One particularly tractable example is when $K$ is the field $k((t^{\mathbb{Q}}))$ of Mal'cev–Neumann series over an algebraically closed field $k$ of characteristic $p$, in which case we can write $x$ as a formal sum $\sum_{i \in \mathbb{Q}} c_i t^i$ and for any $j > 0$, the map $x \mapsto \sum_{n \in \mathbb{Z}} p^{-n} c_{jp^n}^{p^{-n}}$ is a well-defined functional on the cokernel.

**Lemma 4.7.** *For* $\mathcal{E} \in \mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p\})$, *the natural map* $\mathcal{E}(Y_K \setminus \{x_p\})^\varphi \to \mathcal{E}(Y_K \setminus \{x_p, x_\varpi\})^\varphi$ *is an isomorphism.*

*Proof.* Let $U$ be an affinoid neighborhood of $x_\varpi$ in $Y_K$ on which $\mathcal{E}$ admits a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$; it will suffice to check that $\mathcal{E}(U \setminus \{x_p\})^\varphi = \mathcal{E}(U \setminus \{x_p, x_\varpi\})^\varphi$. Define the $n \times n$ matrix $A$ over $\mathcal{O}(Y_K \setminus \{x_p\})$ by $\varphi(\mathbf{e}_j) = \sum_i A_{ij} \mathbf{e}_i$. For $\mathbf{v} \in \mathcal{E}(U \setminus \{x_p, x_\varpi\})^\varphi$, we can write $\mathbf{v} = \sum_i c_i \mathbf{e}_i$

for some column vector $c$ over $\mathcal{O}(U \setminus \{x_p, x_\varpi\})$ and then compute that $\varphi(c) = A^{-1}c$. By pulling back repeatedly along $\varphi$, we see that the vector $c$ has bounded entries in the sense of Lemma 3.4, so these entries extend across $x_\varpi$. □

**Proposition 4.8.** *The functor* $\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p\}) \to \mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p, x_\varpi\})$ *is an equivalence of categories. In particular, both categories are equivalent to* $\mathbf{Vec}(X_K)$.

*Proof.* Full faithfulness follows from Lemma 4.7 by taking internal Homs. Essential surjectivity follows from Lemma 4.5 because $\mathcal{O}(s)$ can be extended over $x_\varpi$. □

**Corollary 4.9.** *The functor* $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K) \to \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ *is a left exact equivalence of categories (but not right exact; see Example 8.1).*

*Proof.* From the interpretations given in Definition 4.3, we obtain a chain of equivalences

$$\begin{aligned}\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K) &\to \mathbf{F}\text{-}\mathbf{Vec}(Y_K) \\ &\to \mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_\varpi\}) \times_{\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p, x_\varpi\})} \mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p\}) \\ &\to \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K) \times_{\mathbf{Vec}(X_K)} \mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p\}).\end{aligned}$$

We may thus deduce the equivalence from Proposition 4.8. □

**Lemma 4.10.** *Let* $M \to N$ *be an inclusion in* $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ *whose restriction to* $\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_\varpi\})$ *splits. Then* $M/N$ *is a finite projective* $W(\mathfrak{o}_K)[p^{-1}]$*-module, and hence an object of* $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$.

*Proof.* Let $\mathcal{F} \to \mathcal{E}$ be the inclusion in $\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_\varpi\})$ corresponding to $M \to N$. The splitting ensures that $\mathcal{E}/\mathcal{F}$ is itself a vector bundle and hence an object of $\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_\varpi\})$. We thus deduce the claim from Proposition 4.8 as in Corollary 4.9. □

**Remark 4.11.** By Corollary 4.9, $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ is an abelian category. However, this does not itself imply that the formation of kernels and cokernels in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ is compatible with the forgetful functor to $W(\mathfrak{o}_K)[p^{-1}]$-modules. In fact this is only true for kernels; see Example 8.1.

## 5. The reverse slope filtration

We next introduce another filtration, this time on objects of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$.

**Lemma 5.1.** *Let* $M^\dagger$ *be a finite projective* $W^\dagger(K)$*-module equipped with a semilinear action of* $\varphi^{-d}$ *for some positive integer* $d$*. Equip* $M_K := M^\dagger \otimes_{W^\dagger(K)} W(K)$ *with the induced action of* $\varphi^{-d}$*. Then the natural map* $\operatorname{coker}(\varphi^{-d} - 1, M^\dagger) \to \operatorname{coker}(\varphi^{-d} - 1, M_K)$ *is surjective.*

*Proof.* Choose a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of $M^\dagger$. Given an element $\mathbf{v}$ of $M_K$ representing a class in $\operatorname{coker}(\varphi^{-d} - 1, M_K)$, write $\mathbf{v} = \sum_i c_i \mathbf{e}_i$ with $c_i \in W(K)$. We may in turn write each $c_i$ as a convergent sum $\sum_{j=0}^\infty p^j c_{ij}$ where each $c_{ij}$ belongs to $W^\dagger(K)$; we may then write $\mathbf{v}$ as $\sum_{j=0}^\infty p^j \mathbf{v}_j$ with $\mathbf{v}_j = \sum_i c_{ij}\mathbf{e}_i$. For any sequence of nonnegative integers $\{m_j\}_{j=0}^\infty$, the sum $\mathbf{w} := \sum_{j=0}^\infty p^j \varphi^{-m_j d}(\mathbf{v}_j)$ represents the same class as $\mathbf{v}$ in $\operatorname{coker}(\varphi^{-d} - 1, M_K)$. By taking the $m_j$ sufficiently large, we may ensure that $\mathbf{w} \in M^\dagger$. □

**Lemma 5.2.** *Suppose that* $M^\dagger \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ *restricts to* $M_K \in \mathbf{F}\text{-}\mathbf{Isoc}(K)$ *and has smallest Newton slope* $s = \frac{r}{d}$ *in lowest terms. Then every* $\mathbf{v} \in M_K$ *with* $\varphi^d(\mathbf{v}) = p^{-r}\mathbf{v}$ *belongs to* $M^\dagger$.

*Proof.* The slope condition means that we can choose a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of $M^\dagger$ on which $p^{-r}\varphi^{-d}$ acts via a matrix $A$ over $W^\dagger(K)$ whose reduction modulo $p$ has rank equal to the $\mathbb{Q}_p$-dimension of the $p^{-r}\varphi^{-d}$-fixed subspace of $M_K$. Write $\mathbf{v} = \mathbf{v}_0 + p\mathbf{v}_1$ with $\mathbf{v}_0$ in the $W^\dagger(K)$-span of $\mathbf{e}_1, \dots, \mathbf{e}_n$; then by Lemma 5.1, we can write $(p^{-r}\varphi^{-d} - 1)(\mathbf{v}_1) = (p^{-r}\varphi^{-d} - 1)(\mathbf{v}'_1)$ for some $\mathbf{v}'_1$ in the $W^\dagger(K)$-span of $\mathbf{e}_1, \dots, \mathbf{e}_n$. Put $\mathbf{v}' = \mathbf{v}_0 + p\mathbf{v}'_1 \in M^\dagger$; then

$$\begin{aligned}(p^{-r}\varphi^{-d} - 1)(\mathbf{v}') &= (p^{-r}\varphi^{-d} - 1)(\mathbf{v}_0) + p(p^{-r}\varphi^{-d} - 1)(\mathbf{v}'_1) \\ &= (\varphi^{-1} - 1)(\mathbf{v}_0) + p(p^{-r}\varphi^{-d} - 1)(\mathbf{v}_1) \\ &= (\varphi^{-1} - 1)(\mathbf{v}) = 0.\end{aligned}$$

This means that the $p^{-r}\varphi^{-d}$-fixed subspace of $M_K$ contains a $\mathbb{Q}_p$-subspace of the same dimension generated by elements of $M^\dagger$. This proves the claim. □

**Corollary 5.3.** *For each $s \in \mathbb{Q}$, the functor $\mathbf{F\text{-}Isoc}^\dagger(K) \to \mathbf{F\text{-}Isoc}(K)$ induces an equivalence of categories of objects which are isoclinic of slope $s$.*

*Proof.* Full faithfulness follows from Lemma 5.2. Essential surjectivity follows from Lemma 2.4. □

We recover [2, Proposition 5.5].

**Corollary 5.4.** *For $M^\dagger \in \mathbf{F\text{-}Isoc}^\dagger(K)$, there exists a unique filtration*

$$0 = M_0^\dagger \subset \cdots \subset M_l^\dagger = M^\dagger$$

*with the property that each successive quotient $M_i^\dagger / M_{i-1}^\dagger$ is isoclinic (as an object of $\mathbf{F\text{-}Isoc}(K)$) of some slope $s_i$, and $s_1 < \cdots < s_l$ (this being the reverse of the HN filtration).*

*Proof.* This follows by repeated application of Lemma 5.2. □

**Corollary 5.5.** *For $M^\dagger \in \mathbf{F\text{-}Isoc}^\dagger(K)$ restricting to $M_K \in \mathbf{F\text{-}Isoc}(K)$, the natural map $H^1_\varphi(M^\dagger) \to H^1_\varphi(M_K)$ is surjective.*

*Proof.* Using Corollary 5.4 and the five lemma, we may reduce to the case where $M^\dagger$ is isoclinic of some slope $s$. If $s \neq 0$, then $H^1_\varphi(M_K) = 0$ and there is nothing to check. If $s = 0$, then $M^\dagger$ admits a basis on which $\varphi^{-1}$ acts via an invertible matrix over $W^\dagger(K)$; we may thus apply Lemma 5.1 to conclude. □

**Corollary 5.6.** *For $M \in \mathbf{F\text{-}Isoc}(\mathfrak{o}_K)$, the Newton polygon of $M$ at $z$ lies on or above the Newton polygon of $M$ at $\eta$, with the same endpoint.*

*Proof.* By Proposition 4.8, the Newton polygon of $M$ at $z$ agrees with the HN polygon of the restriction of $M$ to $\mathbf{Vec}(X_K)$. The claim thus follows by comparing the HN filtration of $M$ in $\mathbf{Vec}(X_K)$ with the filtration given by Corollary 5.4. □

The following is a form of [2, Corollary 5.7].

**Corollary 5.7.** *For $M^\dagger \in \mathbf{F\text{-}Isoc}^\dagger(K)$ restricting to $M_K \in \mathbf{F\text{-}Isoc}(K)$ and $\mathbf{v} \in H^0_\varphi(M_K)$, let $N^\dagger \in \mathbf{F\text{-}Isoc}^\dagger(K)$ be the smallest subobject of $M^\dagger$ such that $\mathbf{v}$ belongs to the restriction $N_K \in \mathbf{F\text{-}Isoc}(K)$. Then the Newton polygon of $N^\dagger$ has one slope equal to $0$ and the others all negative.*

*Proof.* We may assume that $N^\dagger = M^\dagger$. Set notation as in Corollary 5.4 and let $M_{i,K} \in \mathbf{F}\text{-}\mathbf{Isoc}(K)$ be the restriction of $M_i^\dagger$. Since $N^\dagger = M^\dagger$, $\mathbf{v} \notin M_{l-1,K}$ and so $s_l = 0$. By Corollary 5.3, the image of $\mathbf{v}$ in $M_{l,K}/M_{l-1,K}$ is the restriction of a subobject of $M_l^\dagger/M_{l-1}^\dagger$; since $N^\dagger = M^\dagger$, this is only possible if $\operatorname{rank}(M_l^\dagger/M_{l-1}^\dagger) = 1$. □

**Remark 5.8.** With notation as in Corollary 5.7, suppose that $M^\dagger = M_1^{\dagger\vee} \otimes M_2^\dagger$. For each $\mu \in \mathbb{Q}$, for $i \in \{1,2\}$, let $P_{i,\mu}^\dagger$ be the step of the filtration of $M_i^\dagger$ provided by Corollary 5.4 with the property that every slope of $P_{i,\mu}^\dagger$ is $\leq \mu$ whereas every slope of $M_i^\dagger/P_{i,\mu}^\dagger$ is $> \mu$. Then the submodule $N^\dagger$ is contained in the image of $\bigoplus_{\mu\in\mathbb{Q}} P_{1,\mu}^{\dagger\vee} \otimes P_{2,\mu}^\dagger$ in $M^\dagger$.

## 6. Extending the slope filtration

**Lemma 6.1.** *For all $s > 0$, the map $H^1_\varphi(Y_K \setminus \{x_\varpi\}, \mathcal{O}(s)) \to H^1_\varphi(Y_K \setminus \{x_p, x_\varpi\}, \mathcal{O}(s))$ is injective.*

*Proof.* We follow [11, Proposition 3.3.7(b1)]. Write $s = \frac{r}{d}$ in lowest terms; then the claim is equivalent to showing that

$$\operatorname{coker}(1 - p^r\varphi^d, H^0(Y_K \setminus \{x_\varpi\}), \mathcal{O}) \to \operatorname{coker}(1 - p^r\varphi^d, H^0(Y_K \setminus \{x_p, x_\varpi\}), \mathcal{O})$$

is surjective. This follows by observing that

$$H^0(Y_K \setminus \{x_\varpi\}, \mathcal{O}) + [\varpi]H^0(Y_K \setminus \{x_p\}, \mathcal{O}) \to H^0(Y_K \setminus \{x_p, x_\varpi\}, \mathcal{O})$$

is surjective, $1 - p^r\varphi^d$ acts on $[\varpi]H^0(Y_K \setminus \{x_p\}, \mathcal{O})$, and the latter map admits a section given by

$$x \mapsto \sum_{i=0}^\infty p^{ir}\varphi^{id}(x).$$

This proves the claim. □

**Proposition 6.2.** *Suppose that $(r,s) \in \mathbb{Z}\times\mathbb{Z}$ is a vertex of the Newton polygon of $M \in$ $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ at both $\eta$ and $z$. Then $M$ splits uniquely as $M_1 \oplus M_2$ in such a way that the right endpoint of the Newton polygon of $M_2$ at every point of $X$ equals $(r,s)$.*

*Proof.* In light of Lemma 3.6, this can be deduced using results of [9] (again keeping in mind the discrepancy in sign conventions): namely, we obtain (a) from [9, Theorem 2.3.1] and (b) from [9, Theorem 2.4.2, Theorem 2.5.1]. We instead give a self-contained proof in our present setup.

Let $M^\dagger \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ be the restriction of $M$. By Corollary 5.4, we have an exact sequence

$$0 \to M_2^\dagger \to M^\dagger \to M^\dagger/M_2^\dagger \to 0$$

where the right endpoint of the Newton polygon of $M_2^\dagger$ equals $(r,s)$. We may then restrict $M_2^\dagger$ to $\mathbf{Vec}(X_K)$, apply Proposition 4.8 to promote to $\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p\})$, and glue with $M_2^\dagger$ to obtain $M_2 \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ of the desired form.

Let

$$0 \to \mathcal{E}_2 \to \mathcal{E} \to \mathcal{E}/\mathcal{E}_2 \to 0$$

be the corresponding exact sequence in $\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_\varpi\})$. The condition on $(r,s)$ ensures that $\mathcal{E}_2 \to \mathcal{E}$ remains saturated when this morphism is extended across $x_\varpi$; consequently, the sequence remains exact when taking global sections. That is, $M/M_2 \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$.

Meanwhile, by Lemma 4.5 and Proposition 4.8, in $\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p\})$ we also have an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}/\mathcal{E}_1 \to 0$$

where the right endpoint of the HN polygon of $\mathcal{E}/\mathcal{E}_1$ equals $(r, s)$. Consequently, the surjection $M \to M/M_2$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ admits a splitting in $\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_p\})$.

It remains to show that this splitting extends across $x_p$. For this, we may check the claim after replacing $M$ with an exterior power so as to reduce to the case where $\operatorname{rank} \mathcal{E}_1 = 1$. We may then restrict to $Y_K \setminus \{x_\varpi\}$, filter as per Corollary 5.4, and repeatedly apply Lemma 6.1 to conclude. □

**Definition 6.3.** We say that an object $M$ of $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ has *constant Newton polygon* if the Newton polygons of $M$ at $\eta$ and $z$ coincide. Note that this does *not* imply that $M$ is isoclinic.

**Corollary 6.4.** *The restriction functor from objects to* $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ *with constant Newton polygon to objects of* $\mathbf{F}\text{-}\mathbf{Isoc}(K)$ *is an equivalence. In particular, by Lemma 2.4, every object in either category splits uniquely as a direct sum of isoclinic objects.*

*Proof.* Essential surjectivity follows from Lemma 2.4, so it suffices to check full faithfulness. It suffices to check that for $M \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ with the same Newton polygons at $\eta$ and $z$, the natural map $H^0_\varphi(M) \to H^0_\varphi(M^\dagger)$ is an isomorphism. By Proposition 6.2, this further reduces to the case where $M$ is isoclinic of some slope $s$; in fact both spaces are zero unless $s = 0$, in which case we may apply Corollary 5.3. □

## 7. Extension of morphisms

We now follow the approach of [2] to give a criterion for descending morphisms from $\mathbf{F}\text{-}\mathbf{Isoc}(K)$ to $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$.

**Proposition 7.1.** *Choose* $M \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ *restricting to* $M^\dagger \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ *and* $M_K \in \mathbf{F}\text{-}\mathbf{Isoc}(K)$. *Choose* $\mathbf{v} \in H^0_\varphi(M_K)$ *and define* $N^\dagger$ *as in Corollary 5.7. Then the following are equivalent.*

(a) *We have* $\mathbf{v} \in H^0_\varphi(M)$.
(b) *We have* $\mathbf{v} \in H^0_\varphi(M^\dagger)$.
(c) *The unique object* $N \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ *restricting to* $N^\dagger \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ *(Corollary 4.9) has the property that* $0$ *occurs as a slope of* $M$ *at* $z$.
(d) *The exact sequence*

$$0 \to N_0^\dagger \to N^\dagger \to \mathcal{O} \to 0 \tag{7.2}$$

*provided by Corollary 5.4 and Corollary 5.7 splits in* $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$.

*Proof.* The equivalence of (a) and (b) is given by Corollary 4.9. If (a) or (b) holds, then $\operatorname{rank}(N^\dagger) = 1$ and so (c) is evident.

Let

$$0 \to \mathcal{E}_0 \to \mathcal{E} \to \mathcal{O} \to 0$$

be the exact sequence in $\mathbf{F}\text{-}\mathbf{Vec}(Y_K \setminus \{x_\varpi\})$ corresponding to (7.2). By Corollary 4.9 and Corollary 5.6, the HN polygon of $\mathcal{E}_0$ also has all slopes negative.

Now suppose that (c) holds. Then the map $\mathcal{E} \to \mathcal{O}$ is split canonically by the HN filtration of $\mathcal{E}$ over $Y_K \setminus \{x_p, x_\varpi\}$, and hence over $Y_K \setminus \{x_p\}$ by Lemma 4.7. By this plus Lemma 4.10, (7.2) is the restriction of a sequence

$$0 \to N_0 \to N \to \mathcal{O} \to 0$$

in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$; by Corollary 4.9, the canonical splitting of (7.2) also promotes to $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$, so we have $N \cong N_0 \oplus \mathcal{O}$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$, yielding (d).

Finally, suppose that (d) holds. Then the image of $\mathbf{v}$ under the projection $N_0^\dagger \to N_{0,K}$ must vanish because 0 does not occur as a slope of $N_{0,K}$ over $\eta$, so $\mathbf{v}$ belongs to the other summand. By Corollary 6.4, we deduce (b). □

## 8. Counterexamples

We describe some counterexamples that limit our ability to prove stronger results than what we have already described.

**Example 8.1.** Choose $M \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ of rank 2 with the action of $\varphi$ on some basis given by $\begin{pmatrix} 0 & 1 \\ p^{-1} & [\varpi] \end{pmatrix}$. Then the Newton polygon of $M$ at $\eta$ has slopes $0, 1$ whereas the Newton polygon at $z$ has slopes $1/2, 1/2$.

Let $M^\dagger \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ be the restriction of $M$. By Corollary 5.4, $M^\dagger$ admits a submodule $N^\dagger$ of rank 1 and slope 0, which by Corollary 5.3 and Lemma 2.4 is isomorphic to $\mathcal{O}$; by the same token, $M^\dagger/\mathcal{O}$ is isomorphic to $\mathcal{O}(1)$. We thus have an exact sequence

$$0 \to \mathcal{O} \to M^\dagger \to \mathcal{O}(1) \to 0 \tag{8.2}$$

which by Corollary 4.9 is the restriction of a sequence

$$0 \to \mathcal{O} \to M \to \mathcal{O}(1) \dashrightarrow 0. \tag{8.3}$$

It will follow from that the latter sequence is exact at the left, but it cannot be exact as the right as this would contradict the computation of the Newton polygon at $z$. More concretely, the map $\mathcal{O} \to M$ restricts to zero at $x_\varpi$, so its cokernel is not projective.

A concrete consequence of this is that the category $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ is abelian (because it is equivalent to $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$), but the formation of kernel and cokernels does not commute with the forgetful functor to $W(\mathfrak{o}_K)[p^{-1}]$-modules.

**Example 8.4.** With notation as in Example 8.1, let $M_K \in \mathbf{F}\text{-}\mathbf{Isoc}(K)$ be the restriction of $M$. By Lemma 2.4, there exists a nonzero morphism $\mathcal{O}(1) \to M_K$, but this morphism does not lift to $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ or $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$. Otherwise, the sequence (8.2) would split, which by Lemma 4.10 would yield a splitting of (8.3). In other words, the map $H^0_\varphi(M^\dagger(-1)) \to H^0_\varphi(M_K(-1))$ is not surjective.

**Example 8.5.** With notation as in Example 8.1, put $N := \mathcal{O} \oplus \mathcal{O}(1) \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ and let $N_K \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ be the restriction of $N$. Then Example 8.1 and Example 8.4 together yield an isomorphism $M_K \cong N_K$ which does not come from an isomorphism $M \cong N$. Similarly, there exist automorphisms of $M_K$, and idempotent endomorphisms of $M_K$, which do not preserve $M$.

By the same token, the sequence (8.2) is not split in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ or $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$, but it becomes split in $\mathbf{F}\text{-}\mathbf{Isoc}(K)$. That is, for $N := \mathcal{O}(-1) \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ restricting to

$N^\dagger \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ and $N_K \in \mathbf{F}\text{-}\mathbf{Isoc}(K)$, the map $H^1_\varphi(N) = H^1_\varphi(N^\dagger) \to H^1_\varphi(N_K)$ is not injective.

**Remark 8.6.** Choose $M_0 \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ of rank 6 whose Newton polygon at $\eta$ has slopes $0(\times 2), \frac{1}{2}(\times 2), 1(\times 2)$ and whose Newton polygon at $z$ has slopes $\frac{1}{3}(\times 3), \frac{2}{3}(\times 3)$. For example, there exists an abelian scheme $A$ over $\mathfrak{o}_K$ whose crystalline Dieudonné module has this form.

Now put $M = M_0(-\frac{1}{2})$. Then the Newton polygon of $M$ at $\eta$ has slopes $-\frac{1}{2}(\times 4), 0(\times 4), \frac{1}{2}(\times 4)$ while the Newton polygon at $z$ has slopes $-\frac{1}{6}(\times 6), \frac{1}{6}(\times 6)$.

By Lemma 2.4, we can find a nonzero $\mathbf{v} \in H^0_\varphi(M_K)$. Let $M^\dagger \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ be the restriction of $M$ and define $N^\dagger \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ as in Corollary 5.7. Then the Newton polygon of $N^\dagger$ has the slope 0 with multiplicity 1 and the slope $-\frac{1}{2}$ with some multiplicity in $\{0, \dots, 4\}$; in particular, $1 \leq \operatorname{rank}(N^\dagger) \leq 5$.

By Corollary 4.9, $N^\dagger \to M^\dagger$ descends to an inclusion $N \to M$ in $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$, but we cannot have $N/M \in \mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ because the restriction of $M$ to $\mathbf{F}\text{-}\mathbf{Isoc}(k)$ has no subobject with rank in $\{1, \dots, 5\}$.

## 9. Global consequences

We next translate some of our preceding results into global statements using arc-descent. The key geometric tool is the following lemma.

**Lemma 9.1.** *Every affine scheme admits an arc-covering of the form* $\operatorname{Spec} \prod_{i \in I} R_i$ *where each* $R_i$ *is a complete height-*1 *AIC valuation ring.*

*Proof.* We follow the proof of [1, Proposition 3.30]. Let $\operatorname{Spec} R$ be an affine scheme. We are looking for a homomorphism $R \to \prod_{i\in I} R_i$, in which each $R_i$ is a complete height-1 valuation ring, such that for every homomorphism $R \to V$ to a valuation ring of height $\leq 1$, there exists a valuation ring $V'$ containing $V$ such that $R \to V \to V'$ factors through $\prod_{i \in I} R_i$. By [1, Lemma 3.29], we can choose a set $S$ of morphisms $R \to W$ to AIC valuation rings such that any morphism $R \to V$ as above factors through some $W$. Let $I$ be the subset of $S$ consisting of morphisms $R \to W$ in which $W$ has height $\leq 1$. For each $i = (R \to W) \in I$, if $W$ is of height 1, let $R_i$ be the completion of $W$; otherwise, let $R_i$ be a completed algebraic closure of $W((t))$.

We claim that $R \to \prod_i R_i$ has the desired effect. To see this, start with any morphism $R \to V$ as above. By construction, there is an index $i = (R \to W) \in I$ such that $R \to V$ factors through $W$. If $V$ is of height 1, then the completion $V'$ of $V$ has the property that $R \to V'$ factors through $R_i$. Otherwise, let $V'$ be a completed algebraic closure of $V((t))$; then $R \to V'$ factors through $R_i$ via a map sending $t$ to $t$. ☐

**Theorem 9.2.** *For $X$ a perfect $\mathbb{F}_p$-scheme and $M \in \mathbf{F}\text{-}\mathbf{Isoc}(X)$, the function taking $x \in X$ to the Newton polygon of $M$ at $x$ is upper semicontinuous. Moreover, each level set is a locally closed subspace whose Zariski closure is locally the zero set of a finitely generated ideal.*

*Proof.* We may assume at once that $X = \operatorname{Spec} R$ is affine. In this case, we may deduce the claim from [9, Theorem 2.3.1] provided that the underlying module of $M$ is free. By Lemma 3.6, the latter holds locally on $X$, which suffices. ☐

**Remark 9.3.** Using Proposition 6.2, we can recover a weaker form of Proposition 9.2: the Newton polygon increases under specialization.

**Theorem 9.4.** *Let $X$ be a perfect $\mathbb{F}_p$-scheme. Suppose that $M \in \mathbf{F}\text{-}\mathbf{Isoc}(X)$ has the property that some point $(r,s) \in \mathbb{Z} \times \mathbb{Z}$ occurs as a vertex of the Newton polygon of $M$ at every point of $X$. Then $M$ splits uniquely as $M_1 \oplus M_2$ in such a way that the right endpoint of the Newton polygon of $M_2$ at every point of $X$ equals $(r,s)$.*

*Proof.* By arc-descent plus Lemma 9.1, we may reduce to the case where $X = \operatorname{Spec} R$ and $R = \prod_{i \in I} R_i$ is a product of complete height-1 AIC valuation rings. The claim then reduces immediately to the case $R = \mathfrak{o}_K$. Following [17, Lemma 6.9], this can again be deduced from [9, Theorem 2.4.2] (for the filtration) and [9, Theorem 2.5.1] (for the splitting); alternatively, we may apply Proposition 6.2(b). □

**Remark 9.5.** Let $U \to X$ be an open immersion of perfect $\mathbb{F}_p$-schemes with dense image. On account of Example 8.4, we cannot show that the restriction functor $\mathbf{F}\text{-}\mathbf{Isoc}(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(U)$ is fully faithful.

This has a consequence for replacing smooth schemes over a perfect field with perfect $\mathbb{F}_p$-schemes in the analogue of Drinfeld's lemma for isocrystals [17]: some arguments must be modified, notably the proof of the relative Dieudonné–Manin decomposition [17, Theorem 7.3] (for which see Theorem 10.17) and numerous arguments in [17, §10] (which we do not treat here).

## 10. Relative Dieudonné–Manin

We give an analogue for perfect schemes of the relative Dieudonné–Manin decomposition stated in [17, Theorem 7.3] for smooth schemes. To simplify notation, we only treat the case of two-term products; this is sufficient to recover the corresponding results for longer (finite) products.

**Definition 10.1.** Let $X_1 = \operatorname{Spec} R_1, X_2 = \operatorname{Spec} R_2$ be perfect affine $\mathbb{F}_p$-schemes; put $X := X_1 \times X_2 = \operatorname{Spec} R$ for $R := R_1 \otimes_{\mathbb{F}_p} R_2$; and let $\varphi_1, \varphi_2 \colon X \to X$ be the morphisms induced by absolute Frobenius on $X_1$ and $X_2$, respectively. We then have $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = \varphi$ (the absolute Frobenius on $X$).

By a $\Phi$-*isocrystal* on $X$, we mean a finite projective $W(R)[p^{-1}]$-module equipped with isomorphisms $\varphi_1^* M \cong M$, $\varphi_2^* M \cong M$ which "commute" in the sense that composing them both ways yields the same isomorphism of $(\varphi_1 \circ \varphi_2)^* M \cong (\varphi_2 \circ \varphi_1)^* M$ with $M$. These form a tensor category $\mathbf{\Phi}\text{-}\mathbf{Isoc}(X)$ in which the morphisms are $\langle \varphi_1, \varphi_2 \rangle$-equivariant $W(R)[p^{-1}]$-module morphisms.

The categories $\mathbf{\Phi}\text{-}\mathbf{Isoc}(X)$ form a stack for the Zariski topology, the étale topology, and the arc-topology; we thus obtain corresponding categories when $X_1$ and $X_2$ are not necessarily affine. By forgetting the separate actions of $\varphi_1, \varphi_2$ and retain only the action of $\varphi$, we obtain a natural functor $\mathbf{\Phi}\text{-}\mathbf{Isoc}(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(X)$.

Given objects $\mathcal{E}_i \in \mathbf{F}\text{-}\mathbf{Isoc}(X_i)$ for $i = 1, 2$, the external product $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ belongs to $\mathbf{\Phi}\text{-}\mathbf{Isoc}(X)$. In case $\mathcal{E}_2$ is the unit object, we refer to this external product as the *pullback* of $\mathcal{E}_1$.

We start with the following "double Dieudonné–Manin decomposition" result.

**Proposition 10.2.** *For $X_1 := \operatorname{Spec} k_1, X_2 := \operatorname{Spec} k_2$ with $k_1, k_2$ algebraically closed fields, every object of $\mathbf{\Phi}$-$\mathbf{Isoc}(X)$ decomposes uniquely as a direct sum*

$$\bigoplus_{d_1,d_2 \in \mathbb{Q}} \mathcal{E}_{d_1,d_2}$$

*in which for $d_1, d_2 \in \mathbb{Q}$ with least common denominator $s$, $\mathcal{E}_{d_1,d_2}$ is obtained by pulling back a finite-dimensional $\mathbb{Q}_p$-vector space equipped with commuting endomorphisms $F_1, F_2$ such that $F_i^s = p^{-d_i s}$, which then give the actions of $\varphi_1, \varphi_2$ on the pullback. (This vector space may be recovered from $\mathcal{E}_{d_1,d_2}$ as the joint kernel of $\varphi_i^s - p^{-d_i s}$.)*

*Proof.* Apply [17, Corollary 7.4]. □

**Corollary 10.3.** *For $\mathcal{E} \in \mathbf{\Phi}$-$\mathbf{Isoc}(X)$ and $x \in X$ a point, the Newton polygon of the image of $\mathcal{E}$ under $\mathbf{\Phi}$-$\mathbf{Isoc}(X) \to \mathbf{F}$-$\mathbf{Isoc}(X) \to \mathbf{F}$-$\mathbf{Isoc}(x)$ depends only on the images of $x$ in $X_1$ and $X_2$.*

*Proof.* This immediately reduces to the case where both $X_1$ and $X_2$ are geometric points, in which case we may read off the claim from Proposition 10.2. Alternatively, see [17, Theorem 6.6]. □

**Remark 10.4.** The reader interested in pursuing the relationship between $F$-isocrystals and isoshtukas is hereby warned that there is no analogue of Proposition 10.2 for vector bundles on $X_K$: not every vector bundle on $X_K \times_{\mathbb{Q}_p} X_K$ can be expressed in terms of external products $\mathcal{O}(s_1) \boxtimes \mathcal{O}(s_2)$. See [18] for further discussion.

**Definition 10.5.** For $X_1, X_2$ affine and $M \in \mathbf{\Phi}$-$\mathbf{Isoc}(X)$, define the groups $H^i_\Phi(M)$ for $i = 0, 1, 2$ as the cohomology groups of the totalization of the complex

$$\begin{array}{ccc} M & \xrightarrow{\varphi_1 - 1} & M \\ {\scriptstyle \varphi_2 - 1}\downarrow & & \downarrow{\scriptstyle \varphi_2 - 1} \\ M & \xrightarrow{\varphi_1 - 1} & M. \end{array}$$

The groups $H^0$ and $H^1$ again compute internal Homs and Ext groups in the category $\mathbf{\Phi}$-$\mathbf{Isoc}(X)$. (The group $H^2$ computes a higher Yoneda extension group, but we will not use this.)

**Lemma 10.6.** *Let $R$ be a perfect $\mathbb{F}_p$-algebra and let $\ell$ be an algebraically closed field of characteristic $p$. Then the sequence*

$$0 \to W(R) \to W(R \otimes \ell) \stackrel{\varphi_2 - 1}{\to} W(R \otimes \ell) \to 0$$

*is exact.*

*Proof.* For $R = \mathbb{F}_p$ this is the standard Artin–Schreier exact sequence. The general case follows by identifying $W(R \otimes \ell)$ with the $p$-adic completion of the tensor product $W(R) \otimes_{\mathbb{Z}_p} W(\ell)$. □

**Lemma 10.7.** *For $X_1 := \operatorname{Spec} \mathfrak{o}_K$ and $X_2 := \operatorname{Spec} \ell$ with $\ell$ algebraically closed, for $M \in \mathbf{F}$-$\mathbf{Isoc}(\mathfrak{o}_K)$ pulling back to $M_\Phi \in \mathbf{\Phi}$-$\mathbf{Isoc}(X)$, the natural maps $H^i_\varphi(M) \to H^i_\Phi(M_\Phi)$ are isomorphisms for $i = 0, 1$ and $H^2_\Phi(M_\Phi) = 0$.*

*Proof.* By Lemma 10.6, the sequence

$$0 \to M \to M_\Phi \stackrel{\varphi_2 - 1}{\to} M_\Phi \to 0$$

is exact; this immediately yields the claim. □

We next introduce an analogue of the category $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$.

**Definition 10.8.** Take $X_1 := \mathfrak{o}_K$ and $X_2 := \operatorname{Spec} \ell$ with $\ell$ algebraically closed. Set $L := W(\ell)[p^{-1}]$ and $Y_{K,\ell} := Y_K \times_{\mathbb{Q}_p} L$. Put $X_K := (\operatorname{Spec} K) \times X_2$ and let $W^\dagger(K \otimes \ell)$ be the stalk of $\mathcal{O}_{Y_{K,\ell}}$ at the zero locus of $p$ (identified with a subring of $W(K \otimes \ell)$). We define the category $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_K)$ to consist of finite projective $W^\dagger(K \otimes \ell)[p^{-1}]$-modules equipped with isomorphisms with their pullbacks along $\varphi$. We define the category $\mathbf{\Phi}\text{-}\mathbf{Isoc}^\dagger(X_K)$ to consist of finite projective $W^\dagger(K \otimes \ell)[p^{-1}]$-modules equipped with commuting isomorphisms with their pullbacks along $\varphi_1$ and $\varphi_2$ (in the same sense as in Definition 10.1). There is then a natural functor $\mathbf{\Phi}\text{-}\mathbf{Isoc}^\dagger(X_K) \to \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_K)$.

**Remark 10.9.** In the following discussion, we will study the situation of Definition 10.8 in parallel with our earlier development of properties of $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$ in terms of the Fargues–Fontaine curve $X_K$. However, there is one key difference that complicates the analogy: there is no counterpart of Lemma 4.5 for vector bundles on $X_K \times_{\mathbb{Q}_p} L$.

We have the following partial analogue of Corollary 4.9. See also Corollary 10.16.

**Lemma 10.10.** *With notation as in Definition 10.8, the restriction functor* $\mathbf{\Phi}\text{-}\mathbf{Isoc}(X) \to \mathbf{\Phi}\text{-}\mathbf{Isoc}^\dagger(X_K)$ *is fully faithful.*

*Proof.* This follows by adapting the proof of Corollary 4.9. Further details to follow. □

We have the following analogue of Lemma 5.1.

**Lemma 10.11.** *Let $M^\dagger$ be a finite projective $W^\dagger(K \otimes \ell)$-module equipped with a semilinear action of $\varphi^{-d}$ for some positive integer $d$. Equip $M_K := M^\dagger \otimes_{W^\dagger(K \otimes \ell)} W(K \otimes \ell)$ with the induced action of $\varphi^{-d}$. Then the natural map* $\operatorname{coker}(\varphi^{-d} - 1, M^\dagger) \to \operatorname{coker}(\varphi^{-d} - 1, M_K)$ *is surjective.*

*Proof.* By adding a complementary summand, we may reduce to the case where the underlying module of $M^\dagger$ is free. Then the proof of Lemma 5.1 carries over. □

This yields the following analogue of Lemma 5.2.

**Lemma 10.12.** *With notation as in Definition 10.8, suppose that $M^\dagger \in$* $\mathbf{\Phi}\text{-}\mathbf{Isoc}^\dagger(X_K)$ *restricts to $M_K \in$* $\mathbf{\Phi}\text{-}\mathbf{Isoc}(X_K)$ *and that its image in* $\mathbf{F}\text{-}\mathbf{Isoc}(X_K)$ *has smallest Newton slope $s = \frac{r}{d}$ in lowest terms. Then every $\mathbf{v} \in M_K$ with $\varphi^d(\mathbf{v}) = p^{-r}\mathbf{v}$ belongs to $M^\dagger$.*

*Proof.* The slope condition means that we can choose module generators $\mathbf{e}_1, \dots, \mathbf{e}_n$ of $M^\dagger$ on which $p^{-r}\varphi^{-d}$ acts via a matrix $A$ over $W^\dagger(K \otimes \ell)$ whose reduction modulo $p$ has rank equal to the $\mathbb{Q}_p$-dimension of the $p^{-r}\varphi^{-d}$-fixed subspace of $M_K$. By adding a complementary summand, we may reduce to the case where $\mathbf{e}_1, \dots, \mathbf{e}_n$ form a basis of $M^\dagger$. Then the proof of Lemma 5.2 carries over, using Lemma 10.11 in place of Lemma 5.1. □

This in turn yields the following analogue of Corollary 5.3.

**Corollary 10.13.** *For each $s \in \mathbb{Q}$, the functor $\mathbf{\Phi}\text{-}\mathbf{Isoc}^\dagger(X_K) \to \mathbf{\Phi}\text{-}\mathbf{Isoc}(X_K)$ induces an equivalence of categories of objects which, as objects of $\mathbf{F}\text{-}\mathbf{Isoc}(X_K)$, are isoclinic of slope $s$.*

*Proof.* Full faithfulness follows from Lemma 10.12. Essential surjectivity follows from Proposition 10.2. □

We also obtain an analogue of Corollary 5.4.

**Corollary 10.14.** *With notation as in Definition 10.8, for $M^\dagger \in \mathbf{\Phi}\text{-}\mathbf{Isoc}^\dagger(X_K)$, there exists a unique filtration*

$$0 = M_0^\dagger \subset \cdots \subset M_l^\dagger = M^\dagger$$

*with the property that each successive quotient $M_i^\dagger/M_{i-1}^\dagger$, as an object of $\mathbf{F}\text{-}\mathbf{Isoc}(X_K)$, is isoclinic of some slope $s_i$, and $s_1 < \cdots < s_l$.*

*Proof.* This follows by repeated application of Lemma 10.12. □

We can now establish a relative Dieudonné–Manin decomposition in the local setting.

**Lemma 10.15.** *With notation as in Definition 10.8, every object $M^\dagger \in \mathbf{\Phi}\text{-}\mathbf{Isoc}^\dagger(X_K)$ decomposes uniquely as a direct sum $\bigoplus_{d \in \mathbb{Q}} M_d^\dagger$ in which for $d = \frac{r}{s}$ in lowest terms, $M_d^\dagger$ is obtained by pulling back an object of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ equipped with an endomorphism $F_2$ such that $F_2^s = p^{-r}$, which then gives the action of $\varphi_2$ on the pullback. (The latter may be recovered from $M_d^\dagger$ as the kernel of $\varphi^s - p^{-r}$.)*

*Proof.* Let $M_K$ be the image of $M^\dagger$ in $\mathbf{\Phi}\text{-}\mathbf{Isoc}(X_K)$ By applying Proposition 10.2, we obtain a corresponding direct sum decomposition $M_K = \bigoplus_{d \in \mathbb{Q}} M_{K,d}$. We first check that this decomposition descends to $M$.

Consider the filtration of $M^\dagger$ given by Corollary 10.14, and let $M_{i,K} \in \mathbf{\Phi}\text{-}\mathbf{Isoc}(X_K)$ be the restriction of $M_i^\dagger$. Then the decomposition of $M_K$ must preserve each $M_{i,K}$, so we get an induced decomposition of $M_{i,K}/M_{i-1,K}$. This decomposition preserves $M_i^\dagger/M_{i-1}^\dagger$ by Corollary 10.13, so by induction on $i$ it also preserves $M_i^\dagger$ for each $i$. The case $i = l$ yields the conclusion that the decomposition $M_K = \bigoplus_{d \in \mathbb{Q}} M_{K,d}$ induces a decomposition $M^\dagger = \bigoplus_{d \in \mathbb{Q}} M_d^\dagger$.

For the remainder of the proof, we may assume that $M^\dagger = M_d^\dagger$ for some $d \in \mathbb{Q}$. It remains to check that $M^\dagger$ arises by pulling back an object of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ equipped with an endomorphism $F_2$ such that $F_2^s = p^{-r}$. We first treat the case $d = 0$. Again, we know that $M_K$ arises by pulling back an object $N_K$ of $\mathbf{F}\text{-}\mathbf{Isoc}(K)$ equipped with an endomorphism $F_2$ such that $F_2^s = p^{-r}$, which we may recover from $M_K$ as the kernel of $\varphi_2^s - p^{-r}$. By Corollary 10.14 again, we have a filtration of $M^\dagger$ which is stable under $\varphi_2$; the corresponding filtration of $M_K$ induces a filtration $0 = N_{0,K} \subset \cdots \subset N_{l,K} = N_K$ of $N_K$. For each $i$, by Corollary 10.13, $N_{i,K}/N_{i-1,K}$ is the restriction of an object of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ equipped with an endomorphism $F_2$ such that $F_2^s = p^{-r}$ which pulls back to $M_i^\dagger/M_{i-1}^\dagger$. By Corollary 4.9 (to replace $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ with $\mathbf{F}\text{-}\mathbf{Isoc}(\mathfrak{o}_K)$) and Lemma 10.7 (with $i = 1$), we may deduce by induction on $i$ that $M_i^\dagger$ is the pullback of an object of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$ equipped with an endomorphism $F_2$ such that $F_2^s = p^{-r}$. The case $i = l$ yields the claim.

Suppose now that $M^\dagger = M_d^\dagger$ for some arbitrary $d \in \mathbb{Q}$. By the previous paragraph, $M^\dagger \otimes \mathcal{O}(-d)$ is the pullback of an object of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(K)$; from this we recover the claim. □

As a byproduct, we obtain a full analogue of Corollary 4.9.

**Corollary 10.16.** *With notation as in Definition 10.8, the restriction functor* $\mathbf{\Phi}\text{-}\mathbf{Isoc}(X) \to \mathbf{\Phi}\text{-}\mathbf{Isoc}^{\dagger}(X_K)$ *is an equivalence of categories.*

*Proof.* With notation as in Definition 10.8, the functor is fully faithful by Lemma 10.10. It is essentially surjective by Lemma 10.15 plus Corollary 4.9. □

We finally end up with an analogue of [17, Theorem 7.3].

**Theorem 10.17.** *For* $X_1$ *arbitrary, take* $X_2 := \operatorname{Spec} \ell$ *with* $\ell$ *algebraically closed. Then every object* $\mathcal{E} \in \mathbf{\Phi}\text{-}\mathbf{Isoc}(X)$ *decomposes uniquely as a direct sum* $\bigoplus_{d \in \mathbb{Q}} \mathcal{E}_d$ *in which for* $d = \frac{r}{s}$ *in lowest terms,* $\mathcal{E}_d$ *is obtained by pulling back an object of* $\mathbf{F}\text{-}\mathbf{Isoc}(X_1)$ *equipped with an endomorphism* $F_2$ *such that* $F_2^s = p^{-r}$*, which then gives the action of* $\varphi_2$ *on the pullback. (The latter may be recovered from* $\mathcal{E}_d$ *as the kernel of* $\varphi^s - p^{-r}$*.)*

*Proof.* By arc-descent plus Lemma 9.1, we may reduce to the case $X_1 = \operatorname{Spec} \mathfrak{o}_K$. This is covered by Lemma 10.15. □

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