Quiz 8 Solutions - Calculus 1A

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1a. (6 *points*) In the following table, match each property in the first column with the corresponding property in the second column and also with the corresponding property in the third column. Note that $f(x)$ is assumed to be twice-differentiable ($f''(x)$ exists).

	$f(x)$	$f'(x)$	$f''(x)$
(1)	increasing	> 0	Not determined
(2)	decreasing	< 0	Not determined
(3)	transitions from increasing to decreasing	$= 0$	< 0
(4)	transitions from decreasing to increasing	$= 0$	> 0
(5)	graph concave up	increasing	> 0
(6)	graph concave down	decreasing	< 0
(7)	transitions from concave up to concave down	transitions from increasing to decreasing	$= 0$
(8)	transitions from concave down to concave up	transitions from decreasing to increasing	$= 0$

1b. (2 *points*) Of the properties listed in the first column of the above table, which ones correspond to:

$$f(x) \text{ has a local min} \longleftrightarrow (4)$$
$$f(x) \text{ has a local max} \longleftrightarrow (3)$$
$$f(x) \text{ has an inflection point} \longleftrightarrow (7), (8)$$

2. Let

$$f(x) = \frac{\ln x}{\sqrt{x}}$$

2a. (1 *point*) What is the domain of $f(x)$?
Solution: $\sqrt{x}$ not defined for $x < 0$, while $1/\sqrt{x}$ not defined for $x \leq 0$, and finally $\ln x$ not defined for $x \leq 0$: so $\text{Domain}\big(f\big) = (0, \infty)$.

2b. (3 *points*) What are the points $(x, f(x))$ at which $f'(x) = 0$, and determine if these points are (local) maxima or minima (or neither)?
Solution:

$$\begin{aligned} f'(x) &= \left(x^{-\frac{1}{2}} \ln x\right)' \\ &= (x^{-\frac{1}{2}})'(\ln x) + (x^{-\frac{1}{2}})(\ln x)' \\ &= (-\tfrac{1}{2}x^{-\frac{3}{2}})(\ln x) + (x^{-\frac{1}{2}})(x^{-1}) \\ &= (x^{-\frac{3}{2}})(-\tfrac{1}{2}\ln x + 1) \end{aligned}$$

Note that

$$\begin{aligned} f'(x) > 0 &\iff (-\tfrac{1}{2}\ln x + 1) > 0 \\ &\iff \tfrac{1}{2}\ln x < 1 \\ &\iff \ln x < 2 \\ &\iff x < e^2 \end{aligned}$$

Similarly, $f'(x) = 0$ when $x = e^2$ and $f'(x) < 0$ when $x > e^2$. The fact that $f'(x)$ transitions from positive to negative at $x = e^2$ means that $(e^2, 2e^{-1})$ is a local maximum. It is also an absolute maximum since $f(x)$ is differentiable on the open interval $(0, \infty)$ so there are no endpoints to check!
2c. (3 *points*) Determine the parts of the domain of $f(x)$ where its graph is concave up, and also where it is concave down.
Solution:

$$\begin{aligned} f''(x) &= \left((x^{-\frac{3}{2}})(-\tfrac{1}{2}\ln x + 1\right)' \\ &= (x^{-\frac{3}{2}})'(-\tfrac{1}{2}\ln x + 1) + (x^{-\frac{3}{2}})(-\tfrac{1}{2}\ln x + 1)' \\ &= (-\tfrac{3}{2}x^{-\frac{5}{2}})(-\tfrac{1}{2}\ln x + 1) + (x^{-\frac{3}{2}})(-\tfrac{1}{2}x^{-1}) \\ &= (x^{-\frac{5}{2}})(\tfrac{3}{4}\ln x - 2) \end{aligned}$$

Note that

$$\begin{aligned} f''(x) < 0 &\iff (\tfrac{3}{4}\ln x - 2) < 0 \\ &\iff \tfrac{3}{4}\ln x < 2 \\ &\iff \ln x < \tfrac{8}{3} \\ &\iff x < e^{\frac{8}{3}} \end{aligned}$$

Similarly, $f''(x) > 0$ when $x > e^{\frac{8}{3}}$. So the graph of $f(x)$ is concave up on $(e^{\frac{8}{3}}, \infty)$, and is concave down on $(0, e^{\frac{8}{3}})$.

3. (5 *points*) Show that the equation $f(x) = x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.
Solution: If $f(x)$ had more than one root on $[-2, 2]$, then in particular it would have two roots a and b with $-2 \leq a < b \leq 2$ and $f(a) = f(b) = 0$, and so by Rolle's Theorem there would exist a $c \in (a, b) \subset [-2, 2]$ with $f'(c) = 0$. On the other hand $f'(x) = 3x^2 - 15$, and since $f'(0) = -15$ and does not reach 0 until $|\ x| = \sqrt{5} > 2$, we have $f'(x) < 0$ when restricted to the interval $|\ x| \leq 2$; in particular $f'(x) \neq 0$ for $x \in (a, b) \subset [-2, 2]$. This contradiction to Rolle's Theorem (namely, that no such $c \in (a, b)$ could exist with $f'(c) = 0$) establishes the assertion that $f(x)$ could not have had two roots in the interval $[-2, 2]$.

Extra Credit. (1 *point*) For the $f(x)$ given in problem #2, what is $\lim_{x\to 0} f(x)$?
Solution: Since $\lim_{x\to 0} \ln(x) = -\infty$ and $\lim_{x\to 0} \sqrt{x} = 0^+$ we have an $\frac{-\infty}{0^+}$ situation which is *not* indeterminate:

$$\lim_{x\to 0} \frac{\ln x}{\sqrt{x}} = \frac{-\infty}{0^+} = -\infty$$