Quiz 5 - Calculus 1A

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Solutions

1. In each of the following questions you will prove that $(\cos\theta)' = -\sin\theta$
1a. (1 *points*) Write $\cos\theta = \sin(\frac{\pi}{2} - \theta)$ and use chain rule.
Solution:

$$\begin{aligned}
(\cos\theta)' &= \big(\sin(\tfrac{\pi}{2} - \theta)\big)' \\
\text{(chain rule)} &= \cos(\tfrac{\pi}{2} - \theta)(\tfrac{\pi}{2} - \theta)' \\
&= \cos(\tfrac{\pi}{2} - \theta)(-1) \\
\text{(since } \cos(\tfrac{\pi}{2} - \theta)) = \sin\theta) &= -\sin\theta
\end{aligned}$$

1b. (2 *points*) Write $\cos\theta = \sqrt{1-\sin^2\theta} = (1-\sin^2\theta)^{\frac{1}{2}}$ and use power and chain rule
Solution:

$$\begin{aligned}
(\cos\theta)' &= \big((1-\sin^2\theta)^{\frac{1}{2}}\big)' \\
\text{(chain rule)} &= \frac{1}{2}(1-\sin^2\theta)^{-\frac{1}{2}}(1-\sin^2\theta)' \\
\text{(power rule \& chain rule)} &= \frac{1}{2}(1-\sin^2\theta)^{-\frac{1}{2}}(-2\sin\theta\cos\theta) \\
\text{(since } \sin^2\theta + \cos^2\theta = 1) &= \frac{1}{2}(\cos^2\theta)^{-\frac{1}{2}}(-2\sin\theta\cos\theta) \\
&= -\sin\theta
\end{aligned}$$

1c. (2 *points*) Write $\cos^2\theta + \sin^2\theta = 1$ and use implicit differentiation.
Solution:

$$\begin{aligned}
\cos^2\theta + \sin^2\theta = 1 &\implies (\cos^2\theta + \sin^2\theta)' = 0 \\
\text{(power \& chain rule)} &\implies 2\cos\theta(\cos\theta)' + 2\sin\theta(\sin\theta)' = 0 \\
\text{(since we know } (\sin\theta)' = \cos\theta) &\implies 2\cos\theta(\cos\theta)' + 2\sin\theta(\cos\theta) = 0 \\
&\implies 2\cos\theta(\cos\theta)' = -2\sin\theta(\cos\theta) \\
&\implies (\cos\theta)' = -\sin\theta
\end{aligned}$$

2. In the following questions you will derive a formula for $\cos 2\theta$
2a. (2 *points*) Compute $(\sin 2\theta)'$ using chain rule.

Solution:

$$\begin{aligned}(\sin 2\theta)' &= (\cos 2\theta)(2\theta)' \\ &= 2(\cos 2\theta)\end{aligned}$$

2b. (2 *points*) Write $\sin 2\theta = 2\sin\theta\cos\theta$ and compute its derivative using product rule.
Solution:

$$\begin{aligned}(\sin 2\theta)' &= (2\sin\theta\cos\theta)' \\ &= 2\big((\sin\theta)(\cos\theta)' + (\sin\theta)'(\cos\theta)\big) \\ &= 2(\cos^2\theta - \sin^2\theta)\end{aligned}$$

2c. (1 *point*) Give a formula for $\cos 2\theta$ by combining your results from (2a) and (2b).
Solution: Equating the two results for $(\sin 2\theta)'$ obtained from (2a) and (2b) gives $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

3. (2 *points*) Starting with $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ compute its derivative using quotient rule
Solution: See page 214 of the text.

4. (3 *points*) Compute the following derivative:

$$\left(e^{e^{\tan\sqrt{x}}}\right)'$$

Solution:

$$\begin{aligned}\left(e^{e^{\tan\sqrt{x}}}\right)' &= \left(e^{e^{\tan\sqrt{x}}}\right)\left(e^{\tan\sqrt{x}}\right)' \\ (\text{since } (e^y)' = e^y \cdot y') &= \left(e^{e^{\tan\sqrt{x}}}\right)\left(e^{\tan\sqrt{x}}\right)\left(\tan\sqrt{x}\right)' \\ (\text{since } (\tan y)' = \sec^2 y \cdot y') &= \left(e^{e^{\tan\sqrt{x}}}\right)\left(e^{\tan\sqrt{x}}\right)\left(\sec^2\sqrt{x}\right)\left(\sqrt{x}\right)' \\ (\text{since } (\sqrt{y})' = \tfrac{1}{2\sqrt{y}} \cdot y') &= \left(e^{e^{\tan\sqrt{x}}}\right)\left(e^{\tan\sqrt{x}}\right)\left(\sec^2\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}\right)\end{aligned}$$

5. (3 *points*) Compute the following derivative:

$$\left(\frac{2\tan x}{1-\tan^2 x}\right)'$$

Solution:

$$\begin{aligned}
\left(\frac{2\tan x}{1-\tan^2 x}\right)' &= 2\,\frac{(1-\tan^2 x)(\tan x)' - (\tan x)(1-\tan^2 x)'}{(1-\tan^2 x)^2} \\
&= 2\,\frac{(1-\tan^2 x)(\sec^2 x) - (\tan x)(-2\tan x\sec^2 x)}{(1-\tan^2 x)^2} \\
&= 2\,\frac{(\sec^2 x - \tan^2 x\sec^2 x) + 2(\tan^2 x\sec^2 x)}{(1-\tan^2 x)^2} \\
&= 2\,\frac{\sec^2 x + \tan^2 x\sec^2 x}{(1-\tan^2 x)^2} \\
&= 2\,\sec^2 x\frac{1+\tan^2 x}{(1-\tan^2 x)^2} \\
&= 2\,\sec^2 x\frac{\sec^2 x}{(1-\tan^2 x)^2} \\
&= 2\,\frac{\sec^4 x}{(1-\tan^2 x)^2} \\
\text{(possible to reduce further)} &= \frac{2}{\cos^4 x(1-\tan^2 x)^2} \\
&= \frac{2}{(\cos^2 x - \sin^2 x)^2} = \frac{2}{(\cos 2x)^2} = 2\sec^2 2x
\end{aligned}$$

6. (2 *points*) Evaluate the following limit:

$$\lim_{t\to 0}\frac{\tan^2 3t}{t\sin 2t}$$

Solution:

$$\begin{aligned}
\lim_{t\to 0}\frac{\tan^2 3t}{t\sin 2t} &= \lim_{t\to 0}\frac{\tan 3t\tan 3t}{t\sin 2t} \\
&= \lim_{t\to 0}\frac{1}{\cos^2 3t}\cdot\lim_{t\to 0}\frac{\sin 3t\sin 3t}{t}\,\frac{1}{\sin 2t} \\
&= \lim_{t\to 0}\frac{1}{\cos^2 3t}\cdot\lim_{t\to 0}\frac{\sin 3t\sin 3t}{t\cdot t}\,\frac{t}{\sin 2t} \\
\text{(since } \cos(3\cdot 0)=1) &= (1)\cdot\lim_{t\to 0}\frac{\sin 3t}{t}\lim_{t\to 0}\frac{\sin 3t}{t}\lim_{t\to 0}\frac{t}{\sin 2t} \\
\text{(want } 3t\text{'s in denominator, } 2t \text{ in numerator)} &= \frac{3\cdot 3}{2}\cdot\lim_{3t\to 0}\frac{\sin 3t}{3t}\lim_{3t\to 0}\frac{\sin 3t}{3t}\lim_{2t\to 0}\frac{2t}{\sin 2t} \\
\text{(since } \lim_{x\to 0}\tfrac{\sin x}{x} = \lim_{x\to 0}\tfrac{x}{\sin x} = 1) &= \frac{3\cdot 3}{2}\cdot(1)\cdot(1)\cdot(1) \\
&= \frac{9}{2}
\end{aligned}$$