Quiz 3 - Calculus 1A

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Jonathan Dorfman

Solutions

1. (3 *points*) Given the function $f(x)$ with domain $(-\infty, \infty)$, assume the following three quantities exist and are finite:

$$f(a) = A$$

$$\lim_{x \to a^-} f(x) = B$$

$$\lim_{x \to a^+} f(x) = C$$

Now match each definition with the corresponding property:

$f(x)$ has a removable discontinuity at a (D1)	...	(P1) $A = B \neq C$
$f(x)$ has a jump discontinuity at a (D2)	...	(P2) $A = B = C$
$f(x)$ is continuous at a (D3)	...	(P3) $B = C \neq A$

Solution:

$$\begin{array}{ccc} \text{(D1)} & \longleftrightarrow & \text{(P3)} \\ \text{(D2)} & \longleftrightarrow & \text{(P1)} \\ \text{(D3)} & \longleftrightarrow & \text{(P2)} \end{array}$$

2. (4 *points*) Show there exists an x in the interval $(0, 1)$ which solves the following equation:

$$\sqrt[3]{x} = 1 - x$$

Solution: Define $f(x) = \sqrt[3]{x} - (1 - x)$. Then the assertion

$$x \in (0,1) \text{ is a solution to } \sqrt[3]{x} = 1 - x$$

is equivalent to the assertion

$$x \in (0,1) \text{ is a solution to } f(x) = 0$$

To establish that an $x \in (0,1)$ exists such that $f(x) = 0$, we consider the following 3 properties of $f(x)$:

1. $f(0) = \sqrt[3]{0} - (1 - 0) = -1 < 0$
2. $f(1) = \sqrt[3]{1} - (1 - 1) = 1 > 0$
3. $f(x)$ is a continuous function on $(0, 1)$

By applying the Intermediate Value Theorem we can conclude that for $x \in [0,1]$, $f(x)$ takes on all values between $f(0) = -1$ and $f(1) = 1$; in particular, there exists a $c \in (0,1)$ such that $f(c) = 0$.

3. Let

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

3a. (3 *points*) Show that there is a horizontal asymptote to the graph of f at $y = \frac{\sqrt{2}}{3}$.
Solution: See page 119 of the text.

3b. (3 *points*) Determine the other horizontal asymptote of f, and show your work.
Solution:

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \\
&= \lim_{x\to-\infty} \frac{\sqrt{x^2(2+\frac{1}{x^2})}}{x\ (3-\frac{x}{5})} \\
&= \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{(2+\frac{1}{x^2})}}{x\ (3-\frac{5}{x})} \\
\text{(since } \sqrt{x^2} \text{ is always positive)} &= \lim_{x\to-\infty} \frac{|x|\sqrt{(2+\frac{1}{x^2})}}{x\ (3-\frac{5}{x})} \\
\left(\text{since } \frac{|x|}{x} = -1 \text{ for } x<0\right) &= \lim_{x\to-\infty} (-1)\frac{\sqrt{(2+\frac{1}{x^2})}}{(3-\frac{5}{x})} \\
\left(\text{since } \frac{1}{x^2} \to 0 \text{ and } \frac{1}{x} \to 0\right) &= -\frac{\sqrt{2}}{3}
\end{aligned}$$

4. (5 *points*) A baseball is thrown and its position (feet) as a function of time (seconds) is given by the formula:

$$s(t) = t^3$$

Compute using limits the (instantaneous) velocity of the baseball at $t = 2$ seconds.

Solution: The instantaneous velocity at time $t = 2$ is computed from the following limit:

$$\lim_{h\to0} \frac{s(2+h)-s(2)}{h} = \lim_{h\to0} \frac{(2+h)^3-(2)^3}{h}$$

This limit was computed as problem 1 on Quiz 2.

5. (2 *points*) Determine a value of δ making the following statement true:

$$|x-4| < \delta \Longrightarrow |\sqrt{x}-2| < .4$$

Solution: See Comments on Homework 2.