Quiz 2 - Calculus 1A

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Solutions

1. Evaluate $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$ (§2.3 #20). Notice that when substituting $h = 0$ both numerator and denominator become zero, resulting in the undefined quotient $\frac{0}{0}$. Therefore we must algebraically simplify the numerator as follows:

$$\begin{aligned}\frac{(2+h)^3-8}{h} &= \frac{(2+h)(2+h)(2+h)-8}{h}\\ &= \frac{(8+12h+6h^2+h^3)-8}{h}\\ &= \frac{12h+6h^2+h^3}{h}\\ &= 12+6h+h^2\end{aligned}$$

and so

$$\lim_{h\to 0}\frac{(2+h)^3-8}{h} = \lim_{h\to 0}(12+6h+h^2) = 12$$

We will see later that we are here computing the derivative of $f(x) = x^3$ at $x = 2$

2. Evaluate $\lim_{h\to 0} \frac{(3+h)^{-1}-3^{-1}}{h}$ (§2.3 #28). Notice that when substituting $h = 0$ both numerator and denominator become zero, resulting in the undefined quotient $\frac{0}{0}$. Therefore we must algebraically simplify the numerator as follows:

$$\begin{aligned}(3+h)^{-1}-3^{-1} &= \frac{1}{3+h}-\frac{1}{3}\\ &= \frac{3}{3(3+h)}-\frac{(3+h)}{3(3+h)}\\ &= \frac{3-(3+h)}{3(3+h)}\\ &= \frac{-h}{3(3+h)}\end{aligned}$$

and so

$$\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h} = \lim_{h\to 0}\frac{\frac{-h}{3(3+h)}}{h} = \lim_{h\to 0}\frac{-1}{3(3+h)} = \frac{-1}{3(3+0)} = \frac{-1}{9}$$

We will see later that we are here computing the derivative of $f(x) = (1/x) = x^{-1}$ at $x = 3$

3. Evaluate $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$ (§2.3 #22). Notice that when substituting $h = 0$ both numerator and denominator become zero, resulting in the undefined quotient $\frac{0}{0}$. Therefore we must rationalize the numerator as follows:

$$\begin{aligned} \frac{\sqrt{1+h}-1}{h} &= \left(\frac{\sqrt{1+h}-1}{h}\right)\left(\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}\right) \\ &= \frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)} \\ &= \frac{(\sqrt{1+h})^2-1}{h(\sqrt{1+h}+1)} \\ &= \frac{(1+h)-1}{h(\sqrt{1+h}+1)} \\ &= \frac{h}{h(\sqrt{1+h}+1)} \\ &= \frac{1}{\sqrt{1+h}+1} \end{aligned}$$

and so

$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

We will see later that we are here computing the derivative of $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ at $x = 1$

4. Evaluate $\displaystyle\lim_{x\to 1^-} \frac{x^2-1}{|x-1|}$ (§2.3 #47).

$$\begin{aligned} \text{Let } f(x) &= \frac{x^2-1}{|x-1|} \\ &= \begin{cases} \frac{x^2-1}{(x-1)}, & (x-1) > 0 \\ \frac{x^2-1}{-(x-1)}, & (x-1) < 0 \\ \text{undefined}, & (x-1) = 0 \end{cases} \\ &= \begin{cases} +(x+1), & x > 1 \\ -(x+1), & x < 1 \\ \text{undefined}, & x = 1 \end{cases} \end{aligned}$$

In order to evaluate the left-hand limit of $f(x)$ as $x \to 1^-$ we need only consider the definition of $f(x)$ for $x < 1$, which by the above simplification is $-(x+1)$. Therefore

$$\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^-} [-(x+1)] = -(1+1) = -2$$