Quiz 1 - Calculus 1A
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Solutions

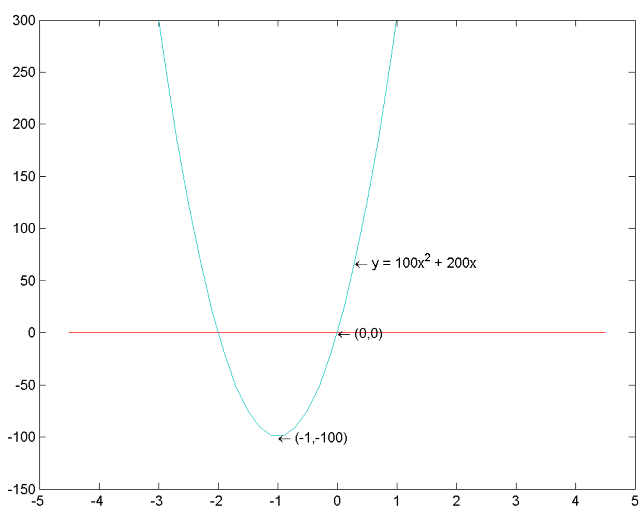
1a. Sketch the graph of the function $y = 100x^2 + 200x$.
Solution: Complete the square as follows:

$$\begin{aligned} y &= 100x^2 + 200x \\ &= 100(x^2 + 2x) \\ &= 100([x+1]^2 - 1) \\ &= 100(x+1)^2 - 100 \end{aligned}$$

so the vertex (lowest point) of the parabola is determined by $x = -1$ and hence $y = -100$, i.e. it is located at $(-1, -100)$. To identify a second point to plot, it is easy to set $x = 0$ resulting in $y = 0$, so $(0, 0)$ is another point on the graph.
(As some students noticed, the following factorization can be made:

$$\begin{aligned} y &= 100x^2 + 200x \\ &= 100x(x+2) \end{aligned}$$

making it clear that $x = -2$ also results in $y = 0$, showing $(-2, 0)$ is another point on the graph.)

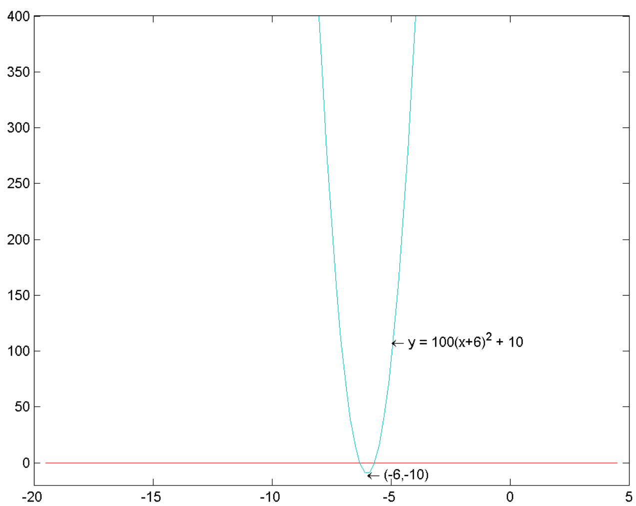


1b. If the parabola in (1a) is shifted so that its vertex (in this case the minimum point) is moved to (-6, -10), what is its resulting equation, and what is its resulting graph?
Solution: The parabola in (1a) is the parabola $y = 100x^2$ shifted so its vertex at $(0, 0)$ becomes repositioned at $(-1, 100)$; so shifting the (1a) so its vertex moves to $(-6, -10)$ is the same as shifting the parabola $y = 100x^2$ so that its vertex at $(0, 0)$ becomes $(-6, -10)$. This results in the equation

$$(y - [-10]) \quad = \quad 100(x - [-6])^2$$

or

$$y \quad = \quad 100(x + 6)^2 + 10$$



2a. A runner running down a football field crosses the 20 yard line when the coach's stopwatch reads 30 seconds, and crosses the 40 yard line when the stopwatch reads 60 seconds. If the runner's motion is uniform (i.e. his position changes linearly with respect to time) then what time will the stopwatch read when he crosses the 30 yard line?
Solution: We make a table as follows:

Position (yards)	20	30	40
Time (secs)	30	?	60

By inspecting the table we can read see that if we are told the motion is uniform then the stopwatch must read halfway between the 30 sec and 60 sec reading when the runner is halfway between the 20 yardline and 40 yardline. Therefore, the answer is 45 sec. Since the position changes linearly with time, we can also use the point-slope formula:

$$
\begin{aligned}
T(d) &= \text{time corresponding to position 'd'} \\
T(20) &= 30\ sec \\
T(40) &= 60\ sec \\
\frac{T(d) - T(20)\ sec}{(d-20)\ yd} &= \frac{T(40) - T(20)\ sec}{(40-20)\ yd} \\
&= \frac{(60-30)\ sec}{(40-20)\ yd} \\
&= \left(\frac{3}{2}\frac{sec}{yd}\right) \\
T(d) - T(20)\ sec &= \left(\frac{3}{2}\frac{sec}{yd}\right)(d-20)\ yd \\
T(d)\ sec &= \left(\frac{3}{2}\frac{sec}{yd}\right)(d\ yds) \\
T(30)\ sec &= \left(\frac{3}{2}\frac{sec}{yd}\right)(30\ yds) \\
&= 45\ sec
\end{aligned}
$$

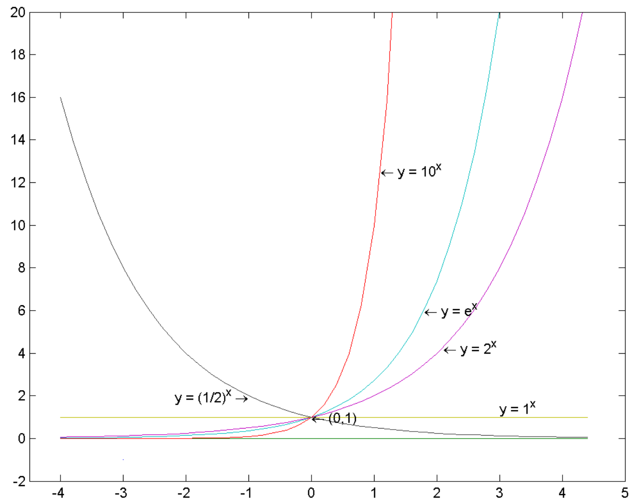
2b. How much time does it take for the runner to travel from the 25 yard line to the 35 yard line?
Solution: From the table we can see that it takes 30 secs to run 20 yards, and therefore (by linearity) 15 secs to run 10 yards, and (again by linearity) this applies to *any 10 yard interval.* Therefore, running the 10 yards between the 25 and 35 yard lines will also take 15 secs. Alternatively, we can use the formula in (2a):

$$
\begin{aligned}
T(35) - T(25) &= \left(\frac{3}{2}\frac{sec}{yd}\right)(35-25)\ yd \\
&= 15\ sec
\end{aligned}
$$

3a. Graph the function $f(x) = a^x$ for each of the following values of a:

$$a = 10, e, 2, 1, \tfrac{1}{2}$$

Solution:



3b. For each of the functions graphed in (3a), use set notation to describe its domain and range.
Solution: The domains and ranges for each of these exponential functions are as follows:

Function	Domain	Range
$y = 10^x$	$(-\infty, \infty)$	$(0, \infty)$
$y = e^x$	$(-\infty, \infty)$	$(0, \infty)$
$y = 2^x$	$(-\infty, \infty)$	$(0, \infty)$
$y = 1^x$	$(-\infty, \infty)$	$[1, 1] = \{1\}$
$y = (\frac{1}{2})^x$	$(-\infty, \infty)$	$(0, \infty)$

4a. If a bacteria population starts with 100 bacteria and doubles every three hours, then what is the size of the population after 9 hours?
Solution: We make a table as follows:

t (hours)	0	3	6	9
N (number bacteria)	100	200	400	800

By inspecting the table we can read off that there will be 800 bacteria after $t = 9$ hours.

4b. Give a formula for the number of bacteria present after t hours.
Solution: Again, from the table:

$t/3$ (dimensionless)	0	1	2	3
$N/100$ (dimensionless)	1	2	4	8

We see that $(N/100)$ increases as $1, 2, 4, 8, ...$ with every $(t/3)$ hours, hence

$$\frac{N(t)}{100} = 2^{(t/3)}$$

so

$$N(t) = 100 \cdot 2^{(t/3)}$$

For those used to working with the form

$$N(t) = N_0 e^{kt}$$

we can proceed as follows: At $t = 0$, $100 = N(0) = N_0 e^{k(0)} = N_0$ and so $N_0 = 100$. Next, to determine k, we use $200 = N(t = 3) = 100e^{k(3)}$ implying $2 = e^{3k}$ implying $k = (\ln 2)/3$, giving $N(t) = 100e^{\frac{\ln 2}{3}t}$.

Notice that $e^{\frac{\ln 2}{3}t} = e^{(\ln 2)(t/3)} = (e^{(\ln 2)})^{(t/3)} = (2)^{(t/3)}$

4c. After how many hours will the population of bacteria have grown to 400?

Solution: Again, inspecting the table in (4a) shows $N(t)$ will equal 400 when $t = 6$ hours.

4d. Give a formula for the number of hours it takes for the population to grow to size N.

Solution: In general, to determine t given $N(t)$, we must invert the function given in (4b). Therefore, we must isolate the t and express it in terms of N as follows:

$$\begin{aligned} N &= 100 \cdot 2^{t/3} \\ N/100 &= 2^{t/3} \\ \log_2(N/100) &= t/3 \\ t &= 3\log_2(N/100) \end{aligned}$$

or equivalently

$$\begin{aligned} N &= 100 \cdot e^{\frac{\ln 2}{3}t} \\ \ln(N/100) &= (\frac{\ln 2}{3})t \\ t &= (3/\ln 2)\ln(N/100) \end{aligned}$$

Note these two formulas agree since

$$\ln(X)/\ln(2) = \log_2(X)$$