Quiz 13 Solutions - Calculus 1A

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1. (10 *points*) Evaluate $\int_1^4 \frac{dt}{\sqrt{t}}$.
Solution:

$$\begin{aligned} \int_1^4 \frac{dt}{\sqrt{t}} &= \int_1^4 t^{-1/2}\ dt \\ \text{by FTC and } \int t^n\ dt = \tfrac{t^{n+1}}{n+1} &= \left[\frac{t^{1/2}}{1/2}\right]_1^4 \\ &= 2\cdot\left[4^{1/2} - 1^{1/2}\right] \\ &= 2(2-1) = 2 \end{aligned}$$

2. (5 *points*) Evaluate $\int_{-1}^1 \frac{\sin(t)}{t^2}\ dt$.
Solution: Since the integrand $\frac{\sin(t)}{t^2}$ is not continuously defined on the interval of integration $[-1, 1]$, this Definite Integral does not exist.

3. (5 *points*) What is $g'(x)$ if $g(x) = \int_0^{\sqrt{x}} \frac{\sin(t)}{1+t^2}\ dt$ for $x \geq 0$?

$$\begin{aligned} \frac{d}{dx}\left(\int_0^{\sqrt{x}} \frac{\sin(t)}{1+t^2}\ dt\right) &= \ldots \\ \text{by FTC and Chain Rule} &= \left[\frac{\sin(t)}{1+t^2}\right]_{t=\sqrt{x}} (\sqrt{x})' \\ &= \frac{\sin(\sqrt{x})}{1+x}\cdot\frac{1}{2\sqrt{x}} \end{aligned}$$

More generally, we have the following formula for $g'(x)$ if $g(x)$ is defined as follows:

$$g(x) = \int_{a(x)}^{b(x)} f(t)\ dt$$

Solution: Pretend we know the anti-derivative of $f(t)$, and call it $F(t)$ (so $F'(t) = f(t)$). Then by FTC

$$g(x) = \left[F(t)\right]_{a(x)}^{b(x)} = F(b(x)) - F(a(x))$$

and so its derivative (using Chain Rule!) is

$$g'(x) = F'(b(x))(b(x))' - F'(a(x))(a(x))'$$

and now using fact that $F'(t) = f(t)$, we get:

$$g'(x) = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$$

1ec. (1 *point*) Evaluate

$$\lim_{n\to\infty} \left(\tfrac{5}{n}\right) \sum_{i=1}^{n} \frac{1}{\sqrt{4+\frac{5i}{n}}}$$

Solution: Recognizing this limit of sums as $\int_4^9 \frac{dt}{\sqrt{t}}$ (by definition of the Definite Integral) we can invoke FTC and assert that it equals $\left[\frac{t^{1/2}}{1/2}\right]_4^9 = 2(\sqrt{9}-\sqrt{4}) = 2$

2ec. (1 *point*) Evaluate $\int_{-1}^{1} \frac{\sin(t)}{1+t^2}\, dt$:
Solution: Since the integrand is an odd function, and the interval of integration is symmetric about $t = 0$, we conclude (from the definition of the Definite Integral) that it equals 0.

3ec. (1 *point*) Evaluate $\int_{-1}^{1} \frac{1}{1+t^2}\, dt$:
Solution:

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{1+t^2}\, dt &= \cdots \\
\text{by FTC and } \left(\tan^{-1}(t)\right)' = \frac{1}{1+t^2} \quad &= \left[\tan^{-1}(t)\right]_{-1}^{1} \\
&= \left[(\pi/4) - (-\pi/4)\right] = \frac{\pi}{2}
\end{aligned}$$

4ec. (1 *point*) If the graph of $f(t)$ is given as follows, sketch $A(x) = \int_0^x f(t)\, dt$.

Solution: $A(x)$ looks roughly like $\{\cos(x \cdot \frac{\pi}{2}) - 1\}$ on the interval $[0, 4]$. Note that $A(x)$ has a minimum when $f(t) = 0$ and that $A(x)$ returns to 0 when the positive area under $f(t)$ cancels the negative area.