Asymptote Problem - Calculus 1A

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Note: this is presented in more detail than would be required in an exam

1. (§2.6, #37) Let

$$f(x) = \frac{x}{x+4}$$

Vertical Asymptote: Notice as $x \to -4$, $f(x)$ becomes infinite - but we must separately evaluate the limits from the left and right sides to see if the graph of $f(x)$ is going to $+\infty$ or $-\infty$.

$$\begin{aligned}
\lim_{x\to -4^-} f(x) &= \left(\lim_{x\to -4^-} x\right)\cdot\left(\lim_{x\to -4^-} \frac{1}{x+4}\right) \\
&= (-4)\left(\frac{1}{-4^- + 4}\right) \\
&= (-4)\left(\frac{1}{0^-}\right) \\
&= (-4)\cdot(-\infty) \\
&= +\infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to -4^+} f(x) &= \left(\lim_{x\to -4^+} x\right)\cdot\left(\lim_{x\to -4^+} \frac{1}{x+4}\right) \\
&= (-4)\left(\frac{1}{-4^+ + 4}\right) \\
&= (-4)\left(\frac{1}{0^+}\right) \\
&= (-4)\cdot(+\infty) \\
&= -\infty
\end{aligned}$$

Horizontal Asymptotes:

$$\begin{aligned}
\lim_{x\to -\infty} f(x) &= \lim_{x\to -\infty} \frac{x}{x+4} \\
&= \lim_{x\to -\infty} \frac{x}{x(1+\frac{4}{x})} \\
&= \lim_{x\to -\infty} \frac{1}{(1+\frac{4}{x})} \\
&= \frac{1}{1+\lim_{x\to -\infty}\left(\frac{4}{x}\right)} \\
&= \frac{1}{1+0} \\
&= 1
\end{aligned}$$

$$
\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{x}{x+4} \\
&= \lim_{x\to+\infty} \frac{x}{x(1+\frac{4}{x})} \\
&= \lim_{x\to+\infty} \frac{1}{(1+\frac{4}{x})} \\
&= \frac{1}{1+\lim_{x\to+\infty}\left(\frac{4}{x}\right)} \\
&= \frac{1}{1+0} \\
&= 1
\end{aligned}
$$

Another approach:

$$
\begin{aligned}
f(x) &= \frac{x}{x+4} \\
&= \frac{x+4-4}{x+4} \\
&= 1+\frac{-4}{x+4} \\
&= 1-\frac{4}{x+4}
\end{aligned}
$$

So graph is your basic $\frac{1}{x}$ (with vertical asymptote at $x=0$ and horizontal asymptote at $y=0$) followed by:

(a) shift graph to the left by 4 (vertical asymptote moved to $x=-4$, $-4^- \to -\infty$, $-4^+ \to +\infty$)

(b) scale vertically by factor of 4

(c) reflect in x-axis to account for minus sign ($-4^- \to +\infty$, $-4^+ \to -\infty$)

(d) shift vertically by $+1$ (horizontal asymptote moved to $y=1$)

2. (§2.6, #38)

$$
\begin{aligned}
f(x) &= \frac{x^2+4}{x^2-1} \\
&= \frac{x^2+4}{(x-1)(x+1)}
\end{aligned}
$$

Vertical Asymptotes - must evaluate limits as $x \to 1^-$,$x \to 1^+$,$x \to -1^-$,$x \to -1^+$:

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{x^2+4}{(x-1)(x+1)} \\
&= \left(\lim_{x\to 1^-} \frac{x^2+4}{(x+1)}\right)\left(\lim_{x\to 1^-} \frac{1}{(x-1)}\right) \\
&= \left(\frac{5}{2}\right)\left(\frac{1}{1^- - 1}\right) \\
&= \left(\frac{5}{2}\right)\left(\frac{1}{0^-}\right) \\
&= -\infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \frac{x^2+4}{(x-1)(x+1)} \\
&= \left(\lim_{x\to 1^+} \frac{x^2+4}{(x+1)}\right)\left(\lim_{x\to 1^+} \frac{1}{(x-1)}\right) \\
&= \left(\frac{5}{2}\right)\left(\frac{1}{1^+ - 1}\right) \\
&= \left(\frac{5}{2}\right)\left(\frac{1}{0^+}\right) \\
&= +\infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} \frac{x^2+4}{(x-1)(x+1)} \\
&= \left(\lim_{x\to -1^-} \frac{x^2+4}{(x-1)}\right)\left(\lim_{x\to -1^-} \frac{1}{(x+1)}\right) \\
&= \left(\frac{5}{-2}\right)\left(\frac{1}{-1^- + 1}\right) \\
&= \left(\frac{5}{-2}\right)\left(\frac{1}{0^-}\right) \\
&= +\infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} \frac{x^2+4}{(x-1)(x+1)} \\
&= \left(\lim_{x\to -1^+} \frac{x^2+4}{(x-1)}\right)\left(\lim_{x\to -1^+} \frac{1}{(x+1)}\right) \\
&= \left(\frac{5}{-2}\right)\left(\frac{1}{-1^+ + 1}\right) \\
&= \left(\frac{5}{-2}\right)\left(\frac{1}{0^+}\right) \\
&= -\infty
\end{aligned}$$

Horizontal Asymptotes:

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x^2+4}{x^2-1} \\
&= \lim_{x\to-\infty} \frac{x^2(1+\frac{4}{x^2})}{x^2(1-\frac{1}{x^2})} \\
&= \lim_{x\to-\infty} \frac{(1+\frac{4}{x^2})}{(1-\frac{1}{x^2})} \\
&= \frac{1+\lim_{x\to-\infty}\left(\frac{4}{x^2}\right)}{1-\lim_{x\to-\infty}\left(\frac{1}{x^2}\right)} \\
&= \frac{1+0}{1-0} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{x^2+4}{x^2-1} \\
&= \lim_{x\to+\infty} \frac{x^2(1+\frac{4}{x^2})}{x^2(1-\frac{1}{x^2})} \\
&= \lim_{x\to+\infty} \frac{(1+\frac{4}{x^2})}{(1-\frac{1}{x^2})} \\
&= \frac{1+\lim_{x\to+\infty}\left(\frac{4}{x^2}\right)}{1-\lim_{x\to+\infty}\left(\frac{1}{x^2}\right)} \\
&= \frac{1+0}{1-0} \\
&= 1
\end{aligned}$$