Math128B
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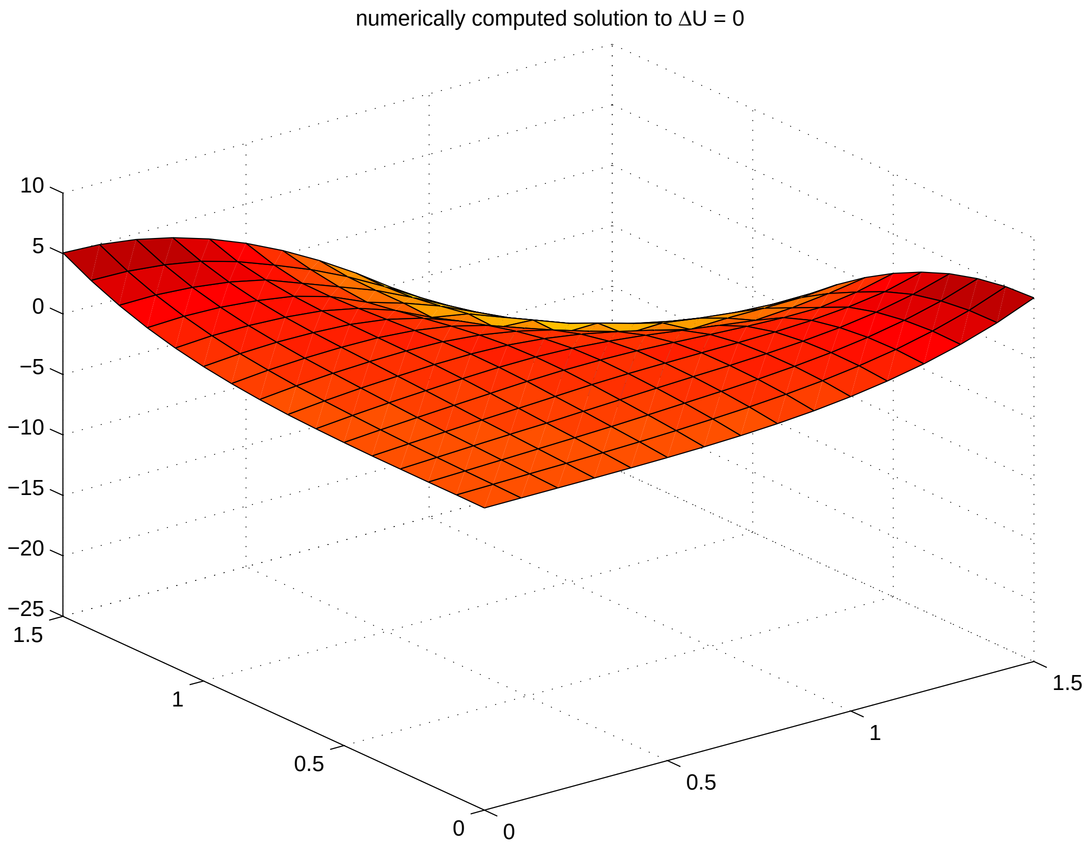
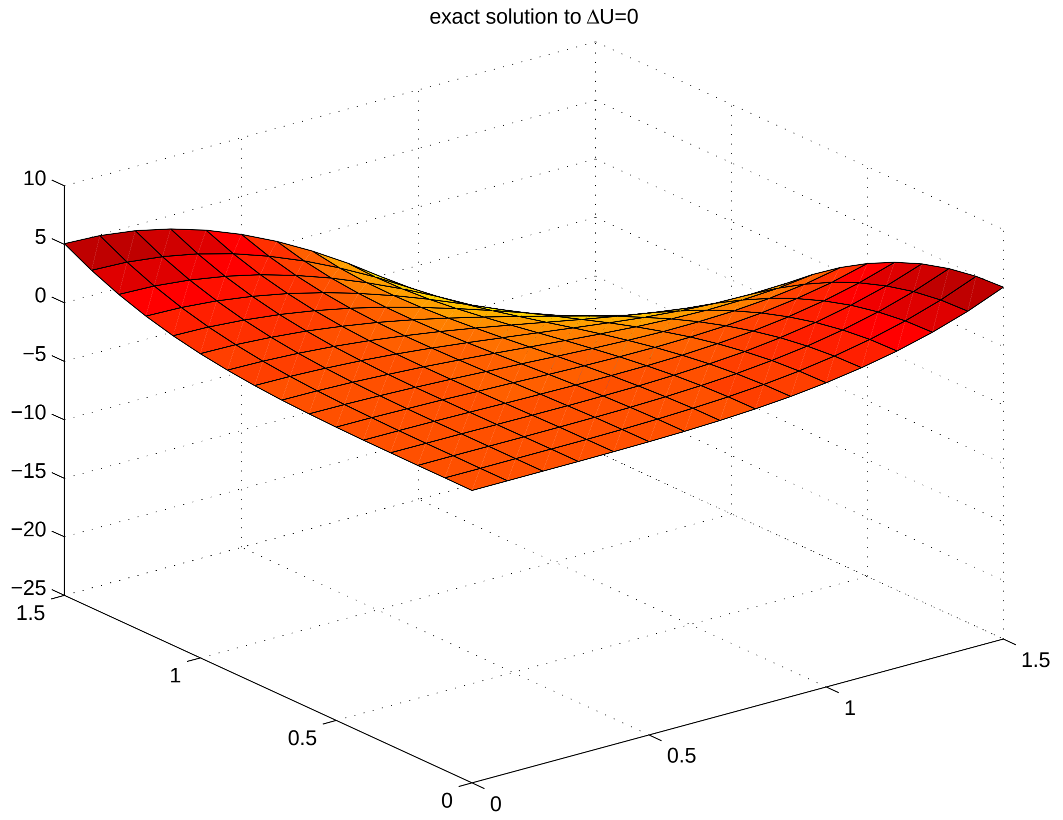
Jonathan Dorfman
Homework 9 Solutions

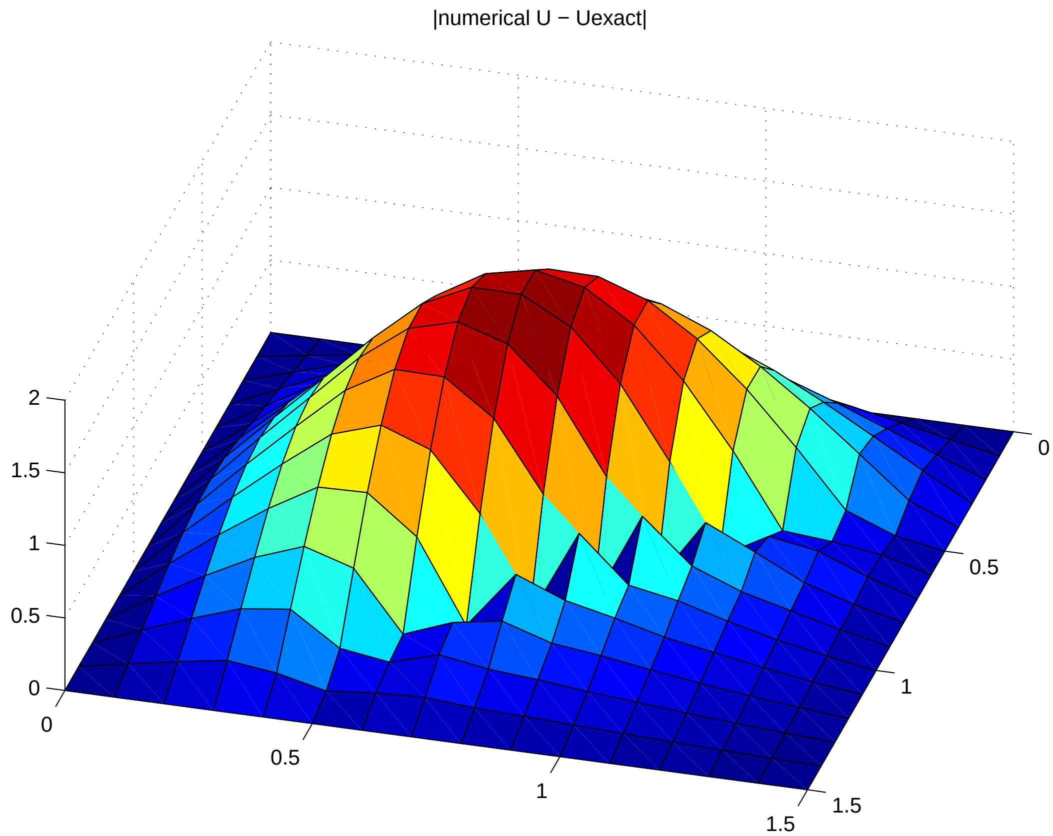
Problem 9.1 & 9.2

file	function
Hmwk9Main.m	calls dirich.m to solve $\Delta u = 0, u(x,0) = x^4, u(0,y) = y^4$ $u(x,1.5) = x^4 - 13.5x^2 + 5.0625, u(1.5,y) = y^4 - 13.5y^2 + 5.0625$ calls BSpline.m and createOmega.m to generate $B_{1,3}$
dirich.m	implements p. 574 (Program 10.4 with corrections) of Burden & Faires
BSpline.m	implements recursion formula for $B_{i,k}$ per p. 5 of deBoor paper
createOmega.m	implements recursion formula for $\omega_{i,k}$ per p. 5 of deBoor paper

Example of graphical output:

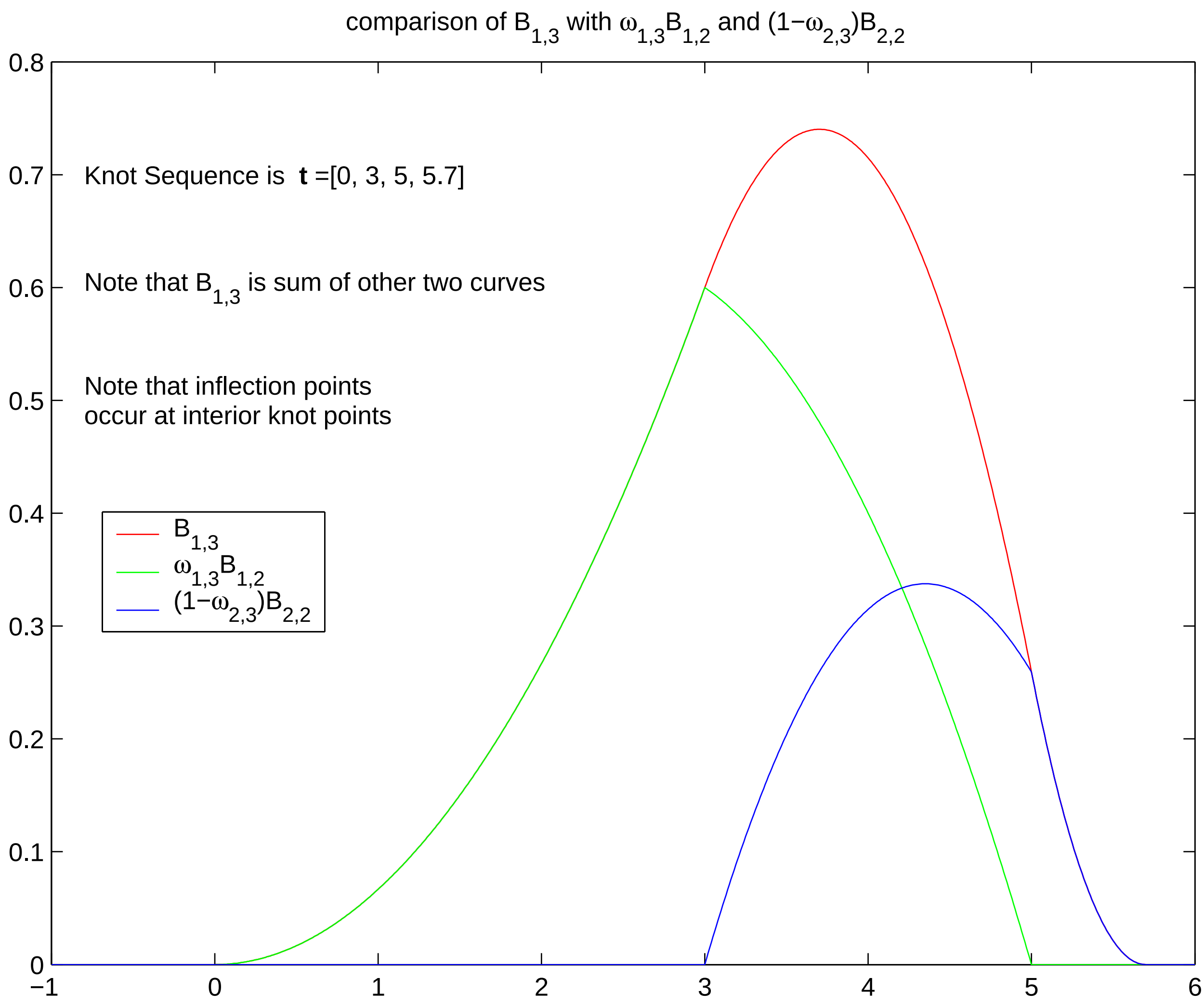



exact solution to ΔU=0
10
5
0
−5
−10
−15
−20
−25
1.5
1
0.5
0
0
0.5
1
1.5


|numerical U − Uexact|
2
1.5
1
0.5
0
0
0.5
1
1.5
0
0.5
1
1.5

Problem 9.2

Example of graphical output:



Problem 9.3

Referring to equation (0.1) on p. 9 of de Boor's paper, we have the following facts about the dual functionals λ_{ik}:

1. $t_i < \tau_i < t_{i+k} \implies \lambda_{ik} : f \longmapsto \sum_{\nu=1}^{k} \frac{(-D)^{(\nu-1)}\psi_{ik}(\tau_i)}{(k-1)!} D^{(k-\nu)} f(\tau_i)$
2. $t_i < \tau_i < t_{i+k} \implies \lambda_{ik} B_{jk} = \delta_{ij} \qquad \forall j$

Now in this exercise we are asked to prove:

$$f = \sum_i a_i B_{ik} \text{ and for some } j,\ f \text{ vanishes outside } [t_{j+1}, t_{j+k}] \implies f \equiv 0$$

Recall that a polynomial which vanishes on an interval is identically 0 (indeed, a polynomial of degree n having more than n roots is necessarily the 0-polynomial). Now this does not apply directly to the present problem since f is only piecewise-polynomial (which in general *will* vanish outside an interval). The key point here is the additional assumption that f vanishes outside a *specific* interval $[t_{j+1}, t_{j+k}]$. In particular, f vanishes on (t_j, t_{j+1}) and hence all its derivatives will vanish at any $\tau \in (t_j, t_{j+1})$, and similarly for $\tau \in (t_{j+k}, t_{j+k+1})$, etc.

Now writing (see p. 5 of de Boor) $f = \sum_j a_j B_{jk}$, and then applying the linear map λ_{ik}:

$$\lambda_{ik}(f) \overset{\text{(linearity)}}{=} \sum_j a_j\ \lambda_{ik}(B_{jk}) \overset{(2)}{=} \sum_j a_j\ \delta_{ij} = a_i$$

gives the 'reconstruction formula':

$$f = \sum_i \lambda_{ik}(f)\ B_{ik} \tag{3}$$

This important identity will allow us to prove the result by establishing $\lambda_{ik}(f) = 0\ \forall i$. The only other ingredient required is to invoke (1) with τ_i chosen appropriately. In what follows j and k are fixed according to the hypotheses of the exercise:

$$\begin{aligned} f\big|_{[t_{j+1},t_{j+k}]^c} = 0 \text{ and } \tau_i \notin [t_{j+1}, t_{j+k}] \quad &\implies \quad D^{(k-\nu)} f(\tau_i) = 0 \qquad \nu = 1, \ldots, k \\ \text{(since for any } i,\ \tau_i \text{ can be chosen in } (t_i, t_{i+k}) \cap [t_{j+1}, t_{j+k}]^c) \quad &\overset{(1)}{\implies} \quad \lambda_{ik}(f) = 0 \qquad \forall i \\ &\overset{(3)}{\implies} \quad f = 0 \end{aligned}$$

∎

As for the second part of the exercise, it suffices to note that for any j the step function B_{j1} vanishes outside the interval $[t_j, t_{j+k}]$ $\forall k$.