Math128B
Mar. 21, 2005
Jonathan Dorfman
Homework 8 Solutions

Problem 8.1 & 8.2

file	function
Hmwk8Main.m	calls NLShooting to solve BVP: $y''(t) = -y'^2 - y + \log x, x \in [1,2], y(1) = 0, y(2) = \log 2$ calls NLFiniteDiff to solve BVP: $y''(t) = -y'^2 - y + \log x, x \in [1,2], y(1) = 0, y(2) = \log 2$ compares with exact solution $y(x) = \log x$
NLShooting.m	calls NLRungeKutta4 adapted to nonlinear method requiring y-values implements p.654-656 of Burden & Faires
NLFiniteDiff.m	implements p.667-668 of Burden & Faires
NLRungeKutta4.m	implements p. 519 of Mathews & Fink (Program 9.9 with NonLinear modifications)
DF8_1.m	implement specifics of ODE $y''(t) = -y'^2 - y + \log x$ (Eq. (11.6) p.653)
DF8_1z.m	implement specifics Jacobian for ODE $y''(t) = -y'^2 - y + \log x$ (Eq. (11.12) p.656)
DF8_2.m	implement specifics of ODE $y''(t) = -y'^2 - y + \log x$ (Eq. (11.6) p.653)
DF8_2Jac.m	implement specifics of Jacobian for ODE $y''(t) = -y'^2 - y + \log x$ (Eq. (11.20) p.668)

A diary of the output follows:

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least squares error in Nonlinear Shooting Method with h=0.1 is 2.2827e-006

least squares error in Nonlinear Finite Difference Method with h=0.1 is 0.00012761
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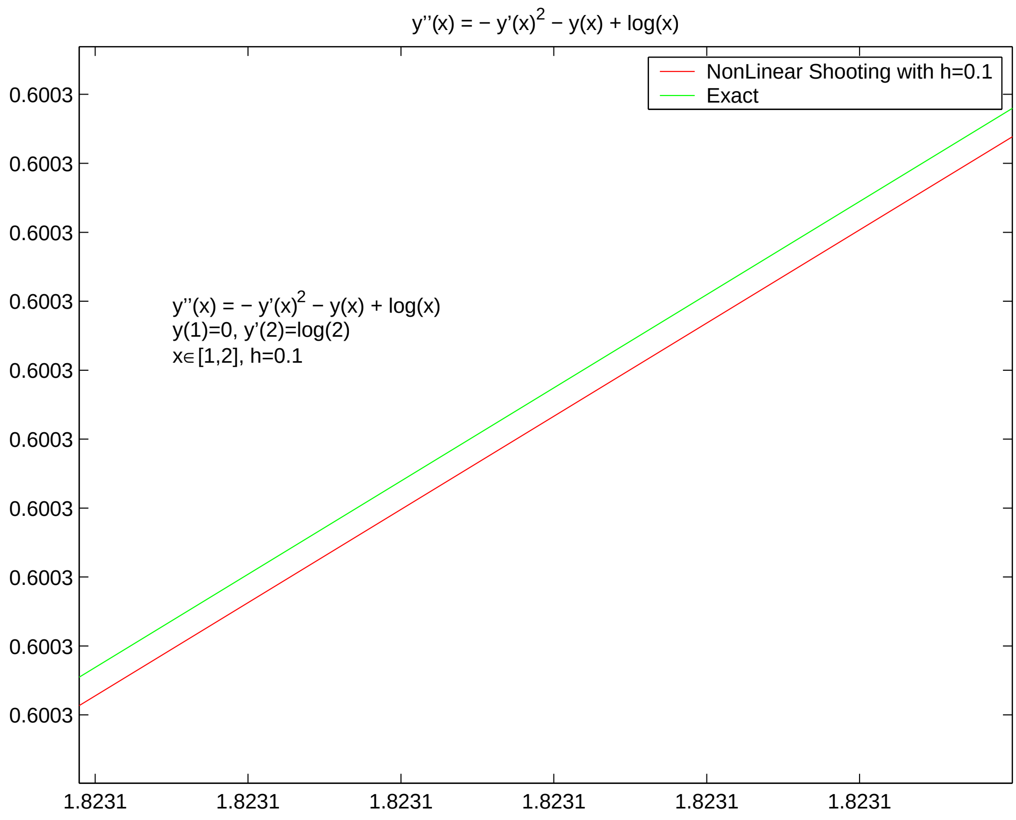
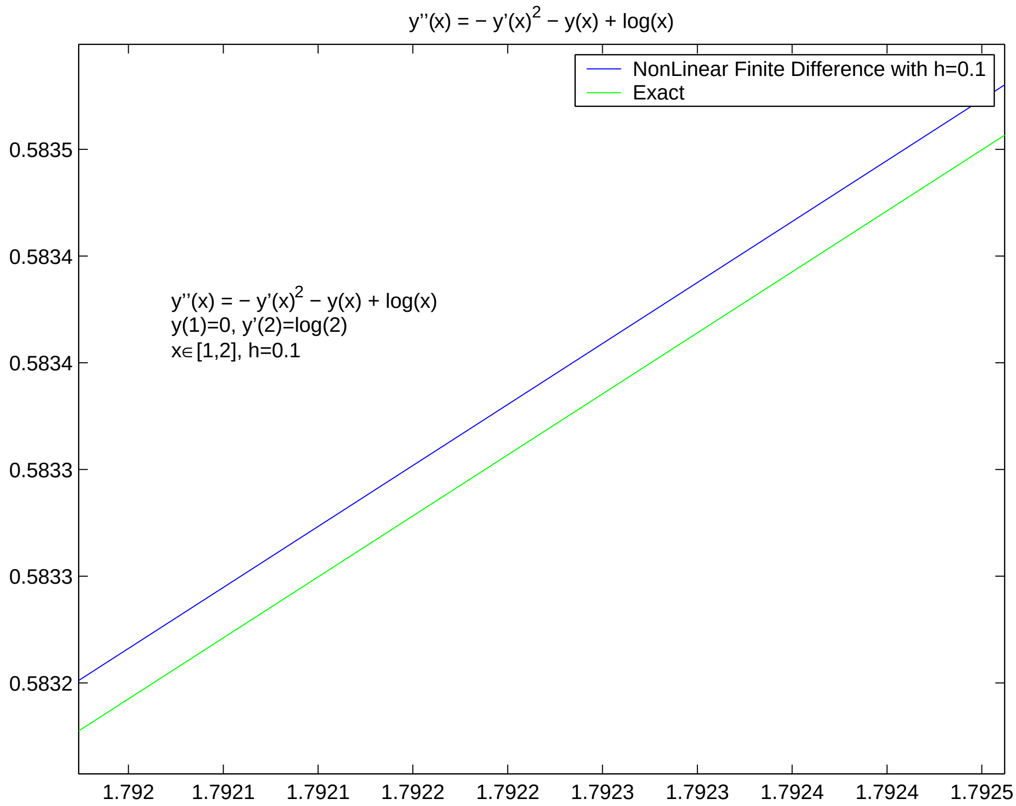
Example of graphical output:



Example of graphical output:



Problem 8.3

(a) Letting $u(x,t) = \sin(n\ \pi\ x)\cos(c\ n\ \pi\ t)$ we compute:

$$\begin{aligned} u_x &= +n\pi\cos(n\,\pi\,x)\cos(c\,n\,\pi\,t) \\ u_{xx} &= -n^2\pi^2\sin(n\,\pi\,x)\cos(c\,n\,\pi\,t) \\ u_t &= -c\,n\pi\sin(n\,\pi\,x)\sin(c\,n\,\pi\,t) \\ u_{tt} &= -c^2n^2\pi^2\sin(n\,\pi\,x)\cos(c\,n\,\pi\,t) \\ u_{tt} - c^2u_{xx} &= 0 \end{aligned}$$

Alternatively, using the trig identity $\sin A\cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$ we could rewrite the above as:

$$u(x,t) = \tfrac{1}{2}\sin(n\pi x + cn\pi t) + \tfrac{1}{2}\sin(n\pi x - cn\pi t)$$

Since this expression for $u(x,t)$ has the D'Alembert form (see Eqn. (14) p. 549 of Mathews & Fink)

$$u(x,t) = F(x+ct) + G(x-ct) \qquad \text{where } F(z) = G(z) = \tfrac{1}{2}\sin(n\ \pi\ z)$$

we conclude it must solve the wave equation.

(b) We have by Eqn. (12.18) on p. 719 of Burden & Faires (or Eqn. (7) p. 550 of Mathews & Fink):

$$w_{i,j+1} = 2(1-\lambda^2)w_{i,j} + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1} \qquad \text{where } \lambda = \alpha k/h$$

Therefore, by setting $\lambda = 1$ we see that the above expression for $w_{i,j+1}$ reduces to:

$$w_{i,j+1} = (w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}$$

as required by the exercise (see also Eqn. (17) p. 550 of Mathews & Fink). Since in the present case $\alpha = \sqrt{9} = 3$ ($u_{tt} - \alpha^2u_{xx} = u_{tt} - 9u_{xx} = 0$ - Eqn (12.16)) , we get $1 = \lambda = \alpha k/h = 3k/h$ whence $h = 3k$.

Problem 8.4

Letting $u(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ we compute:

$$\begin{aligned} u_x &= 2ax + by + d \\ u_{xx} &= 2a \\ u_y &= 2cy + bx + e \\ u_{yy} &= 2c \\ u_{xx} + u_{yy} &= 2a + 2c \end{aligned}$$

(a)

$$u_{xx} + u_{yy} = 0 \implies 2a + 2c = 0 \implies a + c = 0$$

(b)

$$u_{xx} + u_{yy} = -1 \implies 2a + 2c = -1 \implies a + c = -\tfrac{1}{2}$$