Math128B

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Homework 4 Solutions (#1-#4)

Problem 4.1

We wish to establish the following inequality where A is invertible, $K(A) = \|A\| \cdot \|A^{-1}\|$, and B is singular:

$$\frac{1}{K(A)} \leq \frac{\|A-B\|}{\|A\|} \tag{1}$$

First, by definition of $K(A)$, the above can be rearranged to give:

$$1 \leq \|A^{-1}\| \cdot \|A-B\| \tag{2}$$

Next, let's prove this when A is the simplest invertible matrix, namely $A = I = \text{Identity}$:

$$\begin{aligned}
B \text{ is singular (B has 0 as eigenvalue)} &\implies \exists\mathbf{x} \text{ with } \mathrm{Bx} = 0 \cdot \mathrm{x} \\
\text{(so (I-B) has 1 as eigenvalue)} &\implies \exists\mathbf{x} \text{ with } (\mathrm{I}-\mathrm{B})\mathrm{x} = 1 \cdot \mathrm{x} \\
\text{(since } \rho(X) \geq \text{any eigenvalue of } X\text{)} &\implies \rho(I-B) \geq 1 \\
\text{(since } \|X\| \geq \rho(X)\text{)} &\implies \|I-B\| \geq 1
\end{aligned} \tag{3}$$

Now if A is invertible, then A^{-1} exists, and so does $A^{-1}B$. Furthermore, note that if B is singular then so must be $A^{-1}B$ (since if X were a left-inverse to $A^{-1}B$, then $X \cdot A^{-1}$ would be a left inverse to B - or use properties of determinants). So just apply the above result to $A^{-1}B$ instead of B:

$$\begin{aligned}
A^{-1}B \text{ is singular} &\overset{(3)}{\implies} \|I - A^{-1}B\| \geq 1 \\
&\implies \|A^{-1}(A-B)\| \geq 1 \\
\text{(since } \|X \cdot Y\| \leq \|X\| \cdot \|Y\|\text{)} &\implies \|A^{-1}\| \cdot \|(A-B)\| \geq 1
\end{aligned}$$

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Problem 4.2 & 4.3

file	function
Hmwk4Main.m	receives matrix **A**, row vector **x0**, tolerance **tol**, and row vector **b** and (inverse of) PreConditioning matrix **Cinv** as input
ConjGrad.m	implements ConjugateGradient Method
ConjGradCond.m	implements ConjugateGradient Method with preconditioning parameter

The input matrix **A**, vector **b** and tolerance and PreConditioning matrix are taken from page 477 of Burden & Faires for comparison. A diary of the output follows:

```
A =
    0.2000    0.1000    1.0000    1.0000         0
    0.1000    4.0000   -1.0000    1.0000   -1.0000
    1.0000   -1.0000   60.0000         0   -2.0000
    1.0000    1.0000         0    8.0000    4.0000
         0   -1.0000   -2.0000    4.0000  700.0000
b =
     1     2     3     4     5
x0 =
     1     0     0     0     0
tol =
    0.0100
C = diag(diag(A).^(.5))
    0.4472         0         0         0         0
         0    2.0000         0         0         0
         0         0    7.7460         0         0
         0         0         0    2.8284         0
         0         0         0         0   26.4575
Cinv = diag(1./diag(A).^(.5))
    2.2361         0         0         0         0
         0    0.5000         0         0         0
         0         0    0.1291         0         0
         0         0         0    0.3536         0
         0         0         0         0    0.0378

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Hmwk4Main(A,x0,tol,b,Cinv)
%%%%%%%%%%%%Conjugate Gradient method%%%%%%%%%%%%
tolerance achieved in 5 steps
solution is x=[7.8597     0.42293   -0.073592    -0.54064    0.010626]

residual vector is res=[-7.0249e-013 2.0355e-010 4.3509e-010-8.2529e-010-1.4282e-007]

norm of final residual |Ax-b|=1.4282e-007

%%%%%%%%%%%%Conjugate Gradient method with PreConditioning%%%%%%%%%%%%
tolerance achieved in 4 steps
solution is x=[7.8597     0.42291   -0.073594    -0.54064    0.010632]

residual vector is res=[1.4037e-006 6.6852e-005  0.00013335-3.1081e-005  -0.0042223]

norm of final residual |Ax-b|=0.004225
```

Problem 4.4

We are asked to establish the following equality, where $v^{(j)}$ is an A-orthogonal basis:

$$\left\langle v^{(j)}, Ax^{(k)} - b \right\rangle = 0 \qquad\qquad j = 0, 1, \ldots, k-1$$

Slightly restated (still following notation from lecture) this asserts, for $j = 0, 1, \ldots, k$:

$$\left\langle v^{(j)}, Ax^{(k+1)} - b \right\rangle = 0$$

We first prove it for ($j = k = 0$), and then for ($j = k$, where $k > 0$), and finally for ($j < k$, where $k > 0$).

Start with $x^{(0)}$ and $v^{(0)}$, and construct

$$x^{(1)} = x^{(0)} + t_0 v^{(0)} \tag{1}$$

where t_0 is determined by:

$$t_0 = \frac{\left\langle v^{(0)}, b - Ax^{(0)} \right\rangle}{\left\langle v^{(0)}, Av^{(0)} \right\rangle} \tag{2}$$

Now

$$\begin{aligned} Ax^{(1)} - b &\overset{(1)}{=} A\big(x^{(0)} + t_0 v^{(0)}\big) - b \\ &= Ax^{(0)} - b + t_0 Av^{(0)} \end{aligned} \tag{3}$$

So dotting the above with $v^{(0)}$ gives

$$\begin{aligned} \left\langle Ax^{(1)} - b, v^{(0)} \right\rangle &\overset{(3)}{=} \left\langle Ax^{(0)} - b, v^{(0)} \right\rangle + t_0 \left\langle Av^{(0)}, v^{(0)} \right\rangle \\ (\text{definition of } t_0) &\overset{(2)}{=} \left\langle Ax^{(0)} - b, v^{(0)} \right\rangle + \left\langle v^{(0)}, b - Ax^{(0)} \right\rangle \\ &= 0 \qquad\qquad\qquad\qquad (0 = j = k) \quad \blacksquare \end{aligned}$$

Now consider when $k > 0$. In this case the above argument proceeds essentially unchanged (just replace '0' with 'k' and '1' with '$k+1$') and thereby proves $\left\langle v^{(j)}, Ax^{(k+1)} - b \right\rangle$ for $j = k$. $(0 < j = k)$ $\blacksquare$

Finally, to prove that the equality also holds for $j = 0, 1, \cdots, k-1$ is even easier - *when* we make the induction hypothesis that it holds for smaller k and $j \leq k$:

$$\begin{aligned} \left\langle Ax^{(k+1)} - b, v^{(j)} \right\rangle &\overset{(3)}{=} \left\langle Ax^{(k)} - b, v^{(j)} \right\rangle + t_k \left\langle Av^{(k)}, v^{(j)} \right\rangle \\ (\text{since by A-orthogonality } j < k \implies \left\langle Av^{(k)}, v^{(j)} \right\rangle = 0) &= \left\langle Ax^{(k)} - b, v^{(j)} \right\rangle + 0 \\ (\text{by induction since } j < k) &= 0 \end{aligned}$$

$(0 \leq j < k)$ $\blacksquare$