UCB Math 110, Spring 2011: Homework 4
Solutions to Graded Problems

4.3.1(d) 2 points. True. By the corollary on page 223 we know that $\det(A) \neq 0$ if and only if A is invertible. From theorem 3.5 we see that A is invertible if and only if Rank(A)=n.

4.3.1(h) 2 points. False. Consider the system of equations

$$\begin{cases} x_1 + & x_2 &= 2 \\ & x_2 &= 1 \end{cases}$$

whose unique solution is clearly

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Applying this problem gives $x_1 = 2$, a contradiction.

4.3.22(a) 2 points. We use the definition of $[T]_\beta^\gamma$ to see for $\beta = \{v_1, v_2, \ldots, v_n\}$

$$[T]_\beta^\gamma = \begin{bmatrix} [T(v_1)]_\gamma & [T(v_2)]_\gamma & \ldots & [T(v_n)]_\gamma \end{bmatrix}.$$

It is easy to see using $T(v_i) = T(x^{i-1})$

$$[T(1)]_\gamma = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad [T(x)]_\gamma = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}, \quad \ldots, \quad [T(x^n)]_\gamma = \begin{bmatrix} c_0^n \\ c_1^n \\ \vdots \\ c_n^n \end{bmatrix}$$

which implies that $M = [T]_\beta^\gamma$.

4.3.22(c) 4 points. We proceed by induction. For $n = 1$ we have

$$\det \begin{pmatrix} 1 & c_0 \\ 1 & c_1 \end{pmatrix} = c_1 - c_0.$$

Assume the result for the n case, otherwise

$$\det \begin{pmatrix} 1 & \ldots & c_0^n \\ \vdots & \ddots & \vdots \\ 1 & \ldots & c_n^n \end{pmatrix} = \prod_{0 \le i < j \le n} (c_j - c_i).$$

For the $n + 1$ case it is easy to see by expansion by cofactors on the last row that

$$\det \begin{pmatrix} 1 & \ldots & c_0^{n+1} \\ \vdots & \ddots & \vdots \\ 1 & \ldots & c_{n+1}^{n+1} \end{pmatrix} = a_0 + a_1 c_{n+1} + \ldots + a_{n+1} c_{n+1}^{n+1}$$

a polynomial in c_{n+1}. Notice if $c_{n+1} = c_i$ with $0 \le i \le n$ the above determinant is 0. This implies that

$$\det \begin{pmatrix} 1 & \ldots & c_0^{n+1} \\ \vdots & \ddots & \vdots \\ 1 & \ldots & c_{n+1}^{n+1} \end{pmatrix} = a_n \prod_{0 \le i \le n} (c_{n+1} - c_i).$$

Finally from our inductive hypothesis we have that

$$a_n = \det \begin{pmatrix} 1 & \dots & c_0^n \\ \vdots & \ddots & \vdots \\ 1 & \dots & c_n^n \end{pmatrix} = \prod_{0 \leq i < j \leq n} (c_j - c_i).$$

Putting this together we get that

$$\det \begin{pmatrix} 1 & \dots & c_0^{n+1} \\ \vdots & \ddots & \vdots \\ 1 & \dots & c_{n+1}^{n+1} \end{pmatrix} = \left(\prod_{0 \leq i < j \leq n} (c_j - c_i) \right) \left(\prod_{0 \leq i \leq n} (c_{n+1} - c_i) \right)$$
$$= \prod_{0 \leq i < j \leq n+1} (c_j - c_i).$$

4.4.3(f) 2 points. A calculation easily gives that

$$\det \begin{pmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix} = 4 + 2i$$