2.4.3d

Using Thrm 2.19 we need only show that $\dim(V) \neq 4$.

Note that Tr is a LT and $V = \mathrm{null}(T)$.

Since $\mathrm{Tr}\begin{pmatrix}1&0\\0&0\end{pmatrix} = 1 \neq 0$, $\begin{pmatrix}1&0\\0&0\end{pmatrix} \notin \mathrm{null}(T)$, showing $V \neq M_2(\mathbb{R})$. Thus $\dim(V) < 4$ as $V \subseteq M_2(\mathbb{R})$.

2.4.16

define T by $A \mapsto BA$
S by $A \mapsto AB^{-1}$

by Thrm 2.12 a,b. S and T are linear

Then $\Phi = T \circ S$ is also linear.

Since Φ is an operator, Rank nullity implies Φ is 1-1 iff Φ is onto.

We show Φ is 1-1.

If $\Phi(A) = BAB^{-1} = 0$ then $A = B^{-1}(BAB^{-1})B = B^{-1}0B = 0$

Since $\mathrm{null}(\Phi) = \{0\}$, Φ is 1-1.

2.5.4

let $[T]$ denote the matrix of T w.r.t the standard basis. By Thrm 2.23

$$[T]_{\beta'} = \begin{pmatrix}2&-1\\-1&1\end{pmatrix}[T]\begin{pmatrix}1&1\\1&2\end{pmatrix}$$

by def, $[T] = \begin{pmatrix} | & | \\ T\begin{pmatrix}1\\0\end{pmatrix} & T\begin{pmatrix}0\\1\end{pmatrix} \\ | & | \end{pmatrix} = \begin{pmatrix}2&1\\1&-3\end{pmatrix}$

So $[T]_{\beta'} = \begin{pmatrix}2&-1\\-1&1\end{pmatrix}\begin{pmatrix}2&1\\1&-3\end{pmatrix}\begin{pmatrix}1&1\\1&2\end{pmatrix} = \begin{pmatrix}8&13\\-5&-9\end{pmatrix}$

2.7.13.b

Let $P = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n$

then $V = \mathrm{null}(P(D))$. Let S be the set of Solutions to the non-homogeneous. let z be a particular solution and define $K = \{z + y \mid y \in \mathrm{null}(P(D))\}$.

We show $S = K$.

$K \subseteq S$: let $z + y \in K$

then $P(D)(z+y) = P(D)(z) + P(D)(y)$
$= x + 0$
$= x$

So $x + y \in S \Rightarrow K \subseteq S$.

$S \subseteq K$: let $s \in S$, so $P(D)(s) = x$

also $P(D)(z) = x$, So $P(D)(z) = P(D)(s)$

$\Rightarrow P(D)(s) - P(D)(z) = 0$

$\Rightarrow P(D)(s - z) = 0$

$\Rightarrow s - z \in \text{null}(P(D))$

So $s = (s-z) + z \in K$

$\therefore S \subseteq K$ ✓