HW 3 Solutions

2.1 #6 Let $A, B \in M_{n\times n}(F)$.

Then $(A+B)_{ij} = A_{ij} + B_{ij}$, so

$$tr(A+B) = \sum_{i=1}^{n} A_{ii} + B_{ii} = \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = trA + trB$$

$$(cA)_{ij} = cA_{ij}$$

so $tr(cA) = \sum_{i=1}^{n} cA_{ii} = c\sum_{i=1}^{n} A_{ii} = c\,trA$.

So $T: M_{n\times n}(F) \longmapsto F$, $T(A) = trA$ is linear

Define E_{ij} to be the matrix with 1 in row i, col j 0 elsewhere

Then if $f \in F$ $tr(fE_{11}) = f$ so T is onto F,

rank $T = 1$, & $\{1\}$ is a basis for $R(T)$.

$$tr(A) = 0 \Rightarrow A_{11} = -\sum_{i=2}^{n} A_{ii}$$

So $A \in N(T) \Leftrightarrow$

$$A = \begin{pmatrix} (-\sum_{i=2}^{n} A_{ii}) & \cdots & A_{1n} \\ \vdots & & \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

$\Rightarrow \{E_{ij}\}_{i\neq j} \cup \{E_{ii} - E_{11}\}_{i=2}^{n}$ is a basis for null T,

So dim null $T = n^2 - 1$, & T is not onto

if $n > 1$.

2.2 #5a) $\alpha = \left\{ \underset{\alpha_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{\alpha_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{\alpha_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{\alpha_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$

$T: M_{2\times 2}(F) \longrightarrow M_{2\times 2}(F) \quad T(A) = A^t$

$$[T]_\alpha = [\,(T(\alpha_1))_\alpha \quad (T(\alpha_2))_\alpha \quad (T(\alpha_3))_\alpha \quad (T(\alpha_4))_\alpha\,]$$

$$= [\,(\alpha_1)_\alpha \quad (\alpha_3)_\alpha \quad (\alpha_2)_\alpha \quad (\alpha_4)_\alpha\,]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.4 #5) In the RRE form of A col 3 is 2 times col 1 - five times col 2,

So by Thm 3.16 (d)

$$A_3 = 2A_1 - 5A_2 = 2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - 5\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = A_3}$$

& similarly $A_5 = -2A_1 - 3A_2 + 6A_4$

$$= -2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + 6\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ -7 \\ -9 \end{pmatrix} = A_5}$$