Series Tips and Examples

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One of the main goals in Chapter 11 is being able to decide whether an infinite series converges or diverges. To do this, there are quite a few tests to try. In order to successfully use these, one needs both to have mastered the details of the application of the individual tests, and to have developed a good feeling for which tests are likely to be useful when.

Below we will go through each of these one by one, and work through a couple examples of varying difficulty for each one.

- **Geometric Series**

  The easiest kind of series is the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1}$$

We can say that this series converges if $|r| < 1$ and diverges if $|r| \geq 1$ (assuming $a \neq 0$); even better, when $|r| < 1$ we even know exactly what it converges to:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{(if } |r| < 1\text{)}$$

This is unusual and lucky...we will rarely be able to say what a series actually converges to, only whether it converges or diverges.

Note the precise form of the series: the sum begins from $n = 1$, and the powers of r are $n-1$. Geometric series should be easy, but the main challenge would be recognizing them and working them into this form. For example, $\sum_{n=1}^{\infty} ar^n$ is still a geometric series, but it has r^n instead of r^{n-1}. To put it into the form $\sum_{n=1}^{\infty} ar^{n-1}$ you can group one of the r's with the a to get

$$\sum_{n=1}^{\infty} ar^n = \sum_{n=1}^{\infty} (ar)r^{n-1} = \frac{ar}{1-r} \quad \text{(again, only if } |r| < 1\text{)}$$

On the other hand, $\sum_{n=0}^{\infty} ar^n$ is exactly the same as $\sum_{n=1}^{\infty} ar^{n-1}$. Why? Look at the first several terms in the sum. In both cases the sigma notation represents $a + ar + ar^2 + ar^3 + \ldots$.

Example 1. $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$

Solution:

$$\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{\pi^{n-1}}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{\pi}{3}\right)^{n-1}$$

By rearranging things a little bit, we recognize this as a geometric series with $a = 1/3$ and $r = \pi/3$. Since $\pi/3 > 1$, this one diverges. □

- **p-series**

  This is closely related to the p-series test for improper integrals:

  $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$.

  Not much more to say here, except to point out especially that $\sum\limits_{n=1}^{\infty} 1/n$ (the 'harmonic series') diverges ($p = 1$).

  Example 2. *For which $c > 0$ does $\sum\limits_{n=1}^{\infty} c^{\ln(n)}$ converge?*

  Solution: The key here, as with many problems involving a log in the exponent, is to rewrite

  $$c^{\ln(n)} = e^{(\ln(c))(\ln(n))} = n^{\ln(c)}.$$

  So,

  $$\sum_{n=1}^{\infty} c^{\ln(n)} = \sum_{n=1}^{\infty} n^{\ln(c)} = \sum_{n=1}^{\infty} \frac{1}{n^{-\ln(c)}}.$$

  This is a p-series which we know converges precisely when $-\ln(c) > 1 \Rightarrow \ln(c) < -1 \Rightarrow c < e^{-1}$. Therefore the series converges when $0 < c < e^{-1}$, diverges for $c \geq e^{-1}$. □

One main purpose of the geometric series and p-series is to provide a healthy stock of simple series that you should automatically be able to recognize. This will come in handy later with the comparison tests, since we will need a supply of known series to which we can compare more complicated ones.

- **Test for Divergence**

  The Test for Divergence is not useful that often, but when it works it is very easy to apply. However, there is a very common mistake associated with it. The Test for Divergence says

  If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

  It does NOT say:

  [WRONG] If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

  In fact, many series have $\lim\limits_{n\to\infty} a_n = 0$, but may still diverge. A simple example of this is $\sum\limits_{n=1}^{\infty} \frac{1}{n}$.

  So be very careful how you use this. If you are given a series, and you notice right away that the limit of the terms is not zero (either because the limit doesn't exist, or the limit is infinity, or the limit is finite but just not equal to zero), you can immediately invoke the Test for Divergence to say the series must diverge. If you notice that the limit of the terms *is* zero, however, you must find a different way of deciding whether the series converges.

Example 3 (Easy). $\sum_{n=1}^{\infty} \arctan(n)$

Solution: Since $\lim_{n\to\infty} \arctan(n) = \pi/2$, this series must diverge. $\square$

Example 4 (Harder). $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$

Solution: This is a bit harder, because although it is true that $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \neq 0$, this isn't exactly obvious.

However, if one happens to know that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$, then it's apparent that

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0.$$

So the series diverges.

The limit $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$ comes up occasionally and may be worth knowing. Why is it true? Again, rewrite it:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n\ln(\frac{n+1}{n})} = e^{\lim_{n\to\infty} n\ln(\frac{n+1}{n})}$$

provided that the limit in the exponent exists. You can use l'Hospital's Rule to see that

$$\lim_{n\to\infty} n\ln(\frac{n+1}{n}) = \lim_{n\to\infty} \frac{\ln(\frac{n+1}{n})}{1/n} = 1.$$

$\square$

- **Integral Test**

  The Integral Test relates the convergence of a series to convergence of a corresponding improper integral. Keep in mind that there are a couple conditions that must be satisfied for this to work.

  If $f(x)$ is a *positive*, *continuous*, *decreasing* function and $a_n = f(n)$ for every n, then

  - If $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
  - If $\int_1^\infty f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

  So if you see a series where the terms can be converted into a function (cases where this might not happen are, for example, if the terms of the series contain $n!$ or $(-1)^n$...since e.g. $x!$ is not defined when x is not an integer), and that function looks easy to integrate, then this test will be useful. You'll just need to remember how to evaluate an improper integral.

  Unless it is completely obvious, part of the work is checking that your function $f(x)$ really is decreasing. This can be done by seeing if the derivative is ≤ 0.

  (Obviously decreasing: $f(x) = 1/x$. Not obviously decreasing: $f(x) = \frac{x^2}{1-2x+x^3}$.)

Example 5 (Medium). $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

Solution: The first thing to recognize is that $f(x) = \frac{1}{x\sqrt{\ln(x)}}$ has a pretty easy integral. Since $x\sqrt{\ln(x)}$ is increasing, $f(x)$ is clearly decreasing. It is positive and continuous as long as $x > 1$. (Note that this is why the sum starts with $n = 2$ on this one. For $n = 1$, we'd have a 0 in the denominator since $\ln(1) = 0$.) The integral is done with the substitution $u = \ln(x)$, $du = \frac{1}{x}dx$. So $\int \frac{1}{x\sqrt{\ln(x)}}dx = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u} + c = 2\sqrt{\ln(x)} + c$, and

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{\ln(x)}}dx = \lim_{t\to\infty} \left(2\sqrt{\ln(x)}\Big|_2^t\right)$$
$$= \lim_{t\to\infty} \left(2\sqrt{\ln(t)} - 2\sqrt{\ln(2)}\right) = +\infty.$$

The improper integral diverges, so the Integral Test tells us that the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ diverges as well. □

- **Comparison Test**

  The Comparison Test for series is similar to the comparison test for improper integrals. If we can find a bigger series that we already know converges, the smaller one converges too; if we can find a smaller series that we already know diverges, the bigger one diverges too.

  Be careful about which direction this goes in! This is another extremely common error. If you find another series that is *bigger* than your series, and is known to *diverge*, this tells you nothing. *Nothing!!* *shakes fist*

  Therefore, if you're angling to use the Comparison Test, decide first whether you think your series converges or diverges. If you think it *converges*, look for ways to make it *bigger*; if you think it *diverges*, look for ways to make it *smaller*.

  Here's the exact statement of the Comparison Test:

  - If $0 \le a_n \le b_n$ for all n, and $\sum_{n=1}^{\infty} b_n$ *converges*, then $\sum_{n=1}^{\infty} a_n$ also converges.
  - If $0 \le b_n \le a_n$ for all n, and $\sum_{n=1}^{\infty} b_n$ *diverges*, then $\sum_{n=1}^{\infty} a_n$ also diverges.

  Notice, also, that like the Integral Test this only works for series with all *positive* terms.

Example 6 (Easiest). $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$

Solution: As mentioned above, we first want to get an intuitive idea for whether this should converge or diverge. In this case, you should imagine that when n is very large, the difference between $n^4 - 1$ and n^4 is relatively extremely small. So for large n, $\frac{n^3}{n^4-1} \approx \frac{n^3}{n^4} = \frac{1}{n}$. Since we all know that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, we therefore expect $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$ to diverge as well.

Now, to make this precise using the Comparison Test, note that for all n,

$$\frac{n^3}{n^4-1} > \frac{n^3}{n^4} = \frac{1}{n}$$

because the denominator n^4 is larger than the denominator n^4-1 on the left.

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges and $\frac{n^3}{n^4-1}$ is *bigger* than $1/n$, $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$ must diverge.

Notice how it was useful that the denominator was n^4-1, something easy to make bigger by eliminating the -1. If it had been $\frac{n^3}{n^4+1}$ instead, our intuitive feeling is still that the series is similar to $\sum_{n=2}^{\infty} 1/n$, but the Comparison Test just doesn't work as nicely. That would be a good case to use the Limit Comparison Test instead. □

Example 7 (Still easy). $\sum_{n=1}^{\infty} \frac{n}{(n+1)3^n}$

Solution: This time, the idea should be that as n gets very large, $\frac{n}{n+1}$ is very close to (and slightly smaller than) 1. Thus $\frac{n}{(n+1)3^n} \approx \frac{1}{3^n}$. $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, as a geometric series with $r = 1/3$.

Since $\frac{n}{n+1} < 1$ for all n,

$$\frac{n}{(n+1)3^n} < \frac{1}{3^n}$$

and $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (to $\frac{1/3}{1-(1/3)} = 1/2$), so $\sum_{n=1}^{\infty} \frac{n}{(n+1)3^n}$ also converges by the Comparison Test. □

Example 8 (Hard). $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{(\ln n)}}$

Solution: This was done in lecture at some point, but it's an interesting problem. Like in Example 2, the first step is to rearrange things to bring the $\ln(n)$ down out of the exponent.

$$\frac{1}{(\ln n)^{(\ln n)}} = \frac{1}{e^{(\ln(\ln n))(\ln n)}} = \frac{1}{n^{\ln(\ln(n))}}.$$

Now, $\ln(\ln(n))$ increases very very slowly, but it is increasing. In particular, at some point it will be larger than 2. In fact, $\ln(\ln(n)) \geq 2 \Leftrightarrow \ln(n) \geq e^2 \Leftrightarrow n \geq e^{e^2} \approx 1618$.

So as long as $n > 1618$, $\ln(\ln(n)) > 2$ and so $\frac{1}{(\ln n)^{(\ln n)}} = \frac{1}{n^{\ln(\ln(n))}} < \frac{1}{n^2}$.

This raises an important point. Although it may not be the case that $\frac{1}{n^{\ln(\ln(n))}} < \frac{1}{n^2}$ for *all* n, it is true for all n past a certain point. The Comparison Test applies to show that

$$\sum_{n=1619}^{\infty} \frac{1}{(\ln n)^{(\ln n)}} < \sum_{n=1619}^{\infty} \frac{1}{n^2}$$

and therefore converges. But

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{(\ln n)}} = \sum_{n=2}^{1618} \frac{1}{(\ln n)^{(\ln n)}} + \sum_{n=1619}^{\infty} \frac{1}{(\ln n)^{(\ln n)}}.$$

The first sum is finite as there are only a finite number of terms; we just decided that the second is convergent. So the two added together will be convergent.

This illustrates a general principal not covered very thoroughly in the textbook, that in dealing with limits and series, we are only interested in what *eventually* happens, and for the most part don't care if there is some weird stuff happening with a finite number of terms at the beginning. □

- **Limit Comparison Test**

  Sometimes we are given a series where we'd like to use the Comparison Test to compare it to a more familiar series, but the inequalities just don't seem to work out quite right.

  For example, we know that $\sum_{n=1}^{\infty} \frac{n}{n^3-1}$ should be compared to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p-series test. But, annoyingly, $\frac{n}{n^3-1} > \frac{1}{n^2}$...our series is bigger than the convergent one we want to compare it to, the inequality is going the wrong way! I suspect it is doing it entirely out of spite.

  The Limit Comparison Test saves the day. It helps us get around tweaking inequalities for the regular Comparison Test when we already know what we want to compare to. And it can work for some much harder problems than just the standard $\frac{n}{n^3-1}$ stuff as well.

  Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ exists, is finite, and greater than zero. Then

  - If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.
  - If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Example 9 (Easy). $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$

Solution: Compare this with Example 7. Just as in that example, the idea is that $\frac{n^3}{n^4+1}$ is similar to $\frac{1}{n}$. But the regular Comparison Test doesn't work so well, since $\frac{n^3}{n^4+1} < \frac{1}{n}$, so we've found something *bigger* which *diverges*, which is useless as far as the Comparison Test goes.

With the Limit Comparison Test we can still make the comparison work. Letting $a_n = \frac{n^3}{n^4+1}$ and $b_n = 1/n$,

$$\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \frac{\left(\frac{n^3}{n^4+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^4}{n^4+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^4}} = 1.$$

Since this limit is > 0 and finite, and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the result is that $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$ diverges as well. □

Example 10 (Extra hard)**.** $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{n^2}\right)$

Solution: The hardest part here is figuring out which series to compare this to. So, think about what $\ln\left(\frac{n^2+1}{n^2}\right)$ looks like when n is very big.

$$\ln\left(\frac{n^2+1}{n^2}\right) = \ln\left(1 + \frac{1}{n^2}\right)$$

and n being very big translates into $\frac{1}{n^2}$ being very small.

So now we ask the question of what $\ln(1+x)$ looks like when x is very small, i.e. very close to 0. The idea here is that for x very close to 0, the graph of $y = \ln(1+x)$ will be very closely approximated by its tangent line at 0.

Since the derivative is $y' = 1/(1+x)$, the slope of that tangent line is 1. The y-intercept is $y(0) = \ln(1) = 0$. So the equation for the tangent line is $y = x$.

What we've discovered here is that for $x \approx 0$, $\ln(1+x) \approx x$. Going back to n now, if n is very huge, then $1/n^2 \approx 0$ and so $\ln(1 + \frac{1}{n^2}) \approx \frac{1}{n^2}$.

That's what we needed to know: at least when n is really big, the terms in our series can be approximated by $1/n^2$. So we're going to set $b_n = 1/n^2$ and use the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln\left(\frac{n^2+1}{n^2}\right)}{\frac{1}{n^2}}.$$

To get this limit, l'Hospital's Rule is needed. The derivative of $\ln\left(\frac{x^2+1}{x^2}\right)$ is

$$\frac{d}{dx}\left(\ln\left(\frac{x^2+1}{x^2}\right)\right) = \frac{x^2}{x^2+1}\frac{-2}{x^3}$$

and the derivative of $1/x^2$ is $\frac{-2}{x^3}$. So

$$\lim_{n\to\infty} \frac{\ln\left(\frac{n^2+1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2+1}\frac{-2}{n^3}}{\frac{-2}{n^3}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1.$$

Finally, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we conclude that $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{n^2}\right)$ converges too. □

- **Alternating Series Test**

  Up till now, we have mostly only seen tests that apply to series with all positive terms: the Integral Test, Comparison Test, and Limit Comparison Test. The next two give us ways of dealing with series that have some negative terms. Therefore, if you see a series that has some negative terms in it, that's a good sign you should think about either the Alternating Series Test, or Absolute Convergence.

On top of that, alternating series are awfully easy to recognize. An alternating series is one in which every term switches signs from the last term: +,-,+,-,+,-,.... To signify this, you will usually see some minor variation on $(-1)^n$ or, if someone's trying to be devious, perhaps $\cos(n\pi)$[1].

Example 11 (Easy). $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

Solution: Clearly this is an alternating series, $\sum_{n=2}^{\infty}(-1)^{n-1}b_n$ with $b_n = 1/(\sqrt{n}-1)$, so it should be no big surprise that we're going to start by trying the Alternating Series Test. It needs to be checked that the terms are decreasing, namely that for all $n \geq 2$ that $b_{n+1} \leq b_n$, and that $\lim_{n\to\infty} b_n = 0$.

It is in fact obvious that $1/(\sqrt{n+1}-1) < 1/(\sqrt{n}-1)$, since $\sqrt{n+1} > \sqrt{n}$. That $\lim_{n\to\infty} 1/(\sqrt{n}-1) = 0$ is also easy to see, since $\lim_{n\to\infty}(\sqrt{n}-1) = +\infty$.

Thus by the Alternating Series Test we've found that $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges.

(By the way, note that this series is *conditionally convergent*: $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ converges, but $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges.) □

Example 12 (Hardish). $\sum_{n=1}^{\infty} \cos(n\pi)\frac{(\arctan(n)-1)^2}{n}$

Solution: It's an alternating series with $b_n = \frac{(\arctan(n)-1)^2}{n}$. It seems like this should work: $(\arctan(n)-1)^2$ is bounded (by $(\pi/2-1)^2$), so as n increases b_n should decrease. At least it is clear that

$$\lim_{n\to\infty} \frac{(\arctan(n)-1)^2}{n} \leq \lim_{n\to\infty} \frac{(\pi/2-1)^2}{n} = 0.$$

Showing that $b_{n+1} \leq b_n$ unfortunately gets a bit complicated. When this is in doubt, generally the best idea is to define a function $f(x) = \frac{(\arctan(x)-1)^2}{x}$ and check if that is decreasing by looking at whether $f'(x) \leq 0$.

$$f'(x) = \frac{2(\arctan(x)-1)\frac{x}{1+x^2} - (\arctan(x)-1)^2}{x^2}$$

Since x^2 is always > 0 ($x \neq 0$), to check where $f'(x) \leq 0$ it is enough to see for which x is

$$2(\arctan(x)-1)\frac{x}{1+x^2} - (\arctan(x)-1)^2 \leq 0$$

[1] Don't be too impressed: $\cos(n\pi) = (-1)^n$.

$$\Updownarrow$$

$$2(\arctan(x)-1)\frac{x}{1+x^2} \leq (\arctan(x)-1)^2$$

$$\Updownarrow$$

$$\frac{2x}{1+x^2} \leq \arctan(x)-1.$$

It is a little bit hard to calculate exactly for which x this last inequality holds. However, as $x \to \infty$ the right hand side is increasing with limit $\pi/2 - 1 > 0$; the left hand side is decreasing with limit 0. Therefore, at some finite point $\frac{2x}{1+x^2}$ must drop permanently below $\arctan(x) - 1$. So let's just say that $\frac{2x}{1+x^2} \leq \arctan(x) - 1$ when $x > k$...we haven't found what k might be, but we've decided that such a k must exist.

The result is that as long as $n > k$, $f(n+1) \leq f(n)$ so $b_{n+1} \leq b_n$. At this point the Alternating Series Test kicks in. As discussed above in Example 8, we can turn a blind eye to annoying things happening for a finite number of terms at the beginning of the series...after n is bigger than k, the Alternating Series Test tells us that the rest of the series converges, so the whole series will converge. □

- **Absolute Convergence**

Not every series with both positive and negative terms is an alternating series. For those that are not, the first thing to do is see if the series might be absolutely convergent: a series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is convergent, i.e. if the series converges when you throw in absolute values. The good news about $\sum_{n=1}^{\infty} |a_n|$ is that it has all positive terms, so it is amenable to all our old tests; and:

> An absolutely convergent series is convergent, i.e. if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ does too.

Note once more that this only goes one way. If you have a series $\sum_{n=1}^{\infty} a_n$ with some negative and some positive terms, and you find out that $\sum_{n=1}^{\infty} |a_n|$ *diverges*, you can't conclude anything... $\sum_{n=1}^{\infty} a_n$ could diverge, or it could converge.

Example 13 (Easy)**.** $\sum_{n=1}^{\infty} e^{-n}\sin(n)$

Solution: The feeling should be that the $\sin(n)$ is fairly insignificant, as sine just bounces between -1 and 1 while the e^{-n} makes the series converge. But the Comparison Test or Integral Test can't be used because some terms of the series are negative. The Alternating Series Test can't be used because, well, it's not really an alternating series.

Letting $b_n = |e^{-n}\sin(n)|$, however, $0 \le b_n \le e^{-n}$ for all n. $\sum_{n=1}^{\infty} e^{-n}$ converges, as it's a geometric series (with $r = 1/e$). So by the Comparison Test, $\sum_{n=1}^{\infty} |e^{-n}\sin(n)|$ converges. This means that $\sum_{n=1}^{\infty} e^{-n}\sin(n)$ converges absolutely.

Conclude that $\sum_{n=1}^{\infty} e^{-n}\sin(n)$ converges. □

- **Ratio Test**

  Suppose $\lim_{n\to\infty} |a_{n+1}/a_n| = L$.

  - If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely (and therefore converges).
  - If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
  - If $L = 1$, the Ratio Test is inconclusive.

The Ratio Test is really very useful in a handful of common situations. By looking at the ratio a_{n+1}/a_n, we can simplify a lot of nasty stuff like $n!$ or x^n, since $(n+1)!/n! = n+1$ and $x^{n+1}/x^n = x$. Therefore, if you see a factorial, and maybe some n^{th} powers, that's a good sign you should use the Ratio Test.

Example 14 (Easy). $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Solution: As promised, as soon as we see that factorial and the 2^n we think, '*Ratio Test!!!*'

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{2^{n+1}}{(n+1)!}\right)}{\left(\frac{2^n}{n!}\right)}\right| = \lim_{n\to\infty}\left|\left(\frac{2^{n+1}}{2^n}\right)\left(\frac{n!}{(n+1)!}\right)\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = 0.$$

Because $0 < 1$, the Ratio Test says that the series converges. □

Example 15 (Somewhat harder). $\sum_{n=1}^{\infty} \frac{(n!)(n^2)}{e^{n^2}}$

Solution: Again, we see factorials and exponents...first instinct should be the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{\left(\frac{(n+1)!(n+1)^2}{e^{(n+1)^2}}\right)}{\left(\frac{n!n^2}{e^{n^2}}\right)}\right| = \lim_{n\to\infty}\left|\left(\frac{(n+1)!}{n!}\right)\left(\frac{(n+1)^2}{n^2}\right)\left(\frac{e^{n^2}}{e^{(n+1)^2}}\right)\right| =$$

$$\lim_{n\to\infty}\left|(n+1)\left(\frac{n+1}{n}\right)^2 e^{n^2-(n+1)^2}\right| = \lim_{n\to\infty}\left|(n+1)\left(\frac{n+1}{n}\right)^2 e^{-2n-1}\right| =$$

$$\lim_{n\to\infty}\left|(n+1)e^{-2n-1}\right|\lim_{n\to\infty}\left|\frac{n+1}{n}\right|^2=\lim_{n\to\infty}\left|(n+1)e^{-2n-1}\right|$$

(since $\lim\limits_{n\to\infty}\frac{n+1}{n}=1$).

This last limit you can check with l'Hospital's Rule...it is 0. So again we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0<1$$

and by the Ratio Test $\sum\limits_{n=1}^{\infty}\frac{(n!)(n^2)}{e^{n^2}}$ converges. □

- **Root Test**

  This is similar to the Ratio Test, but instead of looking at the ratios, we take n^{th} roots. Whoa.

  Suppose $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=L$.

  - If $L<1$, then $\sum\limits_{n=1}^{\infty}a_n$ converges absolutely (and therefore converges).
  - If $L>1$, then $\sum\limits_{n=1}^{\infty}a_n$ diverges.
  - If $L=1$, the Root Test is inconclusive.

The Root Test seems a bit less useful than the Ratio Test. It is mostly used, not surprisingly, for getting around pesky n^{th} powers. So, think of the Root Test when you see a series where all the terms are n^{th} powers. In many such cases, but not all, the Ratio Test would also work.

Example 16 (Easy). $\sum\limits_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$

Solution: A telltale sign that the Root Test might work is that everything is raised to the n^{th} power.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\left|\frac{n^2+1}{2n^2+1}\right|=\lim_{n\to\infty}\left|\frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}}\right|=\frac{1}{2}.$$

Again, $1/2<1$ so the Root Test implies that $\sum\limits_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ converges. □

Example 17 (Harder). $\sum\limits_{n=1}^{\infty}\frac{n^{3n-1}}{3^n}$

Solution: Seeing the n's in the exponent, again we think of the Root Test. We get

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\left|\frac{n^{3n-1}}{3^n}\right|^{1/n}=\lim_{n\to\infty}\left(\frac{n^3}{3}(n^{-1/n})\right)=\frac{1}{3}\lim_{n\to\infty}\frac{n^3}{n^{1/n}}$$

The numerator goes to $+\infty$, but $\lim_{n\to\infty} n^{1/n}$ is a little bit less clear: as the n tries to go to ∞, the n^{th} root makes it smaller again. Who wins?

In fact $\lim_{n\to\infty} n^{1/n} = 1$. This can be computed similarly as $\lim_{n\to\infty} (1+1/n)^n$ in Example 4:

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln(n)/n} = e^{\lim_{n\to\infty} \ln(n)/n} = e^{\lim_{n\to\infty} 1/n} = e^0 = 1$$

(using l'Hospital's Rule in the last step).

Thus we are left with

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{3} \lim_{n\to\infty} \frac{n^3}{n^{1/n}} = +\infty.$$

As the limit here is > 1, the Root Test says the series diverges. $\square$