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OPERATIONS ON CERTAIN NON-COMMUTATIVE OPERATOR-VALUED RANDOM VARIABLES

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An example motivating the study of the addition of free pairs of "non-commutative operator-valued random variables" is provided by the computation of spectra of convolution operators on free groups.

Let G be the (non-commutative) free group on two generators g_1, g_2 and let λ denote the left regular representation on $l^2(G)$. To compute spectra of convolution operators

$$Y = \sum_{g \in G} c_g \lambda(g)$$

with $c_g \neq 0$ only for finitely many $g \in G$ it suffices to be able to decide whether such Y is invertible. This in turn is equivalent to deciding whether a certain operator

$$X = \sum_{k \in \mathbb{Z}} (\alpha_k \otimes \lambda(g_1^k) + \beta_k \otimes \lambda(g_2^k))$$

where $\alpha_k = \alpha_{k'}^*$, $\beta_{-k} = \beta_k^*$ are $n \times n$ matrices, is invertible. If $n = 1$, i.e. if the matrices are scalars, then the spectrum of X can be computed using our results on the addition of free pairs of non-commutative random variables [8]. Thus the computation of the spectrum of Y is reduced to a generalization of the addition of free pairs of non-commutative random variables to the case of "matrix-valued non-commutative random variables". (For a different approach to the question of computing the spectrum of Y see [1].)

The present paper deals with the extension of our previous work ([8],[9]) on addition and multiplication of free pairs of non-commutative random variables to, what might be called, the operator-valued case. This means that the field of complex numbers is replaced by an operator algebra, the free products are with amalgamation over this algebra and the specified states are replaced by specified conditional expectations. Also the natural framework of operator algebras with dual algebraic structure ([10]) for the considered operators in the "scalar" case has a corresponding extension to the "operator-valued" case.

Though our results are meant for applications to operator algebras and spectral theory, most of our considerations will be in a purely algebraic context, since we shall be mainly concerned with finding the formulae for computing the operation on the distributions of the random-variables. Concerning distributions of operator-valued non-commutative random-variables, let us only say that since the scalars $\mathbb{C}$ are replaced by an operator algebra B, the moments of the variable X are the expectation valued of monomials of the form $Xb_1Xb_2\dots Xb_{n-1}X$. It is an important fact for the computation of spectra that the addition of free pairs of B-valued random variables gives an operation among the symmetric parts of the distributions i.e. among the expectation values of monomials of the form $bXbX\dots bXb$. For the symmetric distributions the addition formulae closely resemble those in the scalar case with the generating series viewed as germs of maps $\mathbb{C} \to \mathbb{C}$ replaced by germs of maps $B \to B$.

The paper has eight sections.

The first section discusses free families of non-commutative B-valued random variables and distributions of such random variables.

The second and third section deal with the algebras $A(M)$ and the canonical form of a random variable with a given distribution. This is the analogue for the B-valued case of the special Toeplitz operators which we used in the scalar case for studying the addition of free pairs of non-commutative random variables. We also give formulae for the canonical form of a random variable after multiplication by elements in B.

The fourth section gives the solution to the addition problem for the symmetric parts of distributions of B-valued random variables. It is obtained by studying the differential equation for semigroups with respect to addition. The final formulae closely resemble those in the scalar case.

The fifth section deals with the differential equation for semigroups with respect to the multiplicative operation. We also introduce a corresponding free exponential map. Studying the differential equations we show that multiplicative free convolution is well defined for the symmetric distributions.

The sixth section presents the application to the computation of spectra of convolution operators on free groups.

Section seven is a brief outline of the necessary adaptations to make the operators on B-valued random variables fit in a framework of dual algebraic structures as in the scalar case.

Section eight deals with the free central limit theorem for B-valued random variables generalizing our results from the scalar case [7].

The present paper is an expanded version of our paper with the same title (preliminary version) INCREST Preprint No. 42/1986, Bucarest. This revised version consists of the material of the preliminary version (without changes) to which we have added 3.3.–3.7., 5.4.–5.10. and section 8.

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1. B-valued non-commutative random-variables.

1.1. Throughout B will denote a fixed unital algebra over $\mathbb{C}$ (this choice of the base field is inessential). Let A be another unital algebra over $\mathbb{C}$ containing B as a subalgebra (with the same unit) and let $\varphi : A \to B$ be a conditional expectation i.e. a linear map such that $\varphi(b_1 a b_2) = b_1 \varphi(a) b_2$ if $b_1, b_2 \in B$, $a \in A$ and $\varphi(b) = b$ if $b \in B$. An element $a \in A$, will be viewed as a B-valued random variable.

1.2. Definition. *Let (A, φ) be as in 1.1 and let $B \subset A_i \subset A$ $(i \in I)$ be subalgebras. The family $(A_i)_{i \in I}$ will be called free if*

$$\varphi(a_1 a_2 \dots a_n) = 0$$

whenever $a_j \in A_{i_j}$ with $i_1 \neq i_2 \neq \dots \neq i_n$ and $\varphi(a_j) = 0$ for $1 \leq j \leq n$. A family of subsets $X_i \subset A$ (elements $a_i \in A$) where $i \in I$ will be called free if the family of subalgebras A_i generated by $B \cup X_i$ (respectively $B \cup \{a_i\}$) is free.

Free families of subalgebras arise in the C^*-algebraic context (in which case the conditional expectations are of norm one) from reduced free products with amalgamation (see §5 in [7]).

1.3. Proposition. *Let (A, φ) be as in 1.1 and let $B \subset A_i \subset A$ $(i \in I)$ be subalgebras such that A is generated by $\cup_{i \in I} A_i$ and $(A_i)_{i \in I}$ is a free family. Then φ is completely determined by the $\varphi_i = \varphi \mid A_i$ $(i \in I)$.*

Proof. By linearity it is sufficient to prove that we may compute $\varphi(a_1 \dots a_n)$ whenever $a_j \in A_{i_j}$ $(1 \leq j \leq n)$. We shall proceed by induction on the least non-negative integer such that $\varphi_{i_j}(a_j) = 0$ if $k < j$ and $i_{k+1} \neq i_{k+2} \neq \dots \neq i_n$. If $k = 0$ then $\varphi(a_1 \dots a_n) = 0$. Assume our assertion has been established up to a certain k. Then for $k+1$ if $i_k \neq i_{k+1}$ we have

$$\begin{aligned} \varphi(a_1 \dots a_n) &= \varphi(a_1 \dots a_k (\varphi_{i_{k+1}}(a_{k+1}) a_{k+2}) a_{k+3} \dots a_n) + \\ &+ \varphi(a_1 \dots a_k a'_{k+1} a_{k+2} \dots a_n) \text{ where } a'_{k+1} = a_{k+1} - \varphi_{i_{k+1}}(a_{k+1}), \end{aligned}$$

so that the induction hypothesis applies.

If $i_k = i_{k+1}$ then we write

$$\begin{aligned} \varphi(a_1 \dots a_n) &= \varphi(a_1 \dots a_{k-1} (a_k a_{k+1} - \varphi_{i_k}(a_k a_{k+1})) a_{k+2} \dots a_n) + \\ &+ (a_1 \dots a_{k-1} \varphi_{i_k}(a_k a_{k+1}) a_{k+2} \dots a_n) \end{aligned}$$

which is again a reduction to the induction hypothesis. □

1.4. The algebra freely generated by B and an indeterminate X will be denoted by $B\langle X \rangle$. Let (A, φ) be as in 1.1 and $a \in A$ a B-valued random variable. The distribution of a is the conditional expectation $\mu_a : B\langle X\rangle \to B$ defined by $\mu_a = \varphi \circ \tau_a$ where $\tau_a : B\langle X\rangle \to A$ is the unique homomorphism such that $\tau_a(b) = b$ for $b \in B$ and $\tau_a(X) = a$. Quantities such as $\mu_a(b_0Xb_1X\dots b_{n-1}Xb_n)$ will be called moments. The set of all conditional expectations $\mu : B\langle X\rangle \to B$ will be denoted by Σ_B.

1.5. Let $\mathcal{G}_n$ denote the symmetric group and let

$$S_n(b_1\dots b_n) = \sum_{\sigma\in\mathcal{G}_n} b_{\sigma(1)}Xb_{\sigma(2)}\dots Xb_{\sigma(n)}$$

$S_1(b) = b$ and $S_0 = 1$. Let further

$$\begin{aligned} SB\langle X\rangle =&\ \text{l.s.}\{S_n(b,\dots,b) \mid b \in B, n \geq 0\} = \\ =&\ \text{l.s.}\{S_n(b_1,\dots,b_n) \mid b_j \in B, n \geq 0, n \geq j \geq 1\} \\ \widetilde{SB}\langle X\rangle =&\ \mathbb{C}X + X(SB\langle X\rangle)X \end{aligned}$$

where "l.s." denotes the vector space spanned by the given set.

Lemma. *We have*

$$B(SB\langle X\rangle)B = B + B(\widetilde{SB}\langle X\rangle)B.$$

Proof. The inclusion $\subset$ is obvious. To prove the converse remark that if $n \geq k+1$, $n \geq 3$, we have

$$\begin{aligned} &S_n(\underbrace{b,\dots,b}_{k\text{-times}},1,\dots,1) - (n-k)kbXS_{n-2}(\underbrace{b,\dots,b}_{(k-1)\text{-times}},1,\dots,1)X- \\ &-(n-k)kXS_{n-2}(\underbrace{b,\dots,b}_{(k-1)\text{-times}},1,\dots,1)Xb- \\ &-k(k-1)bXS_{n-2}(\underbrace{b,\dots,b}_{(k-2)\text{-times}},1,\dots)Xb = \\ &=(n-k)(n-k-1)XS_{n-2}(\underbrace{b,\dots,b}_{k\text{-times}},1,\dots,1)X. \end{aligned}$$

Taking into account that

$$n(n-1)XS_{n-2}(1,\dots,1)X = S_n(1,\dots,1)$$

the preceding recurrence relation applied for $k = 1,\dots,n-2$ can be used to prove inductively that for $n \geq 3$ and $1 \leq k \leq n-2$ we have

$$XS_{n-2}(\underbrace{b,\dots,b}_{k\text{-times}},1,\dots,1)X \in B(SB\langle X\rangle)B.$$

Also

$$X = 2^{-1}S_2(1,1) \in B(SB\langle X\rangle)B.$$

□

The above lemma implies that if $\mu \in \Sigma_B$ then $\mu \mid SB\langle X\rangle$ is completely determined by $\mu \mid \widetilde{SB}\langle X\rangle$ and conversely $\mu \mid \widetilde{SB}\langle X\rangle$ is completely determined by $\mu \mid SB\langle X\rangle$. We shall denote by $S\Sigma_B$ the set

$$S\Sigma_B = \{(\mu \mid B(SB\langle X\rangle)B) \mid \mu \in \Sigma_B\}$$

and we shall write $S\mu = \mu \mid B(SB\langle X\rangle)B$ if $\mu \in \Sigma_B$. If $a \in A$ is a random variable, then $S\mu_a$ will be called the symmetric distribution of a and quantities of the type $\mu_a(S(b_1,\dots,b_n))$ or $\mu_a(XS(b_1,\dots,b_n)X)$ will be called symmetric moments of a.

1.6. If $\{a_1,a_2\} \subset A$ is a free pair of B-valued random variables then it follows from Proposition 1.3 that $\mu_{a_1+a_2}$ and $\mu_{a_1a_2}$ depend only on μ_{a_1} and μ_{a_2}. For any given $\mu_1,\dots,\mu_n \in \Sigma_B$ one can find a free family $\{a_1,\dots,a_n\}$ of random-variables in some (A,φ) such that $\mu_{a_j} = \mu_j$. We shall not give an ad-hoc proof for this here since it will follow from our results on the canonical form of a random variable. This implies that there are well-defined operations, $\boxplus$ and $\boxtimes$ on Σ_B such that if $\{a_1,a_2\}$ is a free pair then

$$\mu_{a_1+a_2} = \mu_{a_1} \boxplus \mu_{a_2}$$
$$\mu_{a_1+a_2} = \mu_{a_1} \boxtimes \mu_{a_2}.$$

This gives two semigroup structures on Σ_B.

2. The algebra $A(M)$.

2.1. Let M be a right B-module and let $\mathcal{X}_n(M) = \mathcal{L}(M^{\otimes n}, B)$ be the n-linear B-valued maps of $M \times \ldots \times M$ into B (the $\otimes$ and linearity are over $\mathbb{C}$) and $\mathcal{X}_0(M) = B$. Let further $\mathcal{X}(M) = \oplus_{n\geq 0}\mathcal{X}_n(M)$ with its natural right B-module structure. If $\xi \in \mathcal{X}_n(M)$ we define the endomorphism $\lambda(\xi)$ of the right B-module $\mathcal{X}(M)$ by:

$$\lambda(\xi)\eta \in \mathcal{X}_{n+k}(M)$$
$$(\lambda(\xi)\eta)(m_1 \otimes \cdots \otimes m_{n_k}) =$$
$$= \eta(m_{n+1}\xi(m_1 \otimes \cdots \otimes m_n) \otimes m_{n+2} \otimes \cdots \otimes m_{n+k})$$

if $\deg \eta = k > 0$ where deg refers to the obvious grading of $\mathcal{X}(M)$ and

$$\lambda(\xi)\eta = \xi\eta$$

if $\deg \eta = 0$ i.e. $\eta \in B$. We also define $\lambda^*(m)$, where $m \in M$, by:

$$\lambda^*(m)\eta = 0 \text{ if } \deg \eta = 0$$
$$\deg \lambda^*(m)\eta = \deg \eta - 1$$
$$(\lambda^*(m)\eta)(m_1 \otimes \cdots \otimes m_{k-1}) = \eta(m \otimes m_1 \otimes \cdots \otimes m_{k-1})$$

if $\deg \eta = k > 0$.

$A(M)$ is the algebra of endomorphisms of the right B-module $\mathcal{X}(M)$ generated by

$$\{\lambda(\xi) \mid \xi \in \mathcal{X}_n(M), n \geq 0\} \cup \{\lambda^*(m) \mid m \in M\}.$$

Endowing $A(M)$ with the natural grading corresponding to its action on $\mathcal{X}(M)$ we have $\deg \lambda(\xi) = \deg \xi$ and $\deg \lambda^*(m) = -1$.

2.2. It is easy to check that the following equalities hold

$$\lambda(\xi_1)\lambda(\xi_2) = \lambda(\lambda(\xi_1)\xi_2)$$
$$\lambda^*(m)\lambda(\xi) = \lambda(\lambda^*(m)\xi) \text{ if } \deg \xi > 0$$
$$\lambda^*(m)\lambda(\xi) = \lambda^*(m\xi) \text{ if } \deg \xi = 0.$$

2.3. We define a linear map

$$\gamma : (\oplus_{n\geq 0}\mathcal{X}_n(M)) \oplus (\oplus_{k\geq 0} M^{\otimes k}) \to A(M)$$

by

$$\gamma(\xi \otimes (m_1 \otimes \cdots \otimes m_k)) = \lambda(\xi)\lambda^*(m_1)\dots\lambda^*(m_k).$$

Lemma. *γ is a bijection.*

Proof. Clearly the range of γ contains the $\lambda(\xi)$'s and the $\lambda^*(m)$'s and using the relations 2.2 we easily infer that the range of γ is an algebra, so that γ is onto.

For the injectivity let

$$\alpha = \sum_{k_0 \leq k \leq k_1} \sum_{i \in I_k} \xi_{i,k} \otimes \nu_{i,k} \neq 0$$

where $\xi_{i,k} \in \oplus_{n\geq 0}\mathcal{X}_n(M)$ and $\nu_{i,k} \in M^{\otimes k}$ for $i \in I_k$. Since $\alpha \neq 0$ we may assume the $\nu_{i,k}$'s are linearly independent and the $\xi_{i,k}$'s are non-zero. Then, fixing $i_0 \in I_{k_0}$ there is $\eta \in \mathcal{X}_{k_0}(M)$ such that $\eta(\nu_{i_0,k_0}) = 1 \in B$ and $\eta(\nu_{i,k_0}) = 0$ for $i \in I_{k_0} \setminus \{i_0\}$.

Let $\eta' \in \mathcal{X}_{k_0}(M)$ be defined by

$$\eta'(m_1 \otimes \cdots \otimes m_{k_0}) = \eta(m_{k_0} \otimes \cdots \otimes m_1).$$

We have

$$\gamma(\alpha)\eta' = \gamma\left(\sum_{i\in I_{k_0}} \xi_{i,k_0}\otimes \nu_{i,k_0}\right)\eta' =$$

$$= \sum_{i\in I_{k_0}} \lambda(\xi_{i,k_0})\eta(\nu_{i,k_0}) = \lambda(\xi_{i_0,k_0})1 =$$

$$= \xi_{i_0,k_0} \neq 0.$$

□

2.4. B identifies via $\lambda : \mathcal{X}_0(M) \simeq B. \to A(M)$ with a subalgebra of $A(M)$ and there is a linear map $\varepsilon_M : A(M) \to B$ defined by $\varepsilon_M(\gamma(\xi_n \otimes \nu_k)) = 0$ if $n+k>0$ where $\xi_n \in \mathcal{X}_n(M)$, $\nu_k \in M^{\otimes k}$ and $\varepsilon_M(\gamma(\xi_0\otimes\eta_0)) = \gamma(\xi_0\otimes\eta_0) = \xi_0\nu_0 \in B$ if $\xi_0 \in \mathcal{X}_0(M) = B$ and $\nu_0 \in M^{\otimes 0} \simeq \mathbb{C}$. It is easily seen that ε_M is a conditional expectation i.e. that $\varepsilon_M(\lambda(b_1)a\lambda(b_2)) = b_1\varepsilon_M(a)b_2$ and $\varepsilon_M(\lambda(b)) = b$.

2.5. *Remark.* If $B = \mathbb{C}$ and $M = \mathbb{C}^n$ then $A(\mathbb{C}^n)$ is isomorphic with a certain dense subalgebra of an extension of the C^*-algebra O_n of Cuntz [3] realized on the Fock space for Boltzmann statistics ([6], [5], [4]).

2.6. It will be useful to consider a larger algebra $\bar{A}(M) \supset A(M)$ acting on $\bar{\mathcal{X}}(M) = \prod_{n\geq 0}\mathcal{X}_n(M)$ such that there is a bijection

$$\bar{\gamma} : \left(\prod_{n\geq 0}\mathcal{X}_n(M)\right)\otimes(\oplus_{k\geq 0}M^{\otimes k}) \to \bar{A}(M)$$

extending γ and the multiplication of the formal sums which constitute $A(M)$ is also determined by the formulae 2.2. Thĕ obvious extension of ε_M to $\bar{A}(M)$ will be denoted also by ε_M. We have for $T \in \bar{A}(M)$

$$\varepsilon_M(T) = (T1)_0$$

where $1 \in B = \mathcal{X}_0(M) \subset \bar{\mathcal{X}}(M)$ and $(\cdot)_0$ denotes the component of degree zero. Note also that along the same lines as in the proof of Lemma 2.3 it is easy to show that the representation of $A(M)$ on $\bar{\mathcal{X}}(M)$ is faithful.

2.7. If $M = M_1 \oplus M_2$ there are injections

$$\chi_j : \left(\prod_{n\geq 0}\mathcal{X}_n(M_j)\right)\otimes(\oplus_{k\geq 0}M_j^{\otimes k}) \to \left(\prod_{n\geq 0}\mathcal{X}_n(M)\right)\otimes(\oplus_{k\geq 0}M^{\otimes k})$$

given by

$$\chi_j((\xi_n)_{n\geq 0}\otimes\nu_k)=(\xi_n\circ pr_j^{\otimes n})\otimes(i_j^{\otimes k}\nu_k)$$

where $i_j : M_j \hookrightarrow M$ are the natural inclusions and $pr_j : M \to M_j$ the proejctions onto the two summands and $\xi_0 \circ pr_j^{\otimes 0}$ means just ξ_0. Since the relations 2.2 determine the multiplication in the algebras $\bar{A}(\cdot)$ it is easy to check that the maps $h_j : \bar{A}(M_j) \to \bar{A}(M)$ such that $h_j \circ \bar{\gamma} = \bar{\gamma} \circ \chi_j$ are homomorphisms. Moreover we have $h_j(\lambda(b)) = \lambda(b)$ for $b \in B = \mathcal{X}_0(M_j) = \mathcal{X}_0(M)$ and $\varepsilon_M \circ h_j = \varepsilon_{M_j}$.

Proposition. *If $M = M_1 \oplus M_2$ then with h_1, h_2 as above, the pair of subalgebras $(h_j(\bar{A}(M)))_{j=1,2}$ is B-free in $(\bar{A}(M), \varepsilon_M)$.*

Proof. Write

$$\mathcal{X}(M) = \Gamma_1 \oplus \Gamma_2 \oplus B$$

where

$$\Gamma_j = \prod_{n\geq 1} \mathcal{L}(M_j \otimes M^{\otimes(n-1)}, B)$$

with $\mathcal{L}(M_j \otimes M^{\otimes(n-1)}, B)$ identified with a subspace of $\mathcal{L}(M^{\otimes n}, B)$ via $\eta_n \rightsquigarrow \eta_n \circ (pr_j \otimes id_M \otimes \cdots \otimes id_M)$. If $T \in h_2(\bar{A}(M_1))$ and $\varepsilon_M(T) = 0$ then $T(\Gamma_2 \oplus B) \subset \Gamma_1$. Also the analogue of this with 1 and 2 interchanged holds. This easily implies our assertion. For instance if $T_j \in h_1(\bar{A}(M_1))$ and $S_j \in h_2(\bar{A}(M_j))$ and $\varepsilon_M(T_j) = \varepsilon_M(S_j) = 0$ then $T_1 1 \in \Gamma_1$, $S_1 T_1 1 \in \Gamma_2$ and continuing in this way we get $S_n T_n \ldots S_1 T_1 1 \in \Gamma_2$ so that

$$\varepsilon_M(S_n T_n \ldots S_1 T_1) = 0.$$

□

2.8. If $M = B^m$ we shall denote

$$\bar{A}(M) \text{ by } \bar{A}(m) \text{ and } \varepsilon_M \text{ by } \varepsilon_m.$$

3. The canonical form.

3.1. Elements $a \in \bar{A}(1)$ of the form

$$a = \lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_n)$$

where $\xi_n \in \mathcal{X}_n(B)$, will be called canonical.

Proposition. *Given a distribution $\mu \in \Sigma_B$ there is a unique canonical element*

$$a = \lambda^*(1) + \sum_{n \geq 0} \lambda(\xi_n)$$

such that $\mu_a = \mu$.

Proof. We have $\mu_a(X) = \varepsilon_1(a) = \xi_0$ so that we must put $\xi_0 = \mu(X)$. If $n > 0$ we have

$$\begin{aligned} &\varepsilon_1(a\lambda(b_1)a\lambda(b_2)\dots a\lambda(b_n)a) = \\ &= \varepsilon_1(\lambda^*(1)\lambda(b_1)\lambda^*(1)\lambda(b_2)\dots\lambda^*(1)\lambda(b_n)\lambda(\xi_n)) + \\ &+ E_n(\xi_0, \dots, \xi_{n-1})(b_1 \otimes \dots \otimes b_n) \end{aligned}$$

where $E_n(\xi_0, \dots, \xi_{n-1}) \in \mathcal{L}(B^{\otimes n}, B)$ depends only on $\xi_0, \dots, \xi_{n-1}$. Remark that

$$\varepsilon_1(\lambda^*(1)\lambda(b_1)\dots\lambda^*(1)\lambda(b_n)\lambda(\xi_n)) = \xi_n(b_n \otimes \dots \otimes b_1).$$

We infer that ξ_n satisfies $\xi_n(b_n \otimes \dots \otimes b_1) = \mu(Xb_1Xb_2\dots Xb_nX) - E_n(\xi_0, \dots, \xi_{n-1})(b_1 \otimes \dots \otimes b_n)$ which determines ξ_n inductively. □

The canonical element a in the above proposition will be called the canonical form of a random variable with distribution μ and we shall write $\xi_n = R_{n+1}(\mu)$.

3.2. Proposition. *Let*

$$a_k = \lambda^*(1) + \sum_{n \geq 0} \lambda(\xi_{n,k})$$

$k = 1, 2, 3$ be canonical elements. Then $\mu_{a_3} = \mu_{a_1} \boxplus \mu_{a_2}$ if and only if

$$\xi_{n,3} = \xi_{n,1} + \xi_{n,2}$$

for all $n \geq 0$.

Proof. In view of the uniqueness of the canonical form it will be sufficient to prove that if $\xi_{n,3} = \xi_{n,1} + \xi_{n,2}$ for all $n \geq 0$ then $\mu_{a_3} = \mu_{a_1} \boxplus \mu_{a_2}$. Passing to $\bar{A}(2)$ we have in view of 2.7 that $h_1(a_1) + h_2(a_2)$ has distribution $\mu_{a_1} \boxplus \mu_{a_2}$.

Let

$$\begin{aligned} Y = h_1(a_1) + h_1(a_2) = \lambda^*(1 \oplus 1) + \sum_{n \geq 0} \lambda(\xi_{n,1} \circ pr_1 + \\ + \xi_{n,2} \circ pr_2). \end{aligned}$$

Expanding

$$\begin{aligned} &\varepsilon_2(Y\lambda(b_1)Y\dots Y\lambda(b_n)Y) \text{ and} \\ &\varepsilon_1(a_3\lambda(b_1)a_3\dots a_3\lambda(b_n)a_3) \end{aligned}$$

our assertion is obtained from the following remark. Let

$$\varepsilon_2(S_1\lambda(b_1)S_2\dots S_n\lambda(b_n)S_{n+1})$$

where each S_j is an element of one of the following forms

$$\lambda^*(1\oplus 1),\ \lambda(\beta_n\circ pr_1^{\otimes n}) \text{ or } \lambda(\beta_n\circ pr_2^{\otimes n}).$$

Then replacing S_j by S'_j where S'_j is obtained from S_j by replacing $\lambda^*(1\oplus 1)$ by $\lambda^*(1)$, $\lambda(\beta_n\circ pr_k^{\otimes n})$ $(k=1,2)$ by $\lambda(\beta_n)$ it is easy to see that

$$\begin{aligned}&\varepsilon_1(S'_1\lambda(b_1)S'_2\dots S'_n\lambda(b_n)S'_{n+1})=\\&=\varepsilon_2(S_1\lambda(b_1)S_2\dots S_n\lambda(b_n)S_{n+1}).\end{aligned}$$

□

Thus we have proved that

$$R_n(\mu_1\boxplus\mu_2)=R_n(\mu_1)+R_n(\mu_2)$$

for all $n\geq 1$ and $\mu_j\in\Sigma_B$, $j=1,2$.

3.3. The rest of this section will deal with the effect on the canonical form of the multiplication of a random variable by an element in B. We begin with some definitions this will require.

Since B is a $B-B$-bimodule, in addition to the right multiplication $\xi_n b$ by an element $b\in B$ of $\xi_n\in\mathcal{X}_n(M)$ we also may define $b\xi_n$ by

$$(b\xi_n)(m_1\otimes\cdots\otimes m_n)=b(\xi_n(m_1\otimes\cdots\otimes m_n)).$$

Further, if $M=B$ and $b\in B$, we shall also consider

$$d_n(b),s_n(b),\sigma_n(b):\mathcal{X}_n(B)\to\mathcal{X}_n(B)$$

defined by the formulae

$$\begin{aligned}&(d_n(b)\xi_n)(b_1\otimes\cdots\otimes b_n)=\xi_n(bb_1\otimes\cdots\otimes bb_n)\qquad(n\geq 1)\\&d_0(b)\xi_0=\xi_0\qquad(n=0)\\&(s_n(b)\xi_n)(b_1\otimes\cdots\otimes b_n)=b\xi_n(b_1\otimes b_2b\otimes b_3\otimes\cdots\otimes b_n)\qquad(n\geq 2)\\&(s_1(b)\xi_1)(b_1)=b\xi_1(b_1)\\&s_0(b)\xi_0=\xi_0\\&(\sigma_n(b)\xi_n)(b_1\otimes\cdots\otimes b_n)=b\xi_n(b_1b\otimes\cdots\otimes b_nb).\end{aligned}$$

Let further $d(b),s(b):\mathcal{X}(B)\to\mathcal{X}(B)$ be given by

$$\begin{aligned}d(b)&=\oplus_{n\geq 0}d_n(b)\\s(b)&=\oplus_{n\geq 0}s_n(b).\end{aligned}$$

Note that the somewhat unusual formulae for $s_n(b)$ (when compared to $d_n(b)$) are due to a certain asymmetry in our definition of $A(M)$, $\mathcal{X}(M)$.

3.4. The reader may easily check the following formulae for $\xi_n \in \mathcal{X}_n(B)$ $(n \geq 0)$

$$d(b)\lambda(\xi_n) = \lambda(d_n(b)\xi_n)d(b)$$
$$\lambda^*(1)d(b) = d(b)\lambda^*(b)$$

and

$$\lambda(\sigma_n(b)\xi_n)s(b) = s(b)\lambda(b)\lambda(\xi_n)$$
$$\lambda^*(1)s(b) = s(b)\lambda(b)\lambda^*(1).$$

3.5. Proposition. *Let a be a B-valued random variable and let $b \in B$ and let*

$$\lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_n)$$

be the canonical form of a. Then we have:

(i) *The canonical form of ab is*

$$\lambda^*(1) + \sum_{n\geq 0} \lambda(d_n(b)\xi_n b).$$

(ii) *The canonical form of ba is*

$$\lambda^*(1) + \sum_{n\geq 0} \lambda(\sigma_n(b)\xi_n).$$

Proof. (i) The random variable ab has the same distribution like

$$(\lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_n))\lambda(b) = \lambda^*(b) + \sum_{n\geq 0} \lambda(\xi_n b)$$

where we have used 2.2.

To prove (i) we must show that $T = \lambda^*(b)+\sum_{n\geq 0} \lambda(\xi_n b)$ and $T_1 = \lambda^*(1)+\sum_{n\geq 0} \lambda(d_n(b)\xi_n b)$ have the same distribution in $(\bar{A}(1), \varepsilon_1)$. In view of 3.4 we have

$$d(b)T = T_1 d(b)$$

and hence

$$\begin{aligned}\varepsilon_1(T\lambda(b_1)\dots T\lambda(b_n)) &= (T\lambda(b_1)\dots T\lambda(b_n)1)_0 = \\ &= (d(b)T\lambda(b_1)\dots T\lambda(b_n)1)_0 = \\ &= (T_1 d(b)\lambda(b_1)\dots T\lambda(b_n)1)_0 =\end{aligned}$$

$$= (T_1\lambda(b_1)d(b)T\lambda(b_2)\dots T\lambda(b_n)1)_0 =$$
$$= \dots = (T_1\lambda(b_1)\dots T_1 d(b)\lambda(b_n)1)_0 =$$
$$= \varepsilon_1(T_1\lambda(b_1)\dots T_1\lambda(b_n)).$$

(ii) To prove (ii) we proceed similarly using the second group of formulae in 3.4. Let $T = \lambda(b)(\lambda^*(1) + \sum_{n\geq 0}\lambda(\xi_n))$ and let $T_1 = \lambda^*(1) + \sum_{n\geq 0}\lambda(\sigma_n(b)\xi_n)$. We have

$$s(b)T = T_1 s(b).$$

It is also easy to check that

$$s(b)\lambda(b_1) = \lambda(b_1)s(b).$$

We have

$$\varepsilon_1(T\lambda(b_1)\dots T\lambda(b_n)) = (T\lambda(b_1)\dots T\lambda(b_n)1)_0 =$$
$$= (s(b)T\lambda(b_1)\dots T\lambda(b_n)1)_0 =$$
$$= (T_1\lambda(b_1)s(b)T\lambda(b_2)\dots T\lambda(b_n)1)_0 =$$
$$= \dots = (T_1\lambda(b_1)\dots T_1\lambda(b_n)s(b)1)_0 =$$
$$= \varepsilon_1(T_1\lambda(b_1)\dots T_1\lambda(b_n)).$$

□

3.6. Proposition. *Let (A,φ) be a B-probability space, let $a \in A$ be a random variable and let $e \in B$ be an idempotent $e = e^2 \neq 0$. Let*

$$T = \lambda^*(1) + \sum_{n\geq 0}\lambda(\xi_n) \quad (\xi_n \in \mathcal{X}_n(B))$$

be the canonical form of $eae \in A$ and let

$$S = \lambda^*(e) + \sum_{n\geq 0}\lambda(\eta_n)$$

be the canonical form of the eBe-valued random variable eae (i.e. $eae \in eAe$, where we consider the eBe-probability space $(eAe, \varphi(e\cdot e))$, so that e is the unit of eBe and $\eta_n \in \mathcal{X}_n(eBe)$). Then we have

$$\xi_n(b_1\otimes\dots\otimes b_n) = \eta_n(eb_1e\otimes eb_2e\otimes\dots\otimes eb_ne).$$

Proof. In view of Proposition 3.5 we have

$$\xi_n(b_1\otimes\dots\otimes b_n) = e\xi_n(eb_1e\otimes\dots\otimes eb_ne)e.$$

We identify $\prod_{n\geq 0} \mathcal{X}_n(eBe)$ with a subspace $\mathcal{Y} \subset \prod_{n\geq 0} \mathcal{X}_n(B)$ by identifying $\chi_n \in \mathcal{X}_n(eBe)$ with $\zeta_n \in \mathcal{X}_n(B)$ defined by

$$\zeta_n(b_1 \otimes \cdots \otimes b_n) = \chi_n(eb_1 e \otimes \cdots \otimes eb_n e).$$

Remark that $T\mathcal{Y} \subset \mathcal{Y}$, $\lambda^*(1)\mathcal{Y} \subset \mathcal{Y}$, $\lambda(\xi_n)\mathcal{Y} \subset \mathcal{Y}$ and $\lambda^*(1) \mid \mathcal{Y} = \lambda^*(e) \mid \mathcal{Y}$. Moreover

$$\begin{aligned}
&(T\lambda(b_1)T\lambda(b_2)T \dots \lambda(b_k)T1)_0 = \\
&= \varphi(eaeb_1 eaeb_2 e \dots eb_k eae) = \\
&= (T\lambda(eb_1 e)T\lambda(eb_2 e) \dots \lambda(eb_k e)Te)_0 = \\
&= (S\lambda(eb_1 e)S\lambda(eb_2 e) \dots \lambda(eb_k e)Se)_0.
\end{aligned}$$

To conclude from here that

$$\xi_n(b_1 \otimes \cdots \otimes b_n) = \eta_n(eb_1 e \otimes \cdots \otimes eb_n e)$$

one proceeds by induction on n. Let

$$\begin{aligned}
T_n &= \lambda^*(1) + \sum_{k=0}^{n} \lambda(\xi_k) \\
\tilde{T}_n &= \lambda^*(e) + \sum_{k=0}^{n-1} \lambda(\xi_k) + \lambda(\xi_n) \\
S_n &= \lambda^*(e) + \sum_{k=0}^{n} \lambda(\eta_k).
\end{aligned}$$

Assume we proved $\xi_k(b_1 \otimes \cdots \otimes b_k) = \eta_k(eb_1 e \otimes \cdots \otimes eb_k e)$ for $k < n$. This implies $\lambda(\xi_k) \mid \mathcal{Y} = \lambda(\eta_k) \mid \mathcal{Y}$ for $k < n$. It follows that

$$\begin{aligned}
&(S_n\lambda(eb_1 e) \dots S_n\lambda(eb_n e)S_n e)_0 = \\
&= (S\lambda(eb_1 e) \dots S\lambda(eb_n e)Se)_0 = \\
&= (T\lambda(b_1) \dots T\lambda(b_n)Te)_0 = \\
&= (T_n\lambda(b_1) \dots T_n\lambda(b_n)T_n e)_0 = \\
&= (\tilde{T}_n\lambda(b_1) \dots \tilde{T}_n\lambda(b_n)\tilde{T}_n e)_0.
\end{aligned}$$

Like in 3.1 we get from this equality

$$\xi_n(b_1 \otimes \cdots \otimes b_n) = \eta_n(eb_1 e \otimes \cdots \otimes eb_n e)$$

(since the terms involving $\eta_k (k < n)$ are identical). □

3.7. The next corollary outlines a standard application of the preceding result.

Corollary. *Let X_{ij} $(1 \leq i,j \leq n)$ be free random variables in a $\mathbb{C}$-probability space (A, τ). Consider the $\mathcal{M}_n(\mathbb{C})$-probability space $(A \otimes \mathcal{M}_n(\mathbb{C}), \tau \otimes id_n)$ and the $\mathcal{M}_n(\mathbb{C})$-valued random variable*

$$X = \sum_{1 \leq i,j \leq n} X_{ij} \otimes e_{ij}$$

where e_{ij} $(1 \leq i,j \leq n)$ are the canonical matrix units of $\mathcal{M}_n(\mathbb{C})$. Let

$$R(X_{ij})(z) = \sum_{n=0}^{\infty} R_{n+1}(X_{ij}) z^n$$

be the R-series of X_{ij}. Then the canonical form $\lambda^*(1) + \sum_{k \geq 0} \lambda(\xi_k)$ of T is given by

$$\xi_k(M^{(1)} \otimes \cdots \otimes M^{(k)}) = \sum_{1 \leq i,j \leq n} R_{k+1}(X_{ij}) m_{ji}^{(1)} \ldots m_{ji}^{(k)} e_{ij}$$

where $M^{(s)} = \sum_{1 \leq i,j \leq n} m_{ij}^{(s)} e_{ij} \in \mathcal{M}_n(\mathbb{C})$.

Proof. The random variables $X_{ij} \otimes I_n$ are $\mathcal{M}_n(\mathbb{C})$-free in $(A \otimes \mathcal{M}_n(\mathbb{C}), \tau \otimes id_n)$ and hence $X_{ij} \otimes e_{ij}$ $(1 \leq i,j \leq n)$ are also $\mathcal{M}_n(\mathbb{C})$-free. Using the fact that the R-series gives the canonical form of a $\mathbb{C}$-valued random variable and 3.6, we get that the canonical form

$$\lambda^*(1) + \sum_{n \geq 0} \lambda(\eta_n)$$

of $X_{ij} \otimes e_{ii}$ is given by

$$\eta_n(M^{(1)} \otimes \cdots \otimes M^{(n)}) = R_{n+1}(X_{ij}) m_{ii}^{(1)} \ldots m_{ii}^{(n)} e_{ii}.$$

If $\lambda^*(1) + \sum_{n \geq 0} \lambda(\zeta_n)$ is the canonical form of $X_{ij} \otimes e_{ij}$, then $X_{ij} \otimes e_{ij} = (X_{ij} \otimes e_{ii})(I \otimes e_{ij})$ together with 3.5 gives

$$\zeta_n(M^{(1)} \otimes \cdots \otimes M^{(n)}) = \eta_n(e_{ij} M^{(1)} \otimes \cdots \otimes e_{ij} M^{(n)}) e_{ij}$$

so that

$$\zeta_n(M^{(1)} \otimes \cdots \otimes M^{(n)}) = R_{n+1}(X_{ij}) m_{ji}^{(1)} m_{ji}^{(2)} \ldots m_{ji}^{(n)} e_{ij}.$$

To conclude the proof it suffices to use Proposition 3.2. □

4. The differential equation for $\boxplus$.

4.1. Lemma. *Let $T \in \bar{A}(1)$ and let $\lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_n) \in \bar{A}(1)$ be a canonical element. Then if $Y(\alpha) = h_1(\lambda^*(1) + \alpha \sum_{n\geq 0} \lambda(\xi_n))$, we have*

$$\frac{d}{d\alpha}\varepsilon_2(\lambda(b))(Y(\alpha) + h_2(T))\lambda(b))^m)|_{\alpha=0} =$$
$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\cdots+k_{n+1}=m-n-1\\ k_0\geq 0,\ldots,k_{n+1}\geq 0}} \varepsilon_1(\lambda(b)(T\lambda(b))^{k_0}\lambda(\xi_n((b(\varepsilon_1((T\lambda(b))^{k_n}))\otimes$$
$$\cdots \otimes (b\varepsilon_1((T\lambda(b))^{k_1}))(b)(T\lambda(b))^{k_{n+1}})$$

where $\alpha \in \mathbb{C}$ and $b \in B$.

Proof. The expression the derivative of which must be computed is a polynomial in $\alpha \in \mathbb{C}$ with coefficients in B, which shows also the sense in which this derivative should be understood.

We have $Y(\alpha) = \lambda^*(1 \oplus 0) + \alpha \sum_{n\geq 0} \lambda(\xi'_n)$ where $\xi'_n = \xi_n \circ pr_1^{\otimes n}$. Let $\eta_n = \lambda(\xi'_n)b$, $S = h_2(T\lambda(b))$ and $X(\alpha) = \lambda^*(b \oplus 0) + \alpha \sum_{n\geq 0} \lambda(\eta_n)$.

We have

$$\varepsilon_2(\lambda(b)((Y(\alpha) + h_2(T))\lambda(b))^m) =$$
$$= \varepsilon_2(\lambda(b)(S + X(\alpha))^m)$$

and hence

$$\frac{d}{d\alpha}\varepsilon_2(\lambda(b)((Y(\alpha) + h_2(T))\lambda(b))^m|_{\alpha=0} =$$
$$= \sum_{j=0}^{m-1} \varepsilon_2(\lambda(b)(S + \lambda^*(b \oplus 0))^j \sum_{n\geq 0} \lambda(\eta_n)(S + \lambda^*(b \oplus 0))^{m-1-j}).$$

For the computation which follows one should keep in mind that $\varepsilon_M(R) = (R1)_0$ and the proof of Proposition 2.7. We have

$$\sum_{j=0}^{m-1} \varepsilon_2(\lambda(b)(S + \lambda^*(b \oplus 0))^j \sum_{n\geq 0} \lambda(\eta_n)(S + \lambda^*(b \oplus 0))^{m-1-j}) =$$
$$= \sum_{j=0}^{m-1} \sum_{n=0}^{j} \sum_{\substack{n+k_0+\cdots+k_n=j\\ k_0\geq 0,\ldots,k_n\geq 0}} b\varepsilon_2(S^{k_0}\lambda^*(b \oplus 0)\ldots S^{k_{n-1}}\lambda^*(b \oplus 0)S^{k_n}\lambda(\eta_n)S^{m-1-j}) =$$
$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\cdots+k_{n+1}=m-n-1\\ k_0\geq 0,\ldots,k_{n+1}\geq 0}} b\varepsilon_2(S^{k_0}\lambda^*(b \oplus 0)\ldots S^{k_{n-1}}\lambda^*(b \oplus 0)S^{k_n}\lambda(\eta_n)S^{k_{n+1}}) =$$

$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\cdots+k_{n+1}=m-n-1 \\ k_0\geq 0,\ldots,k_{n+1}\geq 0}} b\varepsilon_2(S^{k_0}\lambda(\eta_n(((b\oplus 0)\varepsilon_2(S^{k_n})\otimes \ldots$$

$$\otimes ((b\oplus 0)\varepsilon_2(S^1)))S^{k_{n+1}})) =$$

$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\cdots+k_{n+1}=m-n-1 \\ k_0\geq 0,\ldots,k_{n+1}\geq 0}} \varepsilon_1(\lambda(b)(T\lambda(b))^{k_0}\lambda(\xi_n((b\varepsilon_1((T\lambda(b))^{k_n})\otimes \ldots$$

$$\otimes (b\varepsilon_1((T\lambda(b))^{k_1}))))\lambda(b)(T\lambda(b))^{k_{n+1}}).$$

□

4.2. It is easy to see that the same computation yields the more general formula

$$\frac{d}{d\alpha}\varepsilon_2(\lambda(b_0)(Y(\alpha)+h_2(T))\lambda(b_1)\ldots\lambda(b_{m-1})(Y(\alpha)+h_2(T))\lambda(b_m))|_{\alpha=0} =$$

$$= \sum_{n=0}^{m-1} \sum_{\substack{k_0+\cdots+k_{n+1}=m-n-1 \\ k_0\geq 0\ldots k_{n+1}\geq 0}} \varepsilon_1(\lambda(b_0)T\lambda(b_1)\ldots T\lambda(b_{k_0})$$

$$\lambda(\xi_n(\varepsilon_1(\lambda(b_{k_0+\cdots+k_{n-1}+n})T\ldots\lambda(b_{k_0+\cdots+k_n+n-1})T\lambda(b_{k_0+\cdots+k_n+n}))\otimes$$

$$\cdots\otimes\varepsilon_1(\lambda(b_{k_0+1})T\ldots\lambda(b_{k_0+k_1+1})T\lambda(b_{k_0+k_1+2})))$$

$$\lambda(b_{k_0+\cdots+k_n+n+1})T\ldots\lambda(b_{k_0+\cdots+k_{n+1}+n})T\lambda(b_{k_0+\cdots+k_{n+1}+n+1}))$$

where $b_0,\ldots,b_m \in B$.

4.3. Before passing to the differential equations we have to discuss certain formal series which are the analogue of formal power series when maps $\mathbb{C}\to\mathbb{C}$ are replaced by maps $B\to B$.

Let $S\mathcal{X}_n(B)\subset\mathcal{X}_n(B)$ be the subspace of symmetric n-linear maps i.e. $\xi_n(b_1\otimes\cdots\otimes b_n) = \xi_n(b_{\sigma(1)}\otimes\cdots\otimes b_{\sigma(n)})$ for all $\sigma\in\mathcal{G}_n$. If $\eta\in\mathcal{X}_n(B)$ we denote by $S\eta\in S\mathcal{X}_n(B)$ the element such that $S\eta(b^{\otimes n}) = \eta(b^{\otimes n})$. Elements of $S\mathcal{X}(B) = \prod_{n\geq 0} S\mathcal{X}_n(B)$ will be written

$$\sum_{n\geq 0}\xi_n$$

$S\mathcal{X}(B)$ is a ring with multiplication such that $(\xi_m\xi_n)(b^{\otimes(m+n)}) = \xi_m(b^{\otimes m})\xi_n(b^{\otimes n})$. $S\mathcal{X}(B)$ has a natural filtering given by the powers of the ideal formed by elements of the form $\sum_{n\geq 1}\xi_n$. If $\varphi = \sum_{n\geq 0}\xi_n$ and $\psi = \sum_{n\geq 1}\eta_n$ then the composition $\varphi\circ\psi$ is easily seen to be well defined as follows $\varphi\circ\psi = \Sigma_{n\geq 0}\zeta_n$ where

$\zeta_0 = \xi_0$ and if $k\geq 1$

$$\zeta_k = \sum_{m\geq 1} \sum_{\substack{k_1+\cdots+k_m=k \\ k_1\geq 1,\ldots,k_m\geq 1}} \xi_m(\eta_{k_1}(b^{\otimes k_1})\otimes\cdots\otimes\eta_{k_m}(b^{\otimes k_m})).$$

The differential of $\varphi \in S\mathcal{X}(B)$ is an element of $\prod_{n\geq 0} S\mathcal{L}(B^{\otimes n}, \mathcal{L}(B,B))$ where $S\mathcal{L}(B^{\otimes n}, E)$ denotes the symmetric n-linear E-valued maps. If $\varphi = \sum_{n\geq 0} \xi_n$ then the differential is

$$D\varphi = \sum_{n\geq 1} D\xi_n$$

where $D\xi_n \in \mathcal{L}(B^{\otimes(n-1)}, \mathcal{L}(B,B))$ is such that $(D\xi_n(b^{\otimes(n-1)}))(\beta) = \sum_{k=0}^{n-1} \xi_n(\underbrace{b\otimes\cdots\otimes b}_{k\text{-times}}\otimes \beta\otimes b\otimes\cdots\otimes b)$.

We shall write formally also $\varphi(b) = \sum_{n\geq 0}\xi_n(b^{\otimes n}), (D\varphi)(b)[\beta]$ or $(D_b\varphi)(b)[\beta]$ and $\varphi(\psi(b))$.

If $\varphi = \sum_{n\geq 0}\xi_n$ and the ξ_n's depend on a parameter then the derivative of φ with respect to this parameter is meant component wise.

4.4. If $\mu \in S\Sigma_b$ we consider the formal series

$$G_\mu(b) = \sum_{n\geq 0} \mu(b(Xb)^n).$$

It will be also useful to consider

$$\Gamma_\mu(b) = \sum_{n\geq 0} \mu((Xb)^n X)$$

so that

$$G_\mu(b) = b + b\Gamma_\mu(b)b.$$

If $\mu \in \Sigma_B$ we shall write also G_μ for $G_{S\mu}$ for Γ_μ for $\Gamma_{S\mu}$. Also if μ is a distribution μ_T we shall write G_T and Γ_T instead of G_{μ_T} and Γ_{μ_T}.

4.5. Proposition. *Let $T \in \bar{A}(1)$ and let $Y(\alpha) = h_1(\lambda^*(1) + \alpha\sum_{n\geq 0}\lambda(\xi_n)) \in h_1(\bar{A}(1))$. Let $T(\alpha) = Y(\alpha) + h_2(T)$. Then we have $G_{T(0)} = G_T$ and*

$$\frac{\partial}{\partial\alpha} G_{T(\alpha)}(b) = (D_b G_{T(\alpha)})(b)[b\Xi(G_{T(\alpha)}(b))b],$$

where $\Xi(b) = \sum_{n\geq 0}\xi_n(b^{\otimes n})$.

Proof. If $\alpha = 0$ the equality of the terms which are of degree $m+1$ in b in the differential equation is precisely the equality established in Lemma 4.1. The general case can be reduced to the case $\alpha = 0$ since in view of 2.7 and 3.2 we have

$$\mu_{T(\alpha)} = \mu_T \boxplus \mu_{Y(\alpha)} = \mu_{T(\alpha_0)} \boxplus \mu_{Y(\alpha-\alpha_0)}.$$

□

4.6. Corollary. *Let $\mu \in \Sigma_B$ and $a = \lambda^{*(1)} + \sum_{n\geq 0} \lambda(\xi_n)$ be the canonical element with distribution μ. Then the $(S\xi_n)_{n\geq 0}$ depend only on $S\mu$ and conversely $S\mu$ depends only on the $(S\xi_n)_{n\geq 0}$. In particular if $\mu_1, \mu_2 \in \Sigma_B$ then $S(\mu_1 \boxplus \mu_2)$ is completely determined by $S\mu_1$ and $S\mu_2$.*

Proof. Let $a(\alpha) = \lambda^*(1) + \alpha \sum_{n\geq 0} \lambda(\xi_n)$.

We have

$$\frac{d}{d\alpha}\varepsilon_1(\lambda(b)(a(\alpha)\lambda(b))^m) = bS\xi_{m-1}(b \otimes \cdots \otimes b)b+ \\ + F(S\xi_j, \varepsilon_1(\lambda(b)(a(\alpha)\lambda(b))^j)b, 0 \leq j \leq m-2)$$

where F is a "polynomial" of the quantities on which it depends. These differential equations with initial condition $\bar{\varepsilon}_1(\lambda(b)(a(0)\lambda(b))^m) = 0$ if $m \geq 1$ can be solved recurrently and we obtain that

$$\varepsilon_1(\lambda(b)(a(\alpha)\lambda(b))^m) = \\ = \alpha bS\xi_{m-1}(b \otimes \cdots \otimes b)b+ \\ + P(\alpha, b, S\xi_j, 0 \leq j \leq m-2)$$

where P is "polynomial". Taking $\alpha = 1$ we see that $S\mu$ completely determines the $(S\xi_n)_{n\geq 0}$ and also that conversely the $(S\xi_n)_{n\geq 0}$ completely determine $S\mu$. The assertion concerning $S(\mu_1 \boxplus \mu_2)$ follows now from 3.2. □

4.7. The differential equation in 4.5 immediately implies the following fact: if $\mu_1, \mu_2 \in S\Sigma_B$ and $\mu(\alpha) \in S\Sigma_B$ is such that $SR_n(\mu(\alpha)) = SR_n(\mu_1) + \alpha SR_n(\mu_2)$ then

$$\frac{\partial}{\partial\alpha}G_{\mu(\alpha)}(b) = (D_b G_{\mu(\alpha)})(b)[b\Xi(G_{\mu(\alpha)}(b))b]$$

where $\Xi = \sum_{n\geq 0} \xi_n$ where $\xi_n = SR_{n+1}(\mu_2)$. Interpreting this equation as a system of ordinary differential equations as in 4.6 we see that with the initial condition $G_{\mu(0)} = G_{\mu_1}$ we have $G_{\mu(1)} = G_{\mu_1 \boxplus \mu_2}$ which is completely determined by the differential equation.

4.8. We shall now assume B is a Banach algebra and Γ_μ and hence G_μ is an analytic function in some neighborhood of $0 \in B$. This implies that the symmetric moments of μ viewed as n-linear maps $B^n \to B$ are continuous and the formal series defining G_μ is absolutely convergent in some neighborhood of 0. For instance if T is a B-valued random-variable $T \in A$ where A is also a Banach algebra, $A \supset B$ with a continuous conditional expectation $\varphi : A \to B$ then $G_T(b) = \sum_{n\geq 0} \varphi(b(Tb)^n) = \varphi(b(1 - Tb)^{-1})$ satisfies these assumptions.

For the lemma which follows we shall denote by $\mathcal{M}$ the set of germs at $0 \in B$ of analytic B-valued maps and we shall use the notation F^{-1} only for multiplicative inverses, not for inverses with respect to composition.

Lemma. *Let $\Gamma, G \in \mathcal{M}$ be such that $G(b) = b + b\Gamma(b)b$ near 0.*

(i) *If $K \in \mathcal{M}$ is such that $K(G(b)) = G(K(b)) = b$ near 0, then there is $Q \in \mathcal{M}$ such that $K(b) = b + bQ(b)b$.*

(ii) *There is $R \in \mathcal{M}$ such that for some neighborhood V of $0 \in B$ we have $(K(b))^{-1} = b^{-1} + R(b)$ if $b \in V \cap GL(B)$. R is unique.*

Proof. (i) If $\|b\|$ is small enough, we have

$$b = G(b)(1 + \Gamma(b)b)^{-1} = G(b)(1 + \Gamma(K(G(b)))K(G(b)))^{-1}$$

so that there is $H \in \mathcal{M}$ for which

$$K(b) = bH(b).$$

Similarly there is $J \in \mathcal{M}$ so that

$$K(b) = J(b)b.$$

We have

$$b = G(b) - b\Gamma(b)b = G(b) - G(b)H(G(b))\Gamma(K(G(b)))J(G(b))G(b)$$

so $Q(b) = -H(b)\Gamma(K(b))J(b)$ will do.

(ii) Choosing V small enough, if $b \in V \cap GL(B)$ we have

$$\begin{aligned} K(b)^{-1} &= b^{-1}(1 + bQ(b))^{-1} = \\ &= b^{-1}Q(b)(1 + bQ(b))^{-1}. \end{aligned}$$

The uniqueness of R is easily seen from the fact that $R \mid (V \cap GL(B))$ determines the germ of R at 0. □

4.9. Theorem. *Assume B is a Banach algebra and $\mu \in S\Sigma_B$ is such that $\Gamma_\mu(b)$ is analytic in some neighborhood of $0 \in B$. Let K and R be germs of B-valued analytic functions at $0 \in B$ such that*

$$K(G_\mu(b)) = G_\mu(K(b)) = b$$

and

$$K(b)^{-1} = b^{-1} + R(b)$$

for $b \in GL(B)$ in some neighborhood of 0. Then we have

$$R(b) = \sum_{n \geq 0} SR_{n+1}(\mu)(b^{\otimes n})$$

where the $SR_{n+1}(\mu)$ are given by the canonical element with distribution μ.

Proof. Let

$$K(\alpha, b) = (b^{-1} + \alpha R(b))^{-1} = (b^{-1}(1 + \alpha b R(b)))^{-1} = (1 + \alpha b R(b))^{-1} b$$

which for $0 \leq \alpha \leq 1$ makes sense in some fixed neighborhood of 0, the last equality making the invertibility of b superfluous. There is a neighborhood of 0 independent of $0 \leq \alpha \leq 1$ for which $K(\alpha, b)$ has an inverse (with respect to composition) $K(\alpha, G(\alpha, b)) = G(\alpha, K(\alpha, b)) = b$.

We have

$$\frac{d}{d\alpha} G(\alpha, K(\alpha, b)) = 0$$

which with $b_1 = K(\alpha, b)$ gives

$$\begin{aligned} 0 &= \frac{\partial}{\partial \alpha} G(\alpha, b_1) + (D_b G)(\alpha, b_1)[-b_1 R(b) b_1] = \\ &= \frac{\partial}{\partial \alpha} G(\alpha, b_1) - (D_b G)(\alpha, b_1)[b_1 R(G(\alpha, b_1)) b_1]. \end{aligned}$$

Moreover $G(0, b) = b$. Thus defining $\mu(\alpha) \in S\Sigma_B$ by $G_{\mu(\alpha)}(b) = G(\alpha, b)$ we have that $\frac{\partial}{\partial \alpha} G_{\mu(\alpha)}(b) = (D_b G_{\mu(\alpha)})(b)[bR(G_{\mu(\alpha)}(b))b]$ and $\mu(0)$ is the distribution of the 0 random-variable. In view of 4.7 this implies that G is the generating series for a symmetric distribution for which the corresponding symmetric parts of the components of the corresponding canonical element yield the series $R(b)$. Thus we have

$$R(b) = \sum_{n \geq 0} SR_{n+1}(\mu(1))(b^{\otimes n}).$$

On the other hand $G_{\mu(1)} = G_\mu$ so that $\mu(1) = \mu$. □

4.10. We have chosen in 4.8 and 4.9 to work in the Banach algebra context where we work with genuine functions since this is the situation for the applications to computations of spectra. On the other hand the reader will not find it difficult to transpose Lemma 4.8 and Theorem 4.9 in the framework of formal series and general B where similar statements hold.

4.11. Thus in the B-valued case the computation of $\mu_1 \boxplus \mu_2$, $\mu_j \in S\Sigma_B$ is done as in the scalar case: one forms G_{μ_j}, then the inverse K_{μ_j} then the multiplicative inverses $b^{-1} + R_{\mu_j}$. Then $R_{\mu_1 \boxplus \mu_2} = R_{\mu_1} + R_{\mu_2}$ and from $R_{\mu_1 \boxplus \mu_2}$ one goes back to $K_{\mu_1 \boxplus \mu_2}$ and $G_{\mu_1 \boxplus \mu_2}$.

5. The differential equation for ⊠.

5.1. Lemma. *Let $T_1,\dots,T_m \in \bar{A}(1)$, let $a(\tau)=\lambda^*(1)+\lambda(1)+\tau\sum_{n\geq 1}\lambda(\xi_n)\in\bar{A}(1)$ be a canonical element and let $Y(\tau)=h_1(a(\tau))$. Then we have*

$$\frac{d}{d\tau}\varepsilon_2(Y(\tau)h_2(T_1)Y(\tau)h_2(T_2)\dots Y(\tau)h_2(T_m))|_{\tau=0}=$$
$$=\sum_{p\geq 1}\sum_{\substack{j_0+\dots+j_{p+1}=m\\ j_0\geq 0\\ j_1\geq 1,\dots,j_{p+1}\geq 1}}\varepsilon_1(T_1\dots T_{j_0}\lambda(\xi_p(\varepsilon_1(T_{j_0+\dots+j_{p-1}+1}\dots T_{j_0+\dots+j_p})\otimes$$
$$\dots\otimes\varepsilon_1(T_{j_0+1}\dots T_{j_0+j_1})))T_{j_0+\dots j_p+1}\dots T_{j_0+\dots+j_{p+1}}).$$

Proof. For the computation it will be convenient to put $\xi'_p=\xi_p\circ pr_1^{\otimes p}$ so that $\lambda(\xi'_p)=h_1(\lambda(\xi_p))$ and $S_k=h_2(T_k)$.

We have

$$\frac{d}{d\tau}\varepsilon_2(Y(\tau)S_1\dots Y(\tau)S_m)|_{\tau=0}=$$
$$=\sum_{j=1}^{m}\varepsilon_2(Y(0)S_1\dots Y(0)S_{m-j}\sum_{p\geq 1}\lambda(\xi'_p)S_{m-j+1}Y(0)S_{m-j+2}\dots Y(0)S_m)=$$
$$=\sum_{j=1}^{m}\varepsilon_2((\lambda^*(1\oplus 0)+\lambda(1))S_1\dots(\lambda^*(1\oplus 0)+\lambda(1))S_{m-j}\sum_{p\geq 1}\lambda(\xi'_p)S_{m-j+1}\dots S_m)=$$
$$=\sum_{j=1}^{m}\sum_{p\geq 1}\sum_{\substack{j_0+\dots+j_p=m-j\\ j_0\geq 0\\ j_1\geq 1,\dots,j_p\geq 1}}\varepsilon_2(S_1\dots S_{j_0}\lambda^*(1\oplus 0)S_{j_0+1}\dots S_{j_0+j_1}\lambda^*(1\oplus 0)$$
$$\lambda^*(1\oplus 0)S_{j_0+\dots+j_{p-1}+1}\dots S_{j_0+\dots+j_p}\lambda(\xi'_p)S_{j_0+\dots+j_p+1}\dots S_m)=$$
$$=\sum_{p\geq 1}\sum_{\substack{j_0+\dots+j_{p+1}=m\\ j_0\geq 0\\ j_1\geq 1,\dots,j_{p+1}\geq 1}}\varepsilon_2(S_1\dots S_{j_0}\lambda(\xi_p(\varepsilon_2(S_{j_0+\dots+j_{p-1}+1}\dots S_{j_0+\dots+j_p})\otimes$$
$$\dots\otimes\varepsilon_2(S_{j_0+1}\dots S_{j_0+j_1})))S_{j_0+\dots+j_{p+1}}\dots S_m)=$$
$$=\sum_{p\geq 1}\sum_{\substack{j_0+\dots+j_{p+1}=m\\ j_0\geq 0\\ j_1\geq 1,\dots,j_{p+1}\geq 1}}\varepsilon_1(T_1\dots T_{j_0}\lambda(\xi_p(\varepsilon_1(T_{j_0+\dots+j_{p-1}+1}\dots T_{j_0+\dots+j_p})\otimes$$
$$\dots\otimes\varepsilon_1(T_{j_0+1}\dots T_{j_0+j_1})))T_{j_0+\dots+j_{p+1}}\dots T_{j_0+\dots+j_{p+1}}).$$

□

5.2. Corollary. *Let* $T \in \bar{A}(1)$, *let*

$$a(\tau) = \lambda^*(1) + \lambda(1) + \tau \sum_{n \geq 1} \lambda(\xi_n) \in \bar{A}(1)$$

be a canonical element and let $Y(\tau) = h_1(a(\tau))$. *Then we have*

$$\frac{d}{d\tau}\varepsilon_2((\lambda(b)Y(\tau)h_2(T))^m)|_{\tau=0} =$$

$$= \sum_{p \geq 1} \sum_{\substack{j_0 + \cdots + j_{p+1} = m \\ j_0 \geq 0 \\ j_1 \geq 1, \ldots, j_{p+1} \geq 1}} \varepsilon_1(\lambda(b)(T\lambda(b))^{j_0} \lambda(\xi_p(\varepsilon_1((T\lambda(b)^{j_p}) \otimes \ldots$$

$$\otimes\, \varepsilon_1((T\lambda(b))^{j_1}))T(\lambda(b)T)^{j_{p+1}-1})$$

where $b \in B$ *and* $\tau \in \mathbb{C}$.

5.3. Proposition. *Let* $T \in \bar{A}(1)$ *and let*

$$Y(\tau) = h_1(\lambda^*(1) + \lambda(1) + \tau \sum_{n \geq 1} \lambda(\xi_n)) \in \bar{A}(1).$$

Let $T(\tau) = Y(\tau)h_2(T)$. *Then we have*

$$\frac{d}{d\tau}(b\Gamma_{T(\tau)}(b))|_{\tau=0} = (D_b(b\Gamma_{T(0)}))(b)[b(\Theta(\Gamma_{T(0)}(b)b))]$$

where $\Theta(b) = \sum_{n \geq 1} \xi_n(b^{\otimes n})$.

Proof. The proposition is obtained from Corollary 5.2 by looking at the terms which are degree m in b. □

5.4. The use of differential equations in the study of the operator ⊠ relies on viewing

$$\Sigma_{1,B} = \{\mu \in \Sigma_B \mid \mu(X) = 1\}$$

as a kind of infinite dimensional Lie group with respect to multiplicative free convolution and to identify the solutions of the differential equations:

$$\frac{d}{d\tau}\mu(\tau)(Xb_1 \ldots Xb_m) =$$

$$(*) \qquad = \sum_{p \geq 1} \sum_{\substack{j_0 + \cdots + j_{p+1} = m \\ j_0 \geq 0 \\ j_1 \geq 1, \ldots, j_{p+1} \geq 1}} \mu(\tau)(Xb_1 \ldots Xb_{j_0}$$

$$\xi_p((\mu(\tau)(Xb_{j_0+\cdots+j_{p-1}+1} \ldots Xb_{j_0+\cdots+j_p})) \otimes \ldots$$

$$\otimes\, (\mu(\tau)(Xb_{j_0+1} \ldots Xb_{j_0+j_1})))$$

$$Xb_{j_0+\cdots+j_p+1} \ldots Xb_{j_0+\cdots+j_{p+1}})$$

with the integral curves of right-invariant vector fields on this group. Here $\mu(\tau) \in \Sigma_{1,B}$, $\xi_p \in \mathcal{X}_p(B)$ and the equations are obtained from those in Lemma 5.1 by taking $T_j = T\lambda(b_j)$ with T a function of τ and $\mu(\tau)$ the distribution of T.

5.5. Before looking at the differential equations let us note a few facts about $(\Sigma_{1,B}, \boxtimes)$. The set $\Sigma_{1,B}$ has an obvious affine space structure. Denoting by $\Sigma_{1,B}^{(n)}$ the set of n-th order distributions (i.e. the restrictions of distributions to noncommutative monomials $b_1Xb_2X\dots b_mXb_{m+1}$ of degree $m \le n$) it is easily seen that $(\Sigma_{1,B}, \boxtimes)$ is the inverse limit of groups $(\Sigma_{1,B}^{(n)}, \boxtimes_n)$. We shall denote by μ_1 the distribution of 1, which is the neutral element of $(\Sigma_{1,B}, \boxtimes)$ and by $\mu_1^{(n)}$ the corresponding n-th order distribution.

It is easily seen that if $\mu, \nu \in \Sigma_{1,B}$ then

$$\begin{aligned}&(\mu \boxtimes \nu)(b_1Xb_2X\dots b_mXb_{m+1}) = \\ &= \mu(b_1Xb_2\dots Xb_{m+1}) + \nu(b_1Xb_2\dots Xb_{m+1}) + \\ &+ F(\mu^{(n-1)}, \nu^{(n-1)})\end{aligned}$$

where $F(\mu^{(n-1)}, \nu^{(n-1)})$ depends only on $\mu^{(n-1)}, \nu^{(n-1)}$. Note also that $\boxtimes_n$ is a polynomial map in the sense, that if $V_1, V_2 \subset \Sigma_{1,B}^{(n)}$ are finite-dimensional affine subspaces then there is an affine subspace $V_3 \subset \Sigma_{1,B}^{(n)}$ such that

$$V_1 \boxtimes_n V_2 \subset V_3$$

and the map

$$V_1 \times V_2 \to V_3$$

defined by $\boxtimes_n$ is a polynomial map of degree $\le n$. A more careful inspection of the map defined by $\boxtimes_n$, also yields the following fact which we record as a lemma, the proof of which is left to the reader as an exercise.

5.6. Lemma. *Let $V_1, V_2 \subset \Sigma_{1,B}^{(n)}$ be finite-dimensional affine subspaces. Then there is a finite-dimensional affine subspace $W \subset \Sigma_{1,B}^{(n)}$ such that*

$$V_2 \subset W \text{ and } V_1 \boxtimes_n W \subset W.$$

With these preparations we are ready to study the free exponential.

5.7. Definition. The free exponential is the map

$$\text{fexp} : \prod_{n\ge1} \mathcal{X}_n(B) \to \Sigma_{1,B}$$

defined by $\text{fexp}((\xi_n)_{n\ge1}) = \mu(1)$ where $\mu(\tau)$ satisfies $(*)$ with initial condition $\mu(0) = \mu_1$.

5.8. Proposition.

a) *The map* fexp *is a bijection.*

b) *If $\mu(\tau)$ satisfies $(*)$ with initial condition $\mu(0) = \nu \in \Sigma_{1,B}$, then*

$$\nu(1) = \operatorname{fexp}((\xi_n)_{n\geq 1}) \boxtimes \nu.$$

Proof. a) Denote by $\mu(\tau;n)$ the element $(b_1 \otimes \cdots \otimes b_n) \to \mu(\tau)(Xb_1 \dots Xb_n)$ of $\mathcal{X}_n(B)$. The system of differential equations $(*)$ for the $\mu(\tau;n)$ can be solved recurrently. It is easily seen by recurrence that the right-hand side of $(*)$ is a polynomial in τ of degree $< m$ with zeroth order term

$$\xi_{m-1}(b_{m-1} \otimes \cdots \otimes b_1)b_m$$

while the higher order terms depend only on $\xi_1, \dots, \xi_{m-2}$. This easily gives that fexp is a bijection.

b) It will suffice to prove that

$$\lim_{n\to\infty} \mu_{Z(n;\tau)} = \mu(\tau)$$

where

$$Z(n;\tau) = h_{n+1}(Y(\tau/n)) \dots h_2(Y(\tau/n))h_1(T),$$

$T \in \overline{A(1)}$ has $\mu_T = \nu$ and $h_1, \dots, h_{n+1}$ are the homomorphisms $\bar{A}(1) \to \bar{A}(n+1)$, corresponding to the canonical summands in B^{n+1}. Note indeed that this means

$$\mu_{Z(n;\tau)} = \underbrace{\mu_{Y(\tau/n)} \boxtimes \cdots \boxtimes \mu_{Y(\tau/n)}}_{n\text{-times}} \boxtimes \nu$$

and that for the special case $T = \lambda(1)$, $\mu_T = \mu_1$ we also get

$$\lim_{n\to\infty} (\mu_{Y(\tau/n)})^{\boxtimes n} = \operatorname{fexp}((\xi_n)_{n\geq 1}).$$

In all these considerations, the limits make sense in view of Lemma 5.6. Indeed, by Lemma 5.6 we have that for fixed $b_1, \dots, b_m$ and all n

$$\mu_{Z(n;\tau)}(Xb_1 \dots Xb_m)$$

takes values in a fixed finite dimensional vector subspace of B. The limit is with respect to the usual topology of this finite-dimensional complex vector space.

To prove the assertion we have to use again Lemma 5.6. Let $V_1, V_2 \subset \Sigma_{1,B}^{(m)}$ be finite-dimensional affine subspaces, such that

$$\mu_{Y(\tau)}^{(m)} \in V_1 \text{ for all } \tau \in \mathbb{C}$$

and $\nu^{(m)}, \mu_1^{(m)} \in V_2$. Let W be as in Lemma 5.4. Then

$$D(\tau) : W \to W$$

is a polynomial map of bounded degree, depending polynomially on $\tau \in \mathbb{C}$ and such that $D(0) = \mathrm{id}_W$. The first m equations $(*)$ describe in view of Lemma 5.1 precisely the integral curves of the vector field on W defined by

$$\frac{d}{d\tau} D(\tau) w|_{\tau=0} \text{ at } w \in W.$$

In this context, where all the maps etc. are polynomial it is immediate that

$$\lim_{n\to\infty} (D(\tau/n))^n \nu^{(m)} = (\mu(\tau))^{(m)}$$

which is the desired result. □

5.9. Proposition. *The symmetric part $S\mu(\tau)$ of a solution of $(*)$ depends only on the symmetric parts $(S\xi_n)_{n\geq 1}$ and on the symmetric part $S\mu(0)$ of the initial data. In particular, there is a map* fexp : $\prod_{n\geq 1} S\mathcal{X}_n(B) \to S\Sigma_{B,1}$ *(the free symmetric exponential) such that*

$$\begin{array}{ccc} \prod_{n\geq 1} \mathcal{X}_n(B) & \xrightarrow{\text{fexp}} & \Sigma_{B,1} \\ \big\downarrow S & & \big\downarrow S \\ \prod_{n\geq 1} S\mathcal{X}_n(B) & \xrightarrow{\text{fexp}} & S\Sigma_{B,1} \end{array}$$

is a commutative diagram and fexp *is a bijection. Moreover $S(\mu_1 \boxtimes \mu_2)$ depends only on $S\mu_1$ and $S\mu_2$.*

Proof. For the first assertion it is clearly sufficient to show that the system of differential equations $(*)$ yields a system of differential equations for $S\mu(\tau)$, which completely determines $S\mu(\tau)$ for a given initial condition and which involves only the symmetric parts of the ξ_n. Such a system of differential equations is provided by the equations $(*)$ with $b_1 = \cdots = b_{m-1} = b$ and $b_m = 1$. To see that this system involves only the symmetric parts of ξ_p, note that the sum inthe right-hand side sum of $(*)$ when restricted to fixed p, j_0 and j_{p+1} is a sum over $j_1 \geq 1, \ldots, j_p \geq 1$, $j_1 + \cdots + j_p = m - j_0 - j_{p+1}$. The permutation group $\mathcal{G}_p$ acts on this set of p-tuples $(j_1, \ldots, j_p)$ and the partial sums over its orbits involve only symmetrizations

$$\sum_{\sigma \in \mathcal{G}_p} \xi_p(\beta_{\sigma(1)} \otimes \cdots \otimes \beta_{\sigma(p)})$$

where $\beta_1, \ldots, \beta_p \in B$, which are completely determined by $S\xi_p$. In turn, the partial sum for fixed $p, j_1, \ldots, j_p$ is a sum of the form

$$\mu(\tau)(\beta X b X b \ldots X) + \mu(\tau)(X b \beta X b X \ldots b X) +$$

$$\cdots + \mu(\tau)(XbXb\ldots Xb\beta X) =$$
$$= \beta\mu(\tau)(XbX\ldots bXbX) +$$
$$\frac{d}{d\varepsilon}\mu(\tau)(Xb(1+\varepsilon\beta)Xb(1+\varepsilon\beta)\ldots Xb(1+\varepsilon\beta)X)|_{\varepsilon=0}.$$

This concludes the proof of the first assertion (an alternative proof could have been based on 5.3 and 4.6). The existence of the map fexp is an immediate consequence.

To see that fexp is a bijection, remark that

$$\frac{d}{d\tau}\mu(\tau)(XbX\ldots XbX) -$$
$$- \xi_{m-1}(b\otimes\cdots\otimes b) =$$
$$= \text{ right-hand side sum of } (*) \text{ restricted to } 1\leq p\leq m-2$$

and hence this difference depends only on $S\xi_1,\ldots,S\xi_{m-2}$. The last assertion is now an immediate corollary in view of 5.8. □

5.10. *Remark.* The differential equations for $S\mu(\tau)$, in view of the preceding proposition and of Proposition 5.3 can also be written in terms of "generating series" in the form:

$$\frac{d}{d\tau}(B\Gamma_{S\mu(\tau)}(b)) =$$
$$= (D_b(b\Gamma_{S\mu(\tau)}))(b)[b(\Theta(\Gamma_{S\mu(\tau)}(b)b))]$$

where

$$\Gamma_{S\mu(\tau)}(b) = \sum_{n\geq 0} S\mu(\tau)(b(Xb)^n)$$

and

$$\Theta(b) = \sum_{n\geq 1}\xi_n(b^{\otimes n}).$$

This differential equation can also be used to compute the symmetric multiplication free convolution

$$S\mu_1 \boxtimes S\mu_2$$

provided we compute $(\text{fexp})^{-1}S\mu_1$.

6. Computation of spectra.

As we mentioned in the introduction the results concerning the operation $\boxplus$ provide a method for computing spectra of left convolution operators in $l^2(G)$ where $G = \mathbb{Z} * \mathbb{Z}$ is the free group on two generators g_1, g_2. Actually the same ideas provide a method for dealing with more complicated groups obtained by taking free products with amalgamation. We shall however stick here to the case of $\mathbb{Z} * \mathbb{Z}$ since we think this particular example will suffice to explain our approach.

6.1. Let $Y = \sum_{g \in G} c_g \lambda(g)$ where $c_g \in \mathbb{C}$, $c_g \neq 0$ only for finitely many $g \in G$ and where λ is the left regular representation of G on $l^2(G)$. To compute the spectrum of Y we have to provide a method for deciding whether $Y - zI$ is invertible for a given $z \in \mathbb{C}$. Since $Y - zI$ is of the same form as Y we may state our problem as deciding whether Y is invertible.

6.2. We recall one of the standard algebraic tricks with matrices.

Let A be a ring and let $c_0, \ldots, c_n$ and $u_1, \ldots, u_n$ be elements in A. Let further $y = c_0 + \sum_{k=1}^{n} c_k u_k \ldots u_1$ and $y_p = c_p + \sum_{k=p+1}^{n} c_k u_k \ldots u_{p+1}$. Then in the ring $\mathcal{M}_{n+1}(A)$ of $(n+1) \times (n+1)$ matrices over A we have:

$$\begin{pmatrix} 1 & y_1 & \cdots & y_n \\ & 1 & & \\ 0 & & \ddots & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} y & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ -u_1 & & \ddots & \\ 0 & & -u_n & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ -u_1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & -u_n & 1 \end{pmatrix}.$$

This identity shows that y is invertible if and only if the matrix

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ -u_1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & -u_n & 1 \end{pmatrix}$$

is invertible.

6.3. Let $C_r^*(G)$ be the reduced C^*-algebra of G. An application of 6.2 to the element Y in 6.1 with $A = C_r^*(G)$, $y = Y$, $c_j \in \mathbb{C}$, $u_j \in \{\lambda(g_1), \lambda(g_2), \lambda(g_1^{-1}), \lambda(g_2^{-1})\}$ shows that given Y there is $q \in \mathbb{N}$ and there are $\alpha_j, \beta_j \in \mathcal{M}_q(\mathbb{C})$, $(j = \pm 1)$ and $\gamma \in \mathcal{M}_q(\mathbb{C})$ with α_j, β_j and q depending on $\{g \in G \mid c_g \neq 0\}$ and γ a first order polynomial function of the c_g, such that: (Y invertible) $\Leftrightarrow$ ($\alpha_{-1} \otimes \lambda(g_1^{-1}) + \alpha_1 \otimes \lambda(g_1) + \beta_{-1} \otimes \lambda(g_2^{-1}) + \beta_1 \otimes \lambda(g_2) + \gamma \otimes 1$ invertible).

6.4. It will be convenient to make one further matrix transformation so as to be in the self-adjoint case. With the notations of 6.3 put

$$a = \alpha_{-1} \otimes \lambda(g_1^{-1}) + \alpha_1 \otimes \lambda(g_1) + \gamma \otimes 1$$
$$b = \beta_{-1} \otimes \lambda(g_2^{-1}) + \beta_1 \otimes \lambda(g_2).$$

Then we have

$$\left(a + b \text{ invertible}\right) \Leftrightarrow \left(\left(\begin{pmatrix} 0 & a+b \\ a^* + b^* & 0 \end{pmatrix}\right) \text{ invertible}\right).$$

So defining $X_1 = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}$ we have

$$(Y \text{ invertible}) \Leftrightarrow (X_1 + X_2 \text{ invertible}).$$

Moreover

$$X_j = X_j^* \in \mathcal{M}_{2q}(\mathbb{C}) \otimes C^*(\lambda(g_j)) \subset \mathcal{M}_{2q}(\mathbb{C}) \otimes C_r^*(G) \quad (j = 1, 2)$$

and if $\{g \in G \mid c_g \neq 0\}$ is a fixed finite set then X_2 is constant and X_1 is a first order polynomial in c_g and $\bar{c}_g (g \in G)$. Also only $\lambda(g_j^{\pm 1})$ appear in the expression of X_j.

6.5. Let $B = \mathcal{M}_{2q}(\mathbb{C})$, $A = \mathcal{M}_{2q}(\mathbb{C}) \otimes C_r^*(G)$, $A_j = \mathcal{M}_{2q}(\mathbb{C}) \otimes C^*(\lambda(g_j)) \subset A$ and $\varphi : A \to \mathcal{M}_{2q}(\mathbb{C})$ the conditional expectation $\varphi = \mathrm{id} \otimes \tau$ where τ is the canonical trace on $C_r^*(G)$. Then $\{A_1, A_2\}$ is a free pair of subalgebras in (A, φ) and hence $\{X_1, X_2\}$ is a free pair of $\mathcal{M}_{2q}(\mathbb{C})$-valued random variables. It is especially easy to compute G_{X_j} since $A_j \simeq \mathcal{M}_{2q}(\mathbb{C}) \otimes C(\mathbb{T})$. Using the results of section 4 we have a method for computing $G_{X_1+X_2}(b) = \varphi(b(I - (X_1 + X_2)b)^{-1})$ for $b \in \mathcal{M}_{2q}(\mathbb{C}) \otimes I \subset A$. Note that $\mathrm{Tr}_{2q} \circ \varphi$ is faithful on A so that taking $b = zI_{2q} \otimes I$, $z \in \mathbb{C}$ and $\mathrm{Tr}_{2q}(G_{X_1+X_2}(zI_{2q} \otimes I))$ gives us the generating series for the moments of $X_1 + X_2$ with respect to a faithful trace on A. Solving this moment problem one gets the spectrum of $X_1 + X_2$ and hence the possibility of deciding whether $X_1 + X_2$ is invertible.

7. Dual algebraic structures.

This section deals with the necessary adaptations that have to be performed in our considerations in [10] in order to fit the B-valued case.

7.1. The basic idea is to replace the category of unital pro-C^*-algebras in [10] by some other category. Corresponding to the two cases: the purely algebraic one when B is just a unital algebra over $\mathbb{C}$ and the C^*-algebraic one when B is a unital C^*-algebra, we will consider the categories $\mathcal{C}_B$ and respectively $\mathcal{C}_B^*$.

$\mathcal{C}_B$ is the category of unital algebras A over $\mathbb{C}$ containing B as a subalgebra $B \hookrightarrow A$, the inclusion being unital and the morphisms are homomorphisms for which the diagrams

$$\begin{array}{ccc} A_1 & \rightarrow & A_2 \\ & \nwarrow \quad \nearrow & \\ & B & \end{array}$$

are commutative.

$\mathcal{C}_B^*$ is the category of B-pro-C^*-algebras, i.e. unital C^*-algebras $(A, \| \quad \|)$ with $1 \in B \subset A$ and endowed with a family of C^*-seminorms $(\| \quad \|_\alpha)$ $\alpha \in I$ indexed by some directed set I so that $\|b\|_\alpha = \|b\|$ if $b \in B$ and $\alpha \leq \beta \Rightarrow \|x\|_\alpha \leq \|x\|_\beta$, $\|x\| = \sup_{\alpha \in I} \|x\|_\alpha$ if $x \in A$ and moreover

$$A_1 = \varprojlim_{\alpha \in I} A_{\alpha_1}$$

where the subscript 1 is for the unit ball and A_α is the quotient of A by the ideal annihilated by $\| \quad \|_\alpha$. Morphisms in $\mathcal{C}_B^*$ are morphisms of unital pro-C^*-algebras (see 1.4 in [10]) $A \to A'$ making the diagram

$$\begin{array}{ccc} A & \rightarrow & A' \\ & \nwarrow \quad \nearrow & \\ & B & \end{array}$$

commutative.

7.2. A dual algebraic structure is an algebraic structure in a category as defined in Chapter IV, §1 of [2]. We examined in [10] what a dual group structure means in the category of unital pro-C^*-algebras. In $\mathcal{C}_B$ and $\mathcal{C}_B^*$ we have a similar situation.

Let μ, j, χ be the binary, unary and nullary operations defining the dual group structure on A. Here

$$\begin{array}{ccc} A & \xrightarrow{\chi} & B \\ & \nwarrow \quad \nearrow & \\ & B & \end{array}$$

is commutative.

Also the free products with amalgamation over $\mathbb{C}$ have to be replaced by free products with amalgamation over B. Thus $\mu : A \to A \underset{B}{*} A$. If $A \in \mathcal{C}_B^*$, this free product is defined as follows: it is the inverse limit of the C^*-algebraic free product with amalgamation $A_\alpha \underset{B}{*} A_\alpha$.

7.3. If $A \in \mathcal{C}_B$ the state space of A denoted by $S(A)$ is the set of conditional expectations $\varphi : A \to B$. If $A \in \mathcal{C}_B^*$ then $S(A)$ is the set of conditional expectations $\varphi : A \to B$ such that $\|\varphi(a)\| \leq \|a\|_\alpha$ for one of the seminorms of A.

If $\varphi_j \in S(A_j)$ $(j = 1,2)$ then there is a unique $\varphi \in S(A_1 \underset{B}{*} A_2)$ such that $\varphi(a_1 \dots a_n) = 0$ whenever $a_k \in A_j(k)$, $j(k) \in \{1,2\}$, $\varphi_{j(k)}(a_k) = 0$, $j(k) \neq j(k+1)$ $(1 \leq k \leq n-1)$ and $\varphi \mid A_j = \varphi_j$.

Uniqueness of φ follows from 1.3 both in the $\mathcal{C}_B$ and $\mathcal{C}_B^*$ cases. Existence of φ in the $\mathcal{C}_B^*$ case is obtained from §5 of [7].

The existence of φ in the $\mathcal{C}_B$-case is seen as follows. Let $\overset{\circ}{A}_j = \ker \varphi_j$ and $D_n = \{(i_1, \dots, i_n) \mid i_j \in \{1,2\}, i_k \neq i_{k+1}, 1 \leq j \leq n, 1 \leq k \leq n-1\}$. Then we have

$$A_1 \underset{B}{*} A_2 \simeq B \oplus \bigoplus_{n \geq 0} \bigoplus_{(i_1, \dots, i_n) \in D_n} \overset{\circ}{A}_{i_1} \otimes_B \overset{\circ}{A}_{i_2} \otimes_B \dots \otimes_B \overset{\circ}{A}_{1_n}$$

and we define φ as the projection onto the B-summand.

We shall denote φ by $\varphi_1 * \varphi_2$.

7.4. If (A, μ, j, χ) is a dual group in $\mathcal{C}_B$ or $\mathcal{C}_B^*$ (actually dual semigroup would suffice) and if $\varphi_1, \varphi_2 \in S(A)$ then $(\varphi_1, \varphi_2) \rightsquigarrow \varphi_1 \Theta \varphi_2 = (\varphi_1 * \varphi_2) \circ \mu$ defines a semigroup structure on $S(A)$ with unit χ.

7.5. In $\mathcal{C}_B$ there is a dual group structure on $B\langle X \rangle$ defined by

$$B\langle X \rangle \xrightarrow{\mu} B\langle X \rangle *_B B\langle X \rangle \simeq B\langle X_1, X_2 \rangle$$
$$\mu(X) = X_1 + X_2, \ j(X) = -X$$

and $\chi(b_0 X b_1 \dots b_{n-1} X b_n) = 0$ if $n \geq 1$ and $\chi(b) = b$. Then $S(B\langle X \rangle) = \Sigma_B$ and Θ is $\boxplus$.

Similarly in $\mathcal{C}_B^*$ a corresopnding dual group is $A = \mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} B$ (with the notations of 5.1 [10]) and since $(\mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} B) *_B (\mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} B) = (\mathbb{R}_{\mathbb{C},nc} *_{\mathbb{C}} \mathbb{R}_{\mathbb{C},nc}) *_{\mathbb{C}} B$ we define μ from the dual operation of $\mathbb{R}_{\mathbb{C},nc}$.

It is easy to construct similar examples for other dual groups considered in [10] by taking free products with B.

7.6. There are also examples of a somewhat different nature involving tensor products. For instance let $B[X]$ be the polynomials in X with coefficients in B (X and $b \in B$ commute) and let

$$\mu(X) = X_1 + X_2$$

where X_j are the two images of X in $B[X] *_B B[X]$, $j(X) = -X$ and $\chi(bX^n) = 0$ if $n > 0$, $\chi(b) = b$.

8. The B-valued Central Limit Theorem.

The B-valued central limit theorem for free random variables is an immediate consequence of the properties of the canonical form.

8.1. Definition. A random variable a is called B-semicircular if its canonical form is

$$\lambda^*(1) + \lambda(\xi_0) + \lambda(\xi_1).$$

The distribution of such a random variable is also called B-semicircular. The B-semicircular random variable is centered if $\varphi(a) = 0$ (equivalently if $\xi_0 = 0$).

8.2. Since this paper concentrates on algebraic aspects, we will use the weakest kind of convergence for distributions. Clearly, there is a lot of room for improving the convergence side in our central limit result.

Definition. Let B be a Banach algebra and $\mu, \mu_n : B\langle X\rangle \to B$ $(n \in \mathbb{N})$ B-valued distributions. We shall say μ_n convergence pointwise to μ if for every $P \in B\langle X\rangle$ we have $\lim_{n\to\infty} \|\mu_n(P) - \mu(P)\| = 0$.

8.2. *Remark.* It is easy to see along the lines of 3.1 if $(\mu_\iota)_{\iota\in I}$ is a family of B-valued distributions (B a Banach algebra) with canonical forms

$$\lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_{n,\iota})$$

then the following two conditions are equivalent

(i) there are constants $C_0, \dots, C_n$ such that

$$\sup_{\iota\in I} \|\mu_\iota(Xb_1X \dots b_kX)\| \leq C_k\|b_1\| \dots \|b_k\|$$

for all $b_1, \dots, b_k \in B$, $0 \leq k \leq n$.

(ii) there are constants $D_0, \dots, D_n$ such that

$$\sup_{\iota\in I} \|\xi_{k,\iota}(b_1 \otimes \dots \otimes b_k)\| \leq D_k\|b_1\| \dots \|b_k\|$$

for all $b_1, \dots, b_k \in B$, $0 \leq k \leq n$.

8.3. *Remark.* Assume B is a Banach algebra, μ_j, μ are B-valued distributions $(j \in \mathbb{N})$ and assume the equivalent conditions of 8.2 are satisfied by $(\mu_j)_{j\in\mathbb{N}}$. Then the following are equivalent:

(i) $$\lim_{j\to\infty} \mu_j(Xb_1X \dots b_kX) = \mu(Xb_1X \dots b_kX)$$

for all $0 \leq k \leq n$ and $b_1, \dots, b_k \in B$.

(ii) $$\lim_{j\to\infty} \xi_{k,j}(b_1 \otimes \dots \otimes b_k) = \xi_k(b_1 \otimes \dots \otimes b_k)$$

for all $0 \leq k \leq n$ and $b_1, \dots, b_k \in B$.

8.4. Theorem. *Assume B is a Banach algebra and a_j ($j \in \mathbb{N}$) is a B-free sequence of random variables, such that*

1°. $\varphi(a_j) = 0$, $j \in \mathbb{N}$

2°. *there is a bounded linear map $\eta : B \to B$ such that*

$$\lim_{n\to\infty} n^{-1} \sum_{1\leq j\leq n} \varphi(a_j b a_j) = \eta(b)$$

3°. *there are constants C_k ($k \geq 1$) such that*

$$\sup_{j\in\mathbb{N}} \|\varphi(a_j b_1 a_j \dots b_k a_j)\| \leq C_k \|b_1\| \dots \|b_k\|.$$

Let $S_n = n^{-1/2}(a_1 + \cdots + a_n)$. Then the distribution of S_n converges pointwise to the semicircular distribution with canonical form

$$\lambda^*(1) + \lambda(\eta).$$

Proof. Let

$$\lambda^*(1) + \sum_{k\geq 0} \lambda(\xi_{k,j})$$

be the canonical form of a_j. We have

$$\xi_{0,j} = 0 \quad (j \in \mathbb{N})$$

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \xi_{1,j}(b) = \eta(b)$$

$$\sup_{j\in\mathbb{N}} \|\xi_{n,j}(b_1 \otimes \cdots \otimes b_n)\| \leq C_n \|b_1\| \dots \|b_n\| \quad (n \geq 1)$$

for some constants C_n ($n \in \mathbb{N}$).

Let

$$\lambda^*(1) + \sum_{k\geq 0} \lambda(\eta_{k,n})$$

be the canonical form of S_n. It follows from 3.2 and 3.5 that

$$\eta_{k,n} = n^{-(k+1)/2}(\xi_{k,1} + \cdots + \xi_{k,n}).$$

Clearly $\eta_{0,n} = 0$,

$$\lim_{n\to\infty} \eta_{1,n}(b) = \eta(b),$$

$$\|\eta_{k,n}(b_1 \otimes \cdots \otimes b_k)\| \leq C_k \|b_1\| \dots \|b_k\| n^{-(k-1)/2}.$$

and

$$\|\eta_{1,n}(b)\| \leq C_1 \|b\|.$$

In view of 8.3 and 8.2 this implies that the distribution of S_n converges pointwise to the distribution of

$$\lambda^*(1) + \lambda(\eta)$$

i.e. to a B-semicircular distribution. □

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