A NON-COMMUTATIVE WEYL-VON NEUMANN THEOREM

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A non-commutative Weyl-von Neumann theorem is proved. Among its consequences: every operator on an infinite-dimensional Hilbert spaces is a norm-limit of reducible operators. A reflexivity theorem for separable norm-closed subalgebras of the Calkin algebra is also obtained.

The starting point, for the present work, was a question raised by P. R. Halmos ([3] problem 8.), for which we provide an affirmative answer: *every operator on a separable infinite-dimensional Hilbert space is a norm-limit of reducible operators*. This led to a theorem on representations of separable C*-algebras, that may be regarded as a non-commutative extension of the classical results of Weyl and von Neumann about the representability of hermitian operators as small compact perturbations of diagonable hermitian operators.

This paper has two sections. In the first section we prove our non-commutative Weyl-von Neumann type theorem (see theorem 1.5 and corollary 1.6). In the sense of Brown, Douglas, Fillmore theory ([1]) we prove that all trivial extensions of the compact operators by a separable C*-algebra are equivalent and that the equivalence class of trivial extensions is a unit in the semigroup of all equivalence classes of extensions (theorem 1.3 and corollary 1.4).

Besides we establish a reflexivity theorem for norm-closed separable subalgebras of the Calkin algebra (theorem 1.8).

The second section outlines some of the consequences of the first section in operator theory: the answer to Halmos' question (proposition 2.2), the characterization of operators with closed unitary orbit (proposition 2.4) and the characterization of operators the derivations of which restricted to the hermitian operators have closed range (theorem 2.5).

§ 1

Let $\mathfrak{H}$ denote a complex separable infinite-dimensional Hilbert space. The set of bounded operators on $\mathfrak{H}$ will be denoted by $\mathscr{L}(\mathfrak{H})$. By $\mathscr{K}(\mathfrak{H})$ we shall denote the compact operators and by p the map form $\mathscr{L}(\mathfrak{H})$ onto the Calkin algebra $\mathscr{L}(\mathfrak{H})/\mathscr{K}(\mathfrak{H})$.

LEMMA 1.1. *Let $\mathscr{A} \subset \mathscr{L}(\mathfrak{H})$ be a separable sub-C*-algebra, $I \in \mathscr{A}$ and let π be a representation of $p(\mathscr{A})$ on a Hilbert space $\mathfrak{H}_\pi$, which admits a cyclic vector $\xi \in \mathfrak{H}_\pi$. Let further $x_1, \ldots, x_n \in \mathscr{A}$ and $\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset$*

REV. ROUM. MATH. PURES ET APPL., TOME XXI, N° 1, P. 97–113, BUCAREST, 1976

$\subset \pi(p(\mathscr{A}))\xi$ *be finite-dimensional subspaces such that* $\pi(p(x_j))\,\mathfrak{V}_1 \subset \mathfrak{V}_2$. *Let also* $\varepsilon > 0$ *and a finite-dimensional subspace* $\mathfrak{W} \subset \mathfrak{H}$ *be given. Then there is a linear isometric map* $L : \mathfrak{V}_2 \to \mathfrak{H}$ *such that* $L(\mathfrak{V}_2) \perp \mathfrak{W}$ *and*

$$\| L\,\pi(p(x_j))\,h - x_j L\,h \| \leqslant \varepsilon \| h \|$$

for $h \in \mathfrak{V}_1$, $j = 1, \ldots, n$.

Proof. Let $\{e_k\}_{1 \leqslant k \leqslant p+q}$ be an orthonormal basis of $\mathfrak{V}_2$ such that $\{e_k\}_{1 \leqslant k \leqslant p}$ be a basis of $\mathfrak{V}_1$. Suppose $\|\xi\| = 1$ and consider $u_k \in \mathscr{A}$ such that

$$\pi(p(u_k))\,\xi = e_k \; (1 \leqslant k \leqslant p+q).$$

Write also :

$$\pi(p(x_j))\,e_i = \sum_{k=1}^{p+q} a_{ki}^{(j)} e_k \quad (1 \leqslant i \leqslant p).$$

Consider the C^*-algebra $\mathscr{B} = \mathscr{A} + \mathscr{K}(\mathfrak{H})$ and φ the state of $\mathscr{B}$ defined by

$$\varphi(y) = \langle \pi(p(y))\,\xi, \; \xi \rangle .$$

Consider also $Q \in \mathscr{K}(\mathfrak{H}) \subset \mathscr{B}$ the orthogonal projection onto $\bigvee_{1 \leqslant k \leqslant p+q} u_k^* \mathfrak{W}$ and $b \in \mathscr{B}$ the positive element :

$$b = \sum_{1 \leqslant j \leqslant n} \sum_{1 \leqslant i \leqslant p} \left(x_j u_i - \sum_{k=1}^{p+q} a_{ki}^{(j)} u_k \right)^* \left(x_j u_i - \sum_{k=1}^{p+q} a_{ki}^{(j)} u_k \right).$$

We have :

$$\varphi(Q) = \varphi(b) = 0$$

$$\varphi(u_i^* u_j) = \delta_{ij} \,(1 \leqslant i, j \leqslant p+q)$$

It is known (see [2], 11.2.1) that φ is a weak limit of vector-states of $\mathscr{B}$. Hence, we can find vectors $\eta_m \in \mathfrak{H}$, $\|\eta_m\| = 1$ such that :

$$\lim_{m \to \infty} \langle Q\eta_m, \; \eta_m \rangle = \lim_{m \to \infty} \langle b\eta_m, \; \eta_m \rangle = 0$$

$$\lim_{m \to \infty} \langle u_j \,\eta_m, u_i \,\eta_m \rangle = \delta_{ij}$$

Then, for $\zeta_m = (I - Q)\,\eta_m$ we have :

$$\zeta_m \perp \bigvee_{1 \leqslant k \leqslant p+q} u_k^* \mathfrak{W}, \; \|\zeta_m\| \leqslant 1$$

$$\lim_{m \to \infty} \left\| x_j u_i \zeta_m - \sum_{k=1}^{p+q} a_{ki}^{(j)} u_k \zeta_m \right\| = 0 \;\; (1 \leqslant j \leqslant n, \; 1 \leqslant i \leqslant p)$$

$$\lim_{m \to \infty} \langle u_s \zeta_m, u_t \zeta_m \rangle = \delta_{s,t}$$

$$(1 \leqslant s, \; t \leqslant p+q)$$

Let us define $X_m : \mathfrak{V}_2 \to \mathfrak{H}$ by $X_m e_k = u_k \zeta_m$. Then if X_m has polar decomposition $X_m = U_m (X_m^* X_m)^{1/2}$ it follows from the preceding relations that :

$$\lim_{m \to \infty} \|X_m - U_m\| = 0$$

$$\lim_{m \to \infty} \| x_j U_m e_i - U_m \pi(p(x_j))\,e_i) \| = 0$$

$$1 \leqslant j \leqslant n, \; 1 \leqslant i \leqslant p)$$

$$U_m(\mathfrak{V}_2) \perp \mathfrak{W}.$$

Thus, we can take $L = U_m$ for m great enough.

Q.E.D.

LEMMA 1.2. *Let* $\mathscr{A} \subset \mathscr{L}(\mathfrak{H})$ *be a separable sub-*C^**-algebra,* $I \in \mathscr{A}$ *and let* $\{\pi_r\}_{r=1}^{\infty}$ *be a sequence of representations of* $p(\mathscr{A})$ *on Hilbert spaces* $\{\mathfrak{H}_r\}_{r=1}^{\infty}$*, which admit cyclic vectors* $\xi_r \in \mathfrak{H}_r$*. Then there are linear isometric maps* $L_r : \mathfrak{H}_r \to \mathfrak{H}$ *such that* $r \neq s \Rightarrow L_r(\mathfrak{H}_r) \perp L_s(\mathfrak{H}_s)$, $L_r \pi_r(p(x)) - x L_r$ *is compact and* $\lim_{r \to \infty} \| L_r \pi_r(p(x)) - x L_r \| = 0$, *for every* $x \in \mathscr{A}$.

Proof. If the representations π_r are finite-dimensional then the lemma is an immediate consequence of the preceding one. We shall restrict the proof to the case when all π_r are infinite-dimensional in order not to complicate the notations. The case when some π_r are finite-dimensional and some infinite-dimensional requires an easy adaptation of the argument below, which we leave to the reader.

Let $\{x_j\}_{j=1}^{\infty}$ be a total sequence of hermitian elements of $\mathscr{A}$. We define inductively $\mathfrak{M}_{1,r} = \mathbb{C}\,\xi_r$ and

$$\mathfrak{M}_{k+1,r} = \mathfrak{M}_{k,r} \vee \left(\bigvee_{j=1}^{k} \pi_r(p(x_j))\,\mathfrak{M}_{k,r} \right);$$

Consider $P_{k,r}$ the orthogonal projection of $\mathfrak{H}_r$ onto $\mathfrak{M}_{k,r}$ and $Q_{1,r} = P_{1,r}$, $Q_{k,r} = P_{k,r} - P_{k-1,r}$ $(k \geqslant 2)$. Then

$$Q_{i,r}\, \pi_r\,(p\,(x_k))\, Q_{j,r} = 0$$

if $|i-j| \geqslant 2$ and $\max\,(i,j) \geqslant k+2$, or equivalently: the operator-valued matrix of $\pi_r\,(p(x_k))$ with respect to the decomposition

$$\mathfrak{H}_r = \mathfrak{M}_{k,r} \oplus Q_{k+1}(\mathfrak{H}_r) \oplus Q_{k+2}(\mathfrak{H}_r) \oplus \cdots$$

is tri-diagonal.

Consider then:

$$R_{1,r} = \sum_{k=1}^{2r} \frac{2r-k}{2r-1}\, Q_{k,r} \quad \text{and}$$

$$R_{j,r} = \sum_{k=(2r)^{j-2}}^{(2r)^{j-1}} \frac{k-(2r)^{j-2}}{(2r)^{j-1}-(2r)^{j-2}}\, Q_{k,r} +$$

$$+ \sum_{k=(2r)^{j-1}+1}^{(2r)^{j}} \frac{(2r)^{j}-k}{(2r)^{j}-(2r)^{j-1}}\, Q_{k,r}$$

for $j \geqslant 2$.

Obviously $\sum_{j=1}^{\infty} R_{j,r} = I$. The tri-diagonality of the $\pi_r\,(p\,(x_k))$, already mentioned, yields the following estimates:

1) If $(2r)^{j-2} > k$ then

$$\|\,[R_{j,r}^{1/2},\ \pi_r\,(p\,(x_k))]\,\| \leqslant 2\,\|x_k\|\,((2r)^{j-1}-(2r)^{j-2})^{-1/2}\,.$$

2) If $2r > k$ and $j = 1,\ 2$ then

$$\|[R_{j,r}^{1/2},\ \pi_r\,(p\,(x_k))]\| \leqslant 2k\,\|x_k\|\,(2r-1)^{-1/2}.$$

It follows that:

$$\sum_{j=1}^{\infty} \|[R_{j,r}^{1/2},\ \pi_r\,(p\,(x_k))]\| < \infty$$

for all k and r. Moreover, if $2r > k$ then

$$\sum_{j=1}^{\infty} \|\,[R_{j,r}^{1/2},\ \pi_r\,(p\,(x_k))]\| \leqslant (4k+3)\,(2r-1)^{-1/2}\,\|x_k\|.$$

Putting the set of pairs of positive integers (r,j) into a sequence, we can apply Lemma 1.1. recurrently to obtain isometries:

$$L_{r,j}:\ \mathfrak{M}_{(2r)^j+1} \to \mathfrak{H}$$

which enjoy the following properties:

$$(r,j) \neq (s,k) \Rightarrow L_{r,j}\,(\mathfrak{M}_{(2r)^j+1}) \perp L_{s,k}\,(\mathfrak{M}_{(2s)^k+1})$$

$$\|L_{r,j}\,\pi_r\,(p\,(x_n))\,h - x_n\,L_{r,j}\,h\| \leqslant 2^{-r-j}\,\|x_n\|\ \|h\|$$

for $n < (2r)^j$ and $h \in \mathfrak{M}_{(2r)^j}$.

Since $R_{r,j}^{1/2}\,(\mathfrak{H}_r) \subset \mathfrak{M}_{(2r)^j}$ the composition $L_{r,j}\,R_{r,j}^{1/2}$ makes sense, and in case $n < (2r)^j$ we have:

$$\|L_{r,j}\,R_{r,j}^{1/2}\,\pi_r\,(p\,(x_n)) - x_n\,L_{r,j}\,R_{r,j}^{1/2}\| \leqslant \|[R_{r,j}^{1/2},\ \pi_r\,(p\,(x_n))] + 2^{-r-j}\,\|x_n\|.$$

It follows for all r and n:

$$(*) \qquad \sum_{j=1}^{\infty} \|L_{r,j}\,R_{r,j}^{1/2}\ \pi_r\,(p\,(x_n)) - x_n\,L_{r,j}\,R_{r,j}^{1/2}\| < \infty.$$

Moreover, in case $2r > n$ we have:

$$(**) \qquad \sum_{j=1}^{\infty} \|L_{r,j}\,R_{r,j}^{1/2}\,\pi_r\,(p\,(x_n)) - x_n\,L_{r,j}\,R_{r,j}^{1/2}\| \leqslant$$

$$\leqslant \|x_n\|\,(2^{-r} + (4n+3)\,(2r-1)^{-1/2})$$

Now we define $L_r = \sum_{j=1}^{\infty} L_{r,j}\,R_{r,j}^{1/2}$. Since $\sum_{j=1}^{\infty} R_{r,j} = I$ and the $L_{r,j}$ have mutually orthogonal ranges it follows that L_r is an isometry. Relation $(*)$ shows that $L_r\,\pi_r\,(p\,(x_n)) - x_n\,L_r$ is compact and relation $(**)$ yields

$$\lim_{r\to\infty} \|L_r\,\pi_r\,(p\,(x_n)) - x_n\,L_r\| = 0.$$

Since the sequence $\{x_n\}_{n=1}^{\infty}$ is total in $\mathcal{A}$ it follows that $L_r\,\pi_r\,(p\,(x)) - xL_r$ is compact and $\lim_{r\to\infty} \|L_r\,\pi_r\,(p(x)) - xL_r\| = 0$ for every $x \in \mathcal{A}$.

Q. E. D.

DEFINITION. *Let ρ_1, ρ_2 be representations on separable Hilbert spaces $\mathfrak{H}_1$, $\mathfrak{H}_2$ of a separable C*-algebra with unit $\mathcal{A}$. We shall say that ρ_1, ρ_2 are* *approximately equivalent* ($\rho_1 \underset{a}{\sim} \rho_2$) *if there is a sequence of unitary operators* $u_k : \mathfrak{H}_1 \to \mathfrak{H}_2$ *such that*

$$\rho_1(x) - u_k^* \rho_2(x) u_k \in \mathscr{K}(\mathfrak{H}_1)$$

$$\lim_{k\to\infty} \|\rho_1(x) - u_k^* \rho_2(x) u_k\| = 0$$

for every $x \in \mathscr{A}$.

It is easily seen that approximate equivalence is indeed an equivalence relation. Moreover, clearly $\rho_1 \underset{a}{\sim} \rho_2$ implies $\operatorname{Ker} \rho_1 = \operatorname{Ker} \rho_2$ and $\operatorname{Ker} p \circ \rho_1 = \operatorname{Ker} p \circ \rho_2$. Another simple property of approximate equivalence is that if $\rho_1 \underset{a}{\sim} \rho_2$ and $\rho_3 \underset{a}{\sim} \rho_4$, then $\rho_1 \oplus \rho_3 \underset{a}{\sim} \rho_2 \oplus \rho_4$.

THEOREM. 1.3. *Let* $\mathscr{A}$ *be a separable* C^**-algebra with unit and* ρ *a representation of* $\mathscr{A}$ *on the separable Hilbert space* $\mathfrak{H}$. *Let* π *be a representation of* $p\rho(\mathscr{A})$ *on a separable Hilbert space* $\mathfrak{H}_\pi$. *Then :*

$$\rho \underset{a}{\sim} \rho \oplus \pi \circ p \circ \rho.$$

Proof. The representation π of $p\rho(\mathscr{A})$ is at most a countable direct sum of cyclic representations of $p\ \rho(\mathscr{A})$. We shall restrict the proof to the case when π is an infinite direct sum of cyclic representations of $p\ \rho(\mathscr{A})$, the case of finite direct sum being a corollary.

Thus, suppose $\pi = \bigoplus_{i=1}^{\infty} \pi_i$, where π_i is a representation of $p\ \rho(\mathscr{A})$ on a Hilbert space $\mathfrak{H}_i$ and has a cyclic vector.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a total sequence of hermitian elements in $\mathscr{A}$. Using Lemma 1.2. for a sequence of cyclic representations of $p\rho(\mathscr{A})$ in which every π_i appears an infinity of times, we can then select for each $(i, j) \in \mathbb{N} \times \mathbb{N}$ a linear isometric map

$$L_{i,j} : \mathfrak{H}_i \to \mathfrak{H}$$

such that

$$(i, j) \neq (k, l) \Rightarrow L_{i,j}(\mathfrak{H}_i) \perp L_{k,l}(\mathfrak{H}_k)$$

$L_{i,j}\, \pi_i p\ \rho(x) - \rho(x) L_{ij}$ is compact for every $x \in \mathscr{A}$ and

$$\sum_{s=1}^{i+j} \|L_{ij}\, \pi_i\, p\, \rho(x_s) - \rho(x_s) L_{ij}\| \leq 2^{-i-j}$$

Consider then

$$L_j : \mathfrak{H}_\pi = \bigoplus_{i=1}^{\infty} \mathfrak{H}_i \to \mathfrak{H}$$

defined as $L_j\left(\bigoplus_{i=1}^{\infty} h_i\right) = \sum_{i=1}^{\infty} L_{ij}\, h_i$

We have :

$$\sum_{i=1}^{\infty} \|L_{ij}\, \pi_i\, p\, \rho(x_n) - \rho(x_n) L_{ij}\| \leq$$

$$\leq \sum_{i=1}^{n} \|L_{ij}\, \pi_i\, p\, \rho(x_n) - \rho(x_n) L_{ij}\| + \sum_{i=n+1}^{\infty} 2^{-i-j}.$$

It follows that

$$L_j\, \pi\, p\, \rho(x_n) - \rho(x_n) L_j$$

is compact for every $n \in \mathbb{N}$. Since $\{x_n\}_{n\in\mathbb{N}}$ is total in $\mathscr{A}$,

$$L_j\, \pi p\, \rho(x) - \rho(x) L_j$$

is compact for every $x \in \mathscr{A}$. Moreover, if $j > n$ we have

$$\|L_j\, \pi\, p\, \rho(x_n) - \rho(x_n) L_j\| \leq$$

$$\leq \sum_{i=1}^{\infty} \|L_{ij}\, \pi_i\, p\, \rho(x_n) - \rho(x_n) L_{ij}\| \leq 2^{-j}$$

Again this yields

$$\lim_{j\to\infty} \|L_j\, \pi p\, \rho(x) - \rho(x) L_j\| = 0$$

for every $x \in \mathscr{A}$.

Consider now

$$S_j = I - \sum_{k=j}^{\infty} L_k L_k^* + \sum_{k=j}^{\infty} L_{k+1} L_k^*.$$

We have :

$$[L_k L_k^*,\ \rho(x)] = L_k (L_k^*\ \rho(x) - \pi p\, \rho(x) L_k^*) +$$

$$+ (L_k\, \pi p\, \rho(x) - \rho(x) L_k) L_k^*$$

$$[L_{k+1} L_k^*,\ \rho(x)] = L_{k+1} (L_k^*\ \rho(x) - \pi p\ \rho(x) L_k) +$$

$$+ (L_{k+1}\, \pi\, p\, \rho(x) - \rho(x) L_{k+1}) L_k^*.$$

Hence, $[L_k L_k^*,\ \rho(x)]$ and $[L_{k+1} L_k^*,\ \rho(x)]$ are compact.

Moreover:

$$\sum_{k=j}^{\infty} \|[L_k L_k^*, \rho(x_n)]\| + \sum_{k=j}^{\infty} \|[L_{k+1} L_k^*, \rho(x_n)]\| \leqslant$$

$$\leqslant \sum_{k=j}^{j+n} (\|[L_k L_k^*, \rho(x_n)]\| + \|[L_{k+1} L_k^*, \rho(x_n)]\|) + 4 \sum_{k=j+n}^{\infty} 2^{-k} < \infty.$$

This yields $[S_j, \rho(x)] \in \mathscr{K}(\mathfrak{H})$ for every $x \in \mathscr{A}$.
On the other hand, if $n < j$ we easily find:

$$\|[S_j, \rho(x_n)\| \leqslant 2^{-j+3}$$

This again, yields:

$$\lim_{j\to\infty} \|[S_j, \rho(x)]\| = 0$$

for every $x \in \mathscr{A}$.

Consider now:

$$u_j : \mathfrak{H} \oplus \mathfrak{H}_\pi \to \mathfrak{H}$$

defined by:

$$u_j(h \oplus h_\pi) = S_j h + L_j h_n.$$

Then, clearly u_j is unitary and:

$$u_j(\rho \oplus \pi \circ p \circ \rho)(x) - \rho(x) u_j$$

is compact for every $x \in \mathscr{A}$ and

$$\lim_{j\to\infty} \|u_j(\rho \oplus \pi \circ p \circ \rho)(x) - \rho(x) u_j\| = 0$$

for every $x \in \mathscr{A}$.

Q.E.D.

COROLLARY 1.4. *Let ρ_1, ρ_2 be representations of the separable C^*-algebra with unit $\mathscr{A}$. Suppose* Ker ρ_1 = Ker $p \circ \rho_1$ = Ker ρ_2 = Ker $p \circ \rho_2$. *Then*:

$$\rho_1 \underset{a}{\sim} \rho_2.$$

Proof. Since Ker $p \circ \rho_1$ = Ker ρ_2 there is a representation π_1 of $p\rho_1(\mathscr{A})$ such that $\pi_1 \circ p \circ \rho_1 = \rho_2$. Analogously there is a representation π_2 of $p\rho_2(\mathscr{A})$ such that $\pi_2 \circ p \circ \rho_2 = \rho_1$. It follows from the preceding theorem that:

$$\rho_1 \underset{a}{\sim} \rho_1 \oplus \pi_1 \circ p \circ \rho_1 = \rho_1 \oplus \rho_2$$

$$\rho_2 \underset{a}{\sim} \rho_2 \oplus \pi_2 \circ p \circ \rho_2 = \rho_2 \oplus \rho_1.$$

Q.E.D.

THEOREM 1.5. *Let ρ_1, ρ_2 be representations on separable Hilbert spaces $\mathfrak{H}_1$, $\mathfrak{H}_2$ of a separable C^*-algebra with unit $\mathscr{A}$. Consider $\mathfrak{H}_1' = \rho_1(\text{Ker } p \circ \rho_1)\mathfrak{H}_1$ and $\mathfrak{H}_2' = \rho_2(\text{Ker } p \circ \rho_2)\mathfrak{H}_2$. Then, the following conditions are equivalent*:

(i) $$\rho_1 \underset{a}{\sim} \rho_2$$

(ii) *There is a sequence of unitary operators $u_k : \mathfrak{H}_1 \to \mathfrak{H}_2$ such that*

$$\lim_{k\to\infty} \|u_k^* \rho_2(x) u_k - \rho_1(x)\| = 0$$

for every $x \in \mathscr{A}$.

(iii) Ker ρ_1 = Ker ρ_2, Ker $p \circ \rho_1$ = Ker $p \circ \rho_2$ *and the representations of $\mathscr{A}$ on $\mathfrak{H}_1'$ and $\mathfrak{H}_2'$ induced by ρ_1 and respectively ρ_2 are equivalent.*

(iv) Ker ρ_1 = Ker ρ_2, Ker $p \circ \rho_1$ = Ker $p \circ \rho_2$ *and the representations of* Ker $p \circ \rho_1$ = Ker $p \circ \rho_2$ *on $\mathfrak{H}_1'$ and $\mathfrak{H}_2'$ induced by ρ_1 and respectively ρ_2 are equivalent.*

Proof. Clearly (i) ⇒ (ii) and (iii) ⇒ (iv). Also (iv) ⇒ (iii) follows immediately from the fact that the representations of the ideal Ker $p \circ \rho_1$ = Ker $p \circ \rho_2$ on $\mathfrak{H}_1'$ and respectively $\mathfrak{H}_2'$ are non-degenerate. Hence, we shall have to prove (iii) ⇒ (i) and (ii) ⇒ (iv).

(iii) ⇒ (i) Let $\mathfrak{H}$ be a separable Hilbert space and π_1, π_2 be representations of $p\rho_1(\mathscr{A})$ and, respectively, $p\rho_2(\mathscr{A})$ on $\mathfrak{H}$ such that Ker $p \circ \pi_1 = 0$, Ker $p \circ \pi_2 = 0$. By Theorem 1.3. we have

$$\rho_1 \underset{a}{\sim} \tilde{\rho}_1 = \rho_1 \oplus \pi_1 \circ p \circ \rho_1$$

$$\rho_2 \underset{a}{\sim} \tilde{\rho}_2 = \rho_2 \oplus \pi_2 \circ p \circ \rho_2.$$

Consider also

$$\rho_1' = \tilde{\rho}_1 \mid \mathfrak{H}_1' = \rho_1 \mid \mathfrak{H}_1'$$

$$\rho_1'' = \tilde{\rho}_1 \mid \mathfrak{H} \oplus (\mathfrak{H}_1 \ominus \mathfrak{H}'_1)$$

$$\rho'_2 = \tilde{\rho}_2 \mid \mathfrak{H}'_2 = \rho_2 \mid \mathfrak{H}'_2$$

$$\rho''_2 = \tilde{\rho}_2 \mid \mathfrak{H} \oplus (\mathfrak{H}_2 \ominus \mathfrak{H}'_2)$$

Since Ker ρ_1'' = Ker $p \circ \rho_1''$ = Ker $p \circ \rho_1$ = Ker $p \circ \rho_2$ = Ker ρ_2'' = Ker $p \circ \rho_2''$ it follows from the corollary of Theorem 1.3. that $\rho_1'' \underset{a}{\sim} \rho_2''$. Since ρ_1' and ρ_2' are equivalent, it follows that $\tilde{\rho}_1 \underset{a}{\sim} \tilde{\rho}_2$ and, hence, also $\rho_1 \underset{a}{\sim} \rho_2$.

(ii) ⇒ (iv). That Ker $\rho_1 =$ Ker ρ_2 and Ker $p \circ \rho_1 =$ Ker $p \circ \rho_2$ follows immediately from (ii). To prove the last part of (iv) we may replace the u_k's by a weakly convergent subsequence and thus suppose they have a weak limit u.

Since $\rho_1(x)$, $\rho_2(x)$ are compact for $x \in$ Ker $p \circ \rho_1 =$ Ker $p \circ \rho_2$ it follows that $u_k^* \rho_2(x) u_k$ converges weakly to $u^* \rho_2(x) u$ and $u_k \rho_1(x) u_k^*$ converges weakly to $u \rho_1(x) u^*$ and hence

$$u^* \rho_2(x) u = \rho_1(x)$$

$$u \rho_1(x) u^* = \rho_2(x)$$

for $x \in$ Ker $p \circ \rho_1 =$ Ker $p \circ \rho_2$.

Consider $x_j \in$ Ker $p \circ \rho_1 =$ Ker $p \circ \rho_2 (j \in \mathbb{N})$ an approximate unit of Ker $p \circ \rho_1 =$ Ker $p \circ \rho_2$. Then, denoting by P_1, P_2 the orthogonal projections of $\mathfrak{H}_1$ onto $\mathfrak{H}_1'$ and, respectively, of $\mathfrak{H}_2$ onto $\mathfrak{H}_2'$, we have that $\rho_1(x_j)$ converges strongly to P_1 and $\rho_2(x_j)$ converges strongly to P_2. Hence, we have:

$$\rho_2(x) u = u \rho_1(x)$$

$$u^* P_2 u = P_1, \ u P_1 u^* = P_2$$

Consider $v = P_2 u P_1$. We have

$$v^* v = P_1 u^* P_2 u P_1 = P_1$$

$$v v^* = P_2 u P_1 u^* P_2 = P_2$$

$\rho_2(x) v = v \rho_1(x)$ (for $x \in$ Ker $p \circ \rho_1 =$ Ker $p \circ \rho_2$), which ends the proof.

Q. E. D.

Remark now that if ρ is a representation of $\mathcal{A}$ on $\mathfrak{H}$, then the restriction of ρ to $\mathfrak{H}' = \overline{\rho(\text{Ker } p \circ \rho)(\mathfrak{H})}$ is a direct sum of irreducible representations. Also, there is a representation π of $p\,\rho(\mathcal{A})$, which is a direct sum of irreducible representations such that Ker $p \circ \pi = 0$. This gives the following corollary.

Corollary 1.6. *Every representation of a separable C^*-algebra with unit, on a separable Hilbert space is approximately equivalent to a direct sum of irreducible representations.*

Theorem 1.7. *Let ρ be a representation of a separable C^*-algebra with unit $\mathcal{A}$ on a separable Hilbert space $\mathfrak{H}$. In order that all representations of $\mathcal{A}$ approximately equivalent with ρ be equivalent it is necessary and sufficient that $\rho(\mathcal{A})$ be finite-dimensional.*

Proof. The sufficiency of the condition dim $\rho(\mathcal{A}) < \infty$ is quite easy, so we shall concentrate on the necessity.

Hence, assume that every representation of $\mathcal{A}$ approximately equivalent to ρ is equivalent to ρ. First, we shall prove that the spectrum $\widehat{p\,\rho(\mathcal{A})}$ consists only of isolated points. Suppose $\pi \in \widehat{p\,\rho(\mathcal{A})}$ is not an isolated point and let $\pi_i \in \widehat{p\,\rho(\mathcal{A})} \setminus \{\pi\}$ $(i \in \mathbb{N})$ be a sequence which has π as a cluster point. Let further $\pi^{(j)}$ be equivalent to π and $\pi_i^{(j)}$ be equivalent to $\pi_i (j \in \mathbb{N})$. Let also $\mathfrak{H}_1 \subset \mathfrak{H}$ be the greatest reducing subspace for $\rho(\mathcal{A})$ such that $\pi \circ p \circ \rho$ and $\rho | \mathfrak{H}_1$ be disjoint. Remark that

$$\text{Ker } \pi \circ p \circ \rho \supset \text{Ker } p \circ \big(\bigoplus_{i,j} \pi_i^{(j)} \circ p \circ \rho\big)$$

and hence

$$\rho \underset{a}{\simeq} \rho | \mathfrak{H}_1 \oplus \bigoplus_j \pi^{(j)} \circ p \circ \rho \underset{a}{\simeq} \rho|_{\mathfrak{H}_1} \oplus \bigoplus_j \pi^{(j)} \circ p \circ \rho \oplus \bigoplus_{i,j} \pi_i^{(j)} \circ p \circ \rho$$

$$\underset{a}{\simeq} \rho | \mathfrak{H}_1 \oplus \bigoplus_{i,j} \pi_i^{(j)} \circ p \circ \rho.$$

But $\rho | \mathfrak{H}_1 \oplus \bigoplus_j \pi^{(j)} \circ p \circ \rho$ contains $\pi \circ p \circ \rho$ and cannot hence be equivalent to $\rho | \mathfrak{H}_1 \oplus \bigoplus_{i,j} \pi_i^{(j)} \circ p \circ \rho$ which is disjoint from $\pi \circ p \circ \rho$. This contradiction proves that $\widehat{p\,\rho(\mathcal{A})}$ consists only of isolated points and hence is finite and discrete. It is known that $p\,\rho(\mathcal{A})$ being a C^*-algebra with unit, this implies that $p\,\rho(\mathcal{A})$ is finite-dimensional (use for instance [2] 4.7.2.). What we have to prove next is that (Ker $p \circ \rho$) / Ker ρ is finite-dimensional.

Consider $\mathfrak{H}' = \overline{\rho(\text{Ker } p \circ \rho)(\mathfrak{H})}$. It will be sufficient to prove that $\mathfrak{H}'$ is finite-dimensional. Suppose $\mathfrak{H}'$ is infinite-dimensional and let π be an irreducible representation of $p(\rho | \mathfrak{H}')(\mathcal{A})$ — Consider then $\mathfrak{H}_1 \subset \mathfrak{H} \ominus \mathfrak{H}'$ the greatest reducing subspace for $\rho |_{\mathfrak{H} \ominus \mathfrak{H}'}$ such that $\rho |_{\mathfrak{H}_1}$ be disjoint from $\pi \circ p \circ \rho$. Consider also $\pi^{(j)}$ equivalent to $\pi (j \in \mathbb{N})$. Then, we have

$$\rho |_{\mathfrak{H}'} \oplus \rho |_{\mathfrak{H}_1} \underset{a}{\simeq} \rho \underset{a}{\simeq} \rho \oplus \bigoplus_j \pi^{(j)} \circ p \circ \rho.$$

But $\pi \circ p \circ \rho$ is contained at most a finite number of times in $\rho |_{\mathfrak{H}'} \oplus \rho |_{\mathfrak{H}_1}$ while it has infinite multiplicity in $\rho \oplus \bigoplus_j \pi^{(j)} \circ p \circ \rho$. Thus, these two representations cannot be equivalent. This contradiction ends the proof.

Q.E.D.

Theorem 1.8. *Let $\mathcal{A} \subset \mathcal{L}(\mathfrak{H}) / \mathcal{K}(\mathfrak{H})$ be a separable norm-closed subalgebra containing the identity. Consider $\mathcal{L}at(\mathcal{A})$ the set of all self-adjoint projections $e \in \mathcal{L}(\mathfrak{H}) / \mathcal{K}(\mathfrak{H})$ such that $(1-e) x e = 0$ for every $x \in \mathcal{A}$. Then the algebra*

$$\mathcal{A}lg(\mathcal{L}at(\mathcal{A})) = \{ y \in \mathcal{L}(\mathfrak{H}) / \mathcal{K}(\mathfrak{H}) | (1-e) y e = 0, \quad (\forall) e \in \mathcal{L}at(\mathcal{A}) \}$$

is equal $\mathcal{A}$.

Proof: Suppose $T \in \mathcal{L}(\mathfrak{H})$ is such that $p(T) \notin \mathcal{A}$, we have to prove the existence of a projection $P \in \mathcal{L}(\mathfrak{H})$ such that $(I - P) S P \in \mathcal{K}(\mathfrak{H})$ for every $S \in p^{-1}(\mathcal{A})$ and $(I - P) T P \notin \mathcal{K}(\mathfrak{H})$.

Consider $\mathcal{B} \subset \mathcal{L}(\mathfrak{H})$ the C*-algebra generated by $p^{-1}(\mathcal{A})$ and T. There is a functional f on $p(\mathcal{B})$ such that $f(p(T)) = 1$ and $f(\mathcal{A}) = \{0\}$.

Write $f = f_1 - f_2 + i(f_3 - f_4)$ with f_1, f_2, f_3, f_4 positive functionals. Consider $\varphi = f_1 + f_2 + f_3 + f_4$. Then for $x \in \mathcal{A}$:

$$\varphi((p(T) - x)^* (p(T) - x)) \geqslant$$

$$\geqslant \left(\sum_{j=1}^{4} \left| f_j(p(T) - x) \right|^2\right) \cdot (\max_{1 \leqslant j \leqslant 4} \| f_j \|)^{-1}$$

$$\geqslant \frac{1}{4} (\max_{1 \leqslant j \leqslant 4} \| f_j \|)^{-1}$$

Let ω be the state of $p(\mathcal{B})$ proportional to φ and π the associated representation on $\mathfrak{H}_\pi$ with cyclic vector ξ. Then we have $\pi p(T) \xi \notin \overline{\pi(\mathcal{A}) \xi}$. Consider π_j representations of $p(\mathcal{B})$ equivalent tot π on Hilbert spaces $\mathfrak{H}_{\pi_j}$ and ξ_j the corresponding cyclic vectors. Denote by Q_j the projection of $\mathfrak{H}_{\pi_j}$ onto $\overline{\pi_j(\mathcal{A}) \xi_j}$. The representations of $\mathcal{B}$ on $\mathfrak{H}$ and on $\mathfrak{H} \oplus \bigoplus_j \mathfrak{H}_{\pi_j}$ are approximately equivalent so that there is a unitary operator u: $\mathfrak{H} \to \mathfrak{H} \oplus \bigoplus_j \mathfrak{H}_{\pi_j}$ such that:

$$R - u^* (R \oplus \bigoplus_j \pi_j p(R))\, u \in \mathcal{K}(\mathfrak{H})$$

for every $R \in \mathcal{B}$. Consider $\tilde{P}$ the projection $0 \oplus \bigoplus_j Q_j$. Then, we have:

$$(I - \tilde{P})(S \oplus \bigoplus_j \pi_j p(S))\, \tilde{P} = 0$$

for every $S \in p^{-1}(\mathcal{A})$ and

$$(I - \tilde{P})(T \oplus \bigoplus_j \pi_j p(T))\, \tilde{P} \notin \mathcal{K}(\mathfrak{H} \oplus \bigoplus_j \mathfrak{H}_{\pi_j}).$$

Hence, we can take $P = u^* \tilde{P} u$.

Q.E.D.

COROLLARY 1.9. *A separable sub-C*-algebra of the Calkin algebra, containing the identity is equal to its bi-commutant.*

Remark. The proof of the preceding theorem can be adapted to obtain the following fact:

Let $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H})$, $1 \in \mathcal{A}_1$ be separable subalgebras then there is a projection $e \in \mathcal{L}(\mathfrak{H})/\mathcal{K}(\mathfrak{H})$ such that

$$\inf \{ \|y - x\| \,|\, x \in \mathcal{A}_1 \} \leqslant \sqrt{2}\, \|(1 - e)\, y\, e\|$$

for all $y \in \mathcal{A}_2$.

§ 2.

Throughout this section $\mathfrak{H}$ is a complex separable infinite-dimensional Hilbert space and $\mathcal{U}(\mathfrak{H})$ is the group of all unitary operators on $\mathfrak{H}$.

Consider $T \in \mathcal{L}(\mathfrak{H})$ and $T_1 \in \overline{\{u\, T\, u^* \mid u \in \mathcal{U}(\mathfrak{H})\}}$.

Let $\mathcal{A}$ be the C*-algebra generated by T and the identity operator. Denote by ρ the identic representation of $\mathcal{A}$, $\rho(x) = x$. Then, since T_1 is in the closure of the unitary orbit of T it is easily seen that there is a unique representation ρ_1 of $\mathcal{A}$ on $\mathfrak{H}$ such that $\rho_1(T) = T_1$.

Clearly ρ and ρ_1 are approximately equivalent. Also, conversely, if ρ_1 is a representation of $\mathcal{A}$ on $\mathfrak{H}$ and ρ_1 is approximately equivalent to ρ, then $\rho_1(T)$ is in the closure of the unitary orbit of $T = \rho(T)$. These remarks show that the results in § 1. have the following direct consequences.

PROPOSITION 2.1. *Given* $T \in \mathcal{L}(\mathfrak{H})$ *and* $\varepsilon > 0$ *there is an operator*

$$T_1 \in \overline{\{u\, T\, u^* \mid u \in \mathcal{U}(\mathfrak{H})\}}$$

such that $T - T_1 \in \mathcal{K}(\mathfrak{H})$, $\|T - T_1\| < \varepsilon$ *and* T_1 *be an infinite direct sum of irreducible operators.*

This clearly implies the following proposition.

PROPOSITION 2.2. *The norm-closure of the set of reducible operators in* $\mathcal{L}(\mathfrak{H})$ *is* $\mathcal{L}(\mathfrak{H})$.

PROPOSITION 2.3. *Let* $T \in \mathcal{L}(\mathfrak{H})$ *and*

$$T_1 \in \overline{\{u\, T\, u^* \mid u \in \mathcal{U}(\mathfrak{H})\}}$$

Then we can find $u_n \in \mathcal{U}(\mathfrak{H})$ *such that* $T_1 - u_n T u_n^* \in \mathcal{K}(\mathfrak{H})$ *and* $\lim_{n \to \infty} \| u_n T u_n^* - T_1 \| = 0$.

PROPOSITION 2.4. *The unitary orbit*

$$\{u\, T\, u^* \mid u \in \mathcal{U}(\mathfrak{H})\}$$

is norm-closed if and only if the C-algebra of* T *is finite-dimensional.*

Remark. For $(T_1, \ldots, T_n) \in (\mathcal{L}(\mathfrak{H}))^n$ consider its unitary orbit

$$\{(u T_1 u^*, \ldots, u T_n u^*) \in (\mathcal{L}(\mathfrak{H}))^n \mid u \in \mathcal{U}(\mathfrak{H})\}.$$

Call also an n-tuple reducible it $T_1, \ldots, T_n$ have a joint reducing subspace. Clearly, in this frame work, the results in § 1. imply the analogs of propositions 2.1, 2.2., 2.3, 2.4, for n-tuples.

Proposition 2.4. leads naturally to the question whether its infinitesimal counterpart concerning the restriction of derivations to the antihermitian operators is still true. An affirmative answer is provided by the following theorem.

THEOREM 2.5. *Let $T_j \in \mathcal{L}(\mathfrak{H})$ $(1 \leqslant j \leqslant n)$ and $\mathcal{L}_h(\mathfrak{H})$ denote the set of hermitian elements in $\mathcal{L}(\mathfrak{H})$. Consider the map*

$\Phi : \mathcal{L}_h(\mathfrak{H}) \to (\mathcal{L}(\mathfrak{H}))^n$ defined by : $\Phi(X) = ([T_1, X], \ldots, [T_n, X])$ (for $X \in \mathcal{L}_h(\mathfrak{H})$). Then the following conditions are equivalent :

(i) the map Φ has closed range.

(ii) the C-algebra generated by $T_1, \ldots, T_n$ is finite-dimensional.*

The proof of the theorem will be based on two lemmas.

LEMMA 2.6.

Let $T_j \in \mathcal{L}(\mathfrak{H})$ and $T'_j \in \mathcal{L}(\mathfrak{H})$ $(1 \leqslant j \leqslant n)$. Suppose there is a sequence of unitary operators $u_k \in \mathcal{U}(\mathfrak{H})$ $(k \in \mathbb{N})$ such that

$$\lim_{k\to\infty} \|u_k T_j u_k^* - T'_j\| = 0 \ (1 \leqslant j \leqslant n).$$

Then the following conditions are equivalent :

(i) the map $\Phi : \mathcal{L}_h(\mathfrak{H}) \to (\mathcal{L}(\mathfrak{H}))^n$ defined by $\Phi(X) = ([T_1, X], \ldots, [T_n, X])$ has closed range.

(ii) the map $\Phi' : \mathcal{L}_h(\mathfrak{H}) \to (\mathcal{L}(\mathfrak{H}))^n$ defined by $\Phi'(X) = ([T'_1, X], \ldots, [T'_n, X])$ has closed range.

Proof : It is clearly sufficient to prove that (i) $\Rightarrow$ (ii).

Since Φ has closed range there is a constant $C > 0$ such that for every $X \in \mathcal{L}_h(\mathfrak{H})$ there is $\widetilde{X} \in \mathcal{L}_h(\mathfrak{H})$ such that $\|\widetilde{X}\| \leqslant C \ \|\Phi(X)\|$ and $\Phi(X) = \Phi(\widetilde{X})$. Thus, for $X \in \mathcal{L}(\mathfrak{H})$ we can find $\widetilde{X}_k$ such that $\|\widetilde{X}_k\| \leqslant C \ \|\Phi(u_k^* X u_k)\|$ and $\Phi(\widetilde{X}_k) = \Phi(u_k^* X u_k)$. Then, we have

$$\limsup_{k\to\infty} \|u_k^* \widetilde{X}_k u_k\| \leqslant$$

$$\leqslant C \limsup_{k\to\infty} \| \Phi(u_k^* X u_k)\| =$$

$$= C \| \Phi'(X)\|.$$

Let Y be the weak limit of a sub-sequence of the $u_k \widetilde{X}_k u_k^*$. Then, we have :

$$\Phi'(Y) = \Phi'(X)$$

and $\|Y\| \leqslant C \| \Phi'(X)\|$.

Q.E.D.

LEMMA 2.7.

Let $\mathfrak{K}$ *be a separable Hilbert space of finite or infinite dimension and $S_1, \ldots, S_n \in \mathcal{L}(\mathfrak{K})$. Then, if* dim $\mathfrak{K} \geqslant ((2n)^{k+1} - 1)/(2n-1)$ *there is $X \in \mathcal{L}_h(\mathfrak{K})$, $\{0,1\} \subset \sigma(X)$ such that*

$$\|[S_j, X]\| \leqslant \frac{2}{k} \|S_j\| \ (1 \leqslant j \leqslant n).$$

Proof. Pick a vector $\xi \in \mathfrak{K}$, $\xi \neq 0$ and consider $\mathfrak{M}_1 = \mathbb{C}\,\xi$ and define by induction

$$\mathfrak{M}_{k+1} = \mathfrak{M}_k \vee \bigvee_{j=1}^{n} S_j \mathfrak{M}_k \vee \bigvee_{j=1}^{n} S_j^* \mathfrak{M}_k$$

Then, dim $\mathfrak{M}_k \leqslant ((2n)^k - 1)/(2n-1)$. Denote by P_k the projection of $\mathfrak{K}$ onto $\mathfrak{M}_k$ and put $Q_1 = P_1$, $Q_k = P_k - P_{k-1}$ $(k \geqslant 2)$. Then,

$$|k - l| \geqslant 2 \Rightarrow Q_l S_j Q_k = 0.$$

If $Q_{k+1} = 0$ then P_k is a reducing projection for all S_j and $P_k \neq 0$, $P_k \neq I$, so we can take $X = P_k$.

If $Q_{k+1} \neq 0$ let us define :

$$X = \sum_{l=1}^{k} \frac{k+1-l}{k} Q_l$$

Consider

$$S_j^{(1)} = \sum_{l=1}^{k} Q_{l+1} S_j Q_l$$

$$S_j^{(2)} = \sum_{l=1}^{k+1} Q_l S_j Q_l$$

$$S_j^{(3)} = \sum_{l=1}^{k} Q_l S_j Q_{l+1}$$

We have $P_{k+1} S_j P_{k+1} = \sum_{i=1}^{3} S_j^{(i)}$ and $\|S_j^{(i)}\| \leqslant \|S_j\|$. Then we have

$$[X, S_j] = [X, \ P_{k+1} S_j P_{k+1}] = [X, S_j^{(1)}] + [X, S_j^{(3)}]$$

and

$$\|[X, S_j^{(i)}]\| \leqslant \frac{1}{k} \|S_j^{(i)}\| \leqslant \frac{1}{k} \|S_j\|$$

for $i = 1,3$.

Q.E.D.

Proof of theorem 2.5. : Implication (ii) $\Rightarrow$ (i) is quite easy and will be left to the reader.

Thus assume (i) and we shall prove (ii). Because of Lemma 2.6. we can replace the n-tuple $(T_1, \ldots, T_n)$ by any other n-tuple in the closure of its unitary orbit. In view of the results in § 1. we can suppose $(T_1, \ldots, T_n)$ is the direct sum of an infinity of irreducible n-tuples. That is, we can suppose there are projections $P_l (l \in \mathbb{N})$ such that $P_l \neq 0$, $P_l \neq I$ and $\sum_{l \in \mathbb{N}} P_l = I$, which are minimal projections in the commutant of the C*-algebra generated by $T_1, \ldots, T_n$. We shall prove that $\sup_{l \in \mathbb{N}} \dim P_l < \infty$.

Indeed, if $\sup_{l \in \mathbb{N}} \dim P_l = \infty$, we can find for each $k \in \mathbb{N}$ a number $l_k \in \mathbb{N}$ such that $\dim P_{l_k}(\mathfrak{H}) > ((2n)^{k+1} - 1)/(2n-1)$ and in view of Lemma 2.7 and operator $X_k \in \mathcal{L}_h(\mathfrak{H})$ such that:

$$P_{l_k} X_k P_{l_k} = X_k, \quad \sigma(X_k) \supset \{0,1\}$$

and

$$\| [T_j, X_k] \| \leqslant \frac{2}{k} \| T_j \|.$$

Now for every $Y = Y^*$ in the commutant of the C*-algebra generated by $T_1, \ldots, T_n$ we have

$$\|X_k - Y\| \geqslant \|X_k - P_{l_k} Y P_{l_k}\| \geqslant \frac{1}{2}$$

since $P_{l_k} Y P_{l_k} = \lambda P_{l_k}$, P_{l_k} being a minimal projection of this commutant and $\{0,1\} \subset \sigma(X_k)$. These facts clearly contradict (i) and so $\sup_{l \in \mathbb{N}} \dim P_l(\mathfrak{H}) < \infty$.

To end the proof it will be sufficient to prove that the center of the commutant of the C*-algebra generated by $T_1, \ldots, T_n$ is finite-dimensional. Suppose, on the contrary, that there are mutually disjoint non-zero minimal central projections $Q_i (i \in \mathbb{N})$ in this commutant. Clearly we may suppose even more, that there are $P_{l_i} \leqslant Q_i (i \in \mathbb{N})$ with $\dim P_{l_i}(\mathfrak{H}) = \dim P_{l_j}(\mathfrak{H})$ $(i, j \in \mathbb{N})$. Consider $v_i \in \mathcal{L}(\mathfrak{H})$ partial isometries

$$v_i^* v_i = P_{l_1}, \quad v_i v_i^* = P_{l_i}.$$

Since $P_{l_1}(\mathfrak{H})$ is finite-dimensional we may suppose, passing to a subsequence, that the $v_i^* T_j v_i$ form a convergent sequence for each $1 \leqslant j \leqslant n$. But then:

$$\lim_{m \to \infty} \|[v_m v_{m+1}^* + v_{m+1} v_m^*, T_j]\| = 0$$

for each $1 \leqslant j \leqslant n$. But, on the other hand, if $Y = Y^*$ belongs to the commutant of the C*-algebra generated by the $T_j (1 \leqslant j \leqslant n)$ then:

$$\|v_m v_{m+1}^* + v_{m+1} v_m^* - Y\| \geqslant$$

$$\geqslant \| Q_{m+1}(v_m v_{m+1}^* + v_{m+1} v_m^* - Y) Q_m \| = \| v_{m+1} v_m^* \| = 1.$$

This clearly contradicts (i) and ends the proof.

Q.E.D.

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