Circular and Semicircular Systems and Free Product Factors

DAN VOICULESCU[1]

Dedicated to Prof. Jacques Dixmier on the occasion of his 65th birthday

Many basic questions about the type II_1 factors of free groups, and more generally of free products of cyclic groups, are still unanswered. On the other hand the free group factors, like the hyperfinite factor, arise naturally from a functor taking Hilbert spaces and contractions to C^*-algebras and completely positive maps. This functor is the analogue of Gaussian measure on Hilbert space in a kind of non-commutative probability theory ([1], [2], [3], [4], [5]) in which free products are given a treatment similar to tensor products, i.e. to independence. One of the aims of the present paper is to apply these probabilistic results to free group factors.

To be more specific, here are two results we obtained in this way. By $F(\mathbf{N})$ we denote the free group with generators indexed by the natural numbers.

1°. *If G is an at most countable free product of cyclic groups, then the II_1 factors of $F(\mathbf{N})$ and of $G * F(\mathbf{N})$ are isomorphic.*

2°. *The fundamental group of the II_1 factor of $F(\mathbf{N})$ contains the positive rational numbers.*

The circular and semicircular systems, we study, are analogues of families of independent complex and respectively real Gaussian random variables.

[1] Research supported in part by a grant from the National Science Foundation.

They provide convenient systems of generators for free group factors. Actually, the main source for our results is the recent realization ([5]) that Gaussian random $N \times N$ matrices in the large N limit, behave like objects in the non-commutative probabilistic context for free products. Also, conversely, the results we obtain here, have consequences for random matrices: the components of the polar decomposition of a Gaussian random matrix are asymptotically free (see Remark 2.7).

The semicircular systems are called so because of the semicircle distribution of its elements. The semicircle law for the eigenvalues of random matrices was discovered by E. Wigner ([6], [7]). In [1] we found that the semicircle distribution is the analogue for free products of the Gaussian distribution. This coincidence was the first clue for the connection between random matrices and free products.

This paper has three sections.

Section 1 is a large section of preliminaries, designed to make the paper self-contained.

Section 2 is devoted to results on circular and semicircular systems. Among these we mention Propositon 2.6 concerning the polar decomposition of a circular element.

Section 3 contains applications to free group factors.

1. Preliminaries

This section is devoted to preliminaries. We recall definitions and facts from ([1], [4], [5]) and present some further definitions and general remarks which will be used in the next sections.

A *"non-commutative probability space"* (A, φ) consists of a unital algebra A over $\mathbb{C}$ equipped with a *state* $\varphi : A \to \mathbb{C}$, i.e. a linear functional φ such that $\varphi(1) = 1$. If A is a $*$-algebra and $\varphi(x^*) = \overline{\varphi(x)}$ we call (A, φ) a *$*$-probability space*. Similarly, a $*$-probability space (A, φ) where A is a C^*-algebra and φ is positive, will be called a *C^*-probability space*. Further, a C^*-probability space (A, φ) with A a W^*-algebra and φ a normal state, will be called a *W^*-probability space*.

If (A, φ) is a non-commutative probability space, elements of A will often be referred to as *non-commutative random variables* (or simply random variables).

1.1. DEFINITION: Let (A, φ) be a non-commutative probability space. A family of subalgebras $1 \in A_\iota \subset A$ $(\iota \in I)$ is called a *free family of subalgebras* if $\varphi(a_1 \ldots a_n) = 0$ whenever $a_j \in A_{\iota(j)}$, with $\iota(j) \neq \iota(j+1)$ $(1 \leq j \leq n-1)$ and $\varphi(a_j) = 0$ $(1 \leq j \leq n)$. A *family of subsets* $(\Omega_\iota)_{\iota \in I}$ *is called free* if the subalgebras A_ι generated by $\{1\} \cup \Omega_\iota$ form a free family of subalgebras. A *family of random variables* $(f_\iota)_{\iota \in I}$ in A *is called free* if the family of subsets $(\{f_\iota\})_{\iota \in I}$ is free. If (A, φ) is a $*$-probability space, *a family of subsets* $(\Omega_\iota)_{\iota \in I}$ *is called $*$-free* if the family of subsets $(\Omega_\iota \cup \Omega_\iota^*)_{\iota \in I}$ is free. A *family of random variables* $(f_\iota)_{\iota \in I}$ *is called $*$-free* if the family of subsets $(\{f_\iota\})_{\iota \in I}$ is $*$-free.

1.2. REMARK: If $(\Omega_\iota)_{\iota \in I}$ is a $*$-free family of subsets in a C^*-probability space (respectively in a W^*-probability space) then the C^*-subalgebras (respectively the W^*-subalgebras) generated by $\{1\} \cup \Omega_\iota$ $(\iota \in I)$ form a free family of subalgebras.

1.3. REMARK: Let (A, φ) be a non-commutative probability space.

(i) If $(A_\iota)_{\iota \in I}$ is a free family of subalgebras in (A, φ) and B is the subalgebra generated by $\bigcup_{\iota \in I} A_\iota$, then $\varphi \mid B$ is completely determined by the $\varphi \mid A_\iota$ $(\iota \in I)$.

(ii) If $(\Omega_\iota)_{\iota \in I}$ is a free family of subsets in (A, φ) and if $(I_j)_{j \in J}$ is a partition of I and if $\Sigma_j = \bigcup_{\iota \in I_j} \Omega_\iota$, then $(\Sigma_j)_{j \in J}$ is also a free family of subsets.

(iii) If $(A_\iota)_{\iota \in I}$ is a free family of subalgebras in (A, φ) and if $(\omega_{\iota k})_{k \in K_\iota}$ is a free family of subsets in $(A_\iota, \varphi \mid A_\iota)$, then $(\omega_{\iota k})_{(\iota, k) \in K}$ where $K = \coprod_{\iota \in I} \{\iota\} \times K_\iota$, is a free family of subsets in (A, φ).

1.4. EXAMPLE: Let G be a discrete group which is the free product $\ast_{\iota \in I} G_\iota$ of its subgroups G_ι, and let $(L(G), \tau)$ be the von Neumann algebra generated by the left regular representation of G in $\ell^2(G)$ and τ the canonical trace state, i.e. $\tau(T) = \langle T\xi, \xi \rangle$ where $\xi(g) = \delta_{g,e}$. Let further $L(G_\iota)$ be identified with the corresponding subalgebra in $L(G)$. Then $(L(G_\iota))_{\iota \in I}$ is a free family of subalgebras in $(L(G), \tau)$.

1.5. Given non-commutative probability spaces (A_k, φ_k) $(k \in J)$ there is a state φ on the free product algebra $A = \ast_{k \in J} A_k$ (amalgamation over $\mathbb{C}1$ is assumed) such that $\varphi \mid A_k = \varphi_k$ (the A_k being canonically identified with subalgebras of A) and such that the $(A_k)_{k \in J}$ form a free family of subalgebras in (A, φ). The state φ is uniquely determined by these conditions and is called the *free product of the states* φ_k, denoted $\ast_{k \in J} \varphi_k$.

If the (A_k, φ_k) are W^*-probability spaces such that the A_k are finite W^*-algebras and the φ_k are faithful trace states, then φ is a trace state on the $*$-algebra A and the GNS-construction applied to (A, φ) yields a von Neumann algebra $\tilde{A}$ with a normal faithful trace state $\tilde{\varphi}$ which will be called the *reduced W^*-free product* of the (A_k, φ_k). Each A_k identifies with a von Neumann subalgebra of $\tilde{A}$ so that $\varphi_k = \tilde{\varphi} \mid A_k$ and the A_k form a free family of subalgebras. If the A_k's are II_1 factors then A is a II_1 factor.

If $(G_\iota)_{\iota\in I}$ is a family of discrete groups, then the reduced W^*-free product of the $(L(G_\iota), \tau_\iota)$ is $(L(G), \tau)$ where $G = \underset{\iota\in I}{*} G_\iota$.

1.6. DEFINITION: If $(f_\iota)_{\iota\in I}$ is a family of random variables in (A, φ), let $\mathbb{C}\langle\{X_\iota \mid \iota \in I\}\rangle$ be the free algebra with unit over $\mathbb{C}$ and generators X_ι $(\iota \in I)$ and let $h : \mathbb{C}\langle\{X_\iota \mid \iota \in I\}\rangle \to A$ be the homomorphism such that $h(X_\iota) = f_\iota$ $(\iota \in I)$. The *joint distribution* of the $(f_\iota)_{\iota\in I}$ is the functional $\mu : \mathbb{C}\langle\{X_\iota \mid \iota \in I\}\rangle \to \mathbb{C}$ defined by $\mu = \varphi \circ h$. The *moments* of $(f_\iota)_{\iota\in I}$ are the numbers $\mu(Y)$ where $Y = X_{\iota_1}X_{\iota_2}\dots X_{\iota_k}$ is a monomial. If $(f_\iota)_{\iota\in I}$ is a family of random variables in a $*$-probability space the distribution of $(f_\iota)_{\iota\in I} \cup (f_\iota^*)_{\iota\in I}$ will be called the *$*$-distribution of $(f_\iota)_{\iota\in I}$*.

1.7. DEFINITION: If $(f_{\iota,n})_{\iota\in I}$ are random variables in (A_n, φ_n) and μ_n is their joint distribution, then μ_n is called the *limit distribution* of these families as $n \to \infty$, if $\lim_{n\to\infty} \mu_n(a) = \mu(a)$ for every $a \in \mathbb{C}\langle\{X_\iota \mid \iota \in I\}\rangle$. If $(f_{\iota,n})_{\iota\in I}$ are families of random variables in (A_n, φ_n) and if $I = \bigcup_{s\in S} I_s$ is a partition of I, then the family of subsets $((f_{\iota,n})_{\iota\in I_s})_{s\in S}$ is called *asymptotically free as $n \to \infty$* if the distributions μ_n of $(f_{\iota,n})_{\iota\in I}$ converge to a limit distribution μ and if the family of subsets $((X_\iota)_{\iota\in I_s})_{s\in S}$ is free in $(\mathbb{C}\langle\{X_\iota \mid \iota \in I\}\rangle, \mu)$.

1.8. REMARK: Let (A, φ), (B, ψ) be two C^*-probability spaces (respectively W^*-probability spaces) such that the GNS representations associated with φ and ψ are faithful. Let further $(f_\iota)_{\iota\in I} \subset A$ and $(g_\iota)_{\iota\in I} \subset B$ be families of random variables which generate A and B as C^*-algebras (respectively as W^*-algebras). If the $*$-distributions of $(f_\iota)_{\iota\in I}$ and $(g_\iota)_{\iota\in I}$ are equal then there is an isomorphism of C^*-algebras (respectively W^*-algebras) $\gamma : A \to B$ such that $\varphi = \psi \circ \gamma$ and $\gamma(f_\iota) = g_\iota$ for $\iota \in I$.

1.9. DEFINITION: A family of random variables $(f_\iota)_{\iota\in I}$ in a $*$-probability space (A, φ) is called a *semicircular family* if it is a free family, $f_\iota = f_\iota^*$ for all $\iota \in I$ and the distribution of each f_ι is given by the semicircle law

$$\varphi(f_\iota^k) = \frac{2}{\pi} \int_{-1}^{1} t^k(1-t^2)^{1/2}\, dt.$$

A family of random variables $(g_\iota)_{\iota\in I}$ is called *circular* if the family $(x_\iota)_{\iota\in I} \cup (y_\iota)_{\iota\in I}$ where $x_\iota = 2^{-1/2}(g_\iota + g_\iota^*)$, $y_\iota = -i2^{-1/2}(g_\iota - g_\iota^*)$, is a semicircular family.

1.10. REMARK: If $(f_\iota)_{\iota\in I}$ is a semicircular system in the $*$-probability space (A, φ) and B is the subalgebra generated by the f_ι's, then $\varphi \mid B$ is completely determined and is a trace. If (A, φ) is a C^*-probability space or a W^*-probability space such that the GNS representation associated with φ is faithful, then the C^*-algebra or respectively the W^*-algebra generated by the $(f_\iota)_{\iota\in I}$ is completely determined. In the W^*-algebra case this W^*-algebra is isomorphic to a type II_1 factor of a free group on card I generators and φ is the canonical trace. This follows from 1.3(i), the fact that a free product of trace states is a trace state and 1.8.

1.11. REMARK: The analogue of the Gaussian functor constructed in [1] provides a canonical model for a semicircular family. We recall only part of the construction. Let H be a Hilbert space, $TH = \mathbb{C}1 \oplus \bigoplus_{n\geq 1} H^{\otimes n}$ the full Fock space and if $h \in H$ let $\ell(h)\xi = h \otimes \xi$ for $\xi \in TH$. If $(e_\iota)_{\iota\in I}$ is an orthonormal basis in H, let A be the C^*-algebra generated by the $s(e_\iota) = 1/2(\ell(e_\iota) + \ell(e_\iota)^*)$, $\iota \in I$ and let $\varphi(a) = \langle a1, 1\rangle$. Then φ is a faithful trace state and $(s(e_\iota))_{\iota\in I}$ is a semicircular family. The von Neumann algebra generated by A is isomorphic to the free group factor on card I generators and $\langle\cdot 1, 1\rangle$ is its normal trace state. Moreover, as an algebra of operators on TH it is in standard form.

We conclude this section with some facts from [5] concerning the connection between Gaussian random matrices and free random variables.

1.12. The natural framework for random matrices is the following. $(\Sigma, d\sigma)$ is a standard non-atomic measure space with a probability measure $d\sigma$ and $L = \bigcap_{p\geq 1} L^p(\Sigma)$ is a $*$-algebra endowed with the state $E : L \to \mathbb{C}$ given by $Ef = \int_\Sigma f(s)d\sigma(s)$. Let further $\mathfrak{M}_n$ be the complex $n \times n$ matrices endowed with the normalized trace $\tau_n : \mathfrak{M}_n \to \mathbb{C}$ (i.e. $\tau_n = 1/n\ Tr$). We shall denote by e_{ij} or $e(i,j;n)$ when n needs to be emphasized, the $n \times n$ matrix with zero entries except for the (i,j)-entry which is 1. The algebra of random $n \times n$ matrices is the algebra $M_n = \mathfrak{M}_n(L) = \mathfrak{M}_n \otimes L$ and is equipped with a trace state $\varphi_n : M_n \to \mathbb{C}$ given by $\varphi_n = \tau_n \otimes E$. $\mathfrak{M}_n$ can be naturally identified with $\mathfrak{M}_n \otimes 1 \subset M_n$, i.e. with the constant matrices in M_n. By $\Delta_n \subset \mathfrak{M}_n$ we shall denote the constant diagonal matrices.

1.13. THEOREM ([5]). *Let $Y(s,n) = \sum_{1\leq i,j\leq n} a(i,j;n,s)e(i,j;n) \in M_n$ be random matrices, $s \in \mathbb{N}$. Assume that $a(i,j;n,s) = \overline{a(j,i;n,s)}$ and that*

$\{\operatorname{Re} a(i,j;n,s) | 1 \leq i \leq j \leq n,\ s \in \mathbb{N}\} \cup \{\operatorname{Im} a(i,j;n,s) | 1 \leq i < j \leq n, s \in \mathbb{N}\}$

are independent Gaussian random variables such that

$$E(a(i,j;n,s)) = 0 \quad \text{for} \quad 1 \le i,j \le n,$$

$$E((\operatorname{Re} a(i,j;n,s))^2) = E((\operatorname{Im} a(i,j;n,s))^2) = (2n)^{-1} \quad \text{for} \quad 1 \le i < j \le n,$$
$$E((a(k,k;n,s)^2) = n^{-1} \quad \text{for} \quad 1 \le k \le n.$$

Let further $(D(j,n))_{j\in\mathbf{N}}$ *be elements in* Δ_n *such that* $\sup_{n\in\mathbf{N}} \|D(j,n)\| < \infty$ *for* $j \in \mathbf{N}$ *and* $(D(j,n))_{j\in\mathbf{N}}$ *has a limit distribution as* $n \to \infty$. *Then the family of subsets of random variables*

$$\{(\{Y(s,n)\}_{s\in\mathbf{N}},\ \{D(j,n) \mid j \in \mathbf{N}\}\}$$

is asymptotically free as $n \to \infty$. *Moreover, the limit distribution of the family* $(1/2\, Y(s,n))_{s\in\mathbf{N}}$ *as* $n \to \infty$ *is that of a semicircular family.*

1.14. Theorem ([5]). *Let* $Y(s,n) = \sum a(i,j;n,s)e(i,j;n,s) \in M_n$ *and* $Z(s,n) = \sum b(i,j;n,s)e(i,j;n) \in M_n$ *be random matrices,* $s \in \mathbf{N}$. *Assume that* $\operatorname{Im} a(i,j;n,s) = 0$, $\operatorname{Re} b(i,j;n,s) = 0$, $a(i,j;n,s) = a(j,i;n,s)$, $b(i,j;n,s) = -b(j,i;n,s)$ *and that* $\{a(i,j;n,s) \mid 1 \le i \le j \le n,\ s \in \mathbf{N}\} \cup \{ib(p,q;n,s) \mid 1 \le p < q \le n,\ s \in \mathbf{N}\}$ *are independent Gaussian random variables such that*

$$E(a(i,j;n,s)) = E(b(i,j;n,s)) = 0 \quad \text{for} \quad 1 \le i,j \le n$$

$$E((a(i,j;n,s))^2) = -E((b(p,q;n,s))^2) = n^{-1} \quad \text{for} \quad 1 \le i,j,p,q \le n$$

and $p \ne q$. *Let further* $(D(j,h))_{j\in\mathbf{N}}$ *be elements in* Δ_n *such that*

$$\sup_{n\in\mathbf{N}} \|D(j,n)\| < \infty$$

for $j \in \mathbf{N}$ *and* $(D(j,n))_{j\in\mathbf{N}}$ *has a limit distribution as* $n \to \infty$. *Then the family of sets of random variables*

$$\{(\{Y(s,n)\})_{s\in\mathbf{N}},\ (\{Z(s,n)\})_{s\in\mathbf{N}},\ \{D(j,n) \mid j \in \mathbf{N}\}\}$$

is asymptotically free as $n \to \infty$. *Moreover, the limit distribution of* $\{1/2\, Y(s,n) \mid s \in \mathbf{N}\} \cup \{1/2\, Z(s,n) \mid s \in \mathbf{N}\}$ *is that of a semicircular family.*

Note that Theorem 1.14 actually implies Theorem 1.13.

2. Circular and Semicircular Systems

This section deals with properties of circular and semicircular systems.

2.1. Proposition. *Let* (A,φ) *be a* C^**-probability space such that* φ *is a faithful trace and let* $(f_\iota)_{\iota\in I}$ *be a semicircular system in* (A,φ). *Let further* $\mathcal{F}$ *be the closed real linear span of the* $(f_\iota)_{\iota\in I}$ *in* A. *Then the following hold:*

a) $\mathcal{F}$ *is a real Hilbert space with orthonormal basis* $(f_\iota)_{\iota\in I}$.
b) *An orthonormal system in* $\mathcal{F}$ *is a semicircular system.*

Proof: Since φ is a faithful trace, the GNS representation of the C^*-subalgebra B of A generated by the $(f_\iota)_{\iota\in I}$. Hence we may apply 1.10 to $(B,\varphi \mid B)$ and to the semicircular system in 1.11. After this identification a) and b) become obvious. □

2.2. Proposition. *Let* (A,φ) *be a* C^**-probability space such that* φ *is a faithful trace and let* $(g_j)_{j\in J}$ *be a circular system in* (A,φ). *Let further* $\mathcal{G}$ *be the closed complex linear span of the* $(g_j)_{j\in J}$ *in* A. *Then the following hold:*

a) $\mathcal{G}$ *is a complex Hilbert space with orthonormal basis* $(g_j)_{j\in J}$.
b) *An orthonormal system in* $\mathcal{G}$ *is a circular system.*

Proof: Using the preceding proposition, let us show that the propositon follows if we know that $\|g\| = 1$ if g is circular in (A,φ). Let $x_j = 2^{-1/2}(g_j + g_j^*)$ and $y_j = -i2^{-1/2}(g_j - g_j^*)$ so that $\{x_j \mid j \in J\} \cup \{y_j \mid j \in J\}$ is semicircular. We will show first that $\|\sum_{j\in J} \lambda_j g_j\| = (\sum_{j\in J} |\lambda_j|^2)^{1/2}$ and $\sum_{j\in J} \lambda_j g_j$ is circular. Note first that it is sufficient to prove this for finite J. Next note that the family $(e^{i\theta_j} g_j)_{j\in J}$ is $*$-free and that if g is circular then $e^{i\theta} g$ is also circular. Indeed, if $2^{-1/2}(g+g^*) = x$, $-i2^{-1/2}(g-g^*) = y$ then $\{x,y\}$ is circular and hence $\{(\cos\theta)x - (\sin\theta)y,\ (\cos\theta)y + (\sin\theta)x\}$ is also semicircular, since it is an orthonormal system in the Hilbert space with basis $\{x,y\}$. Hence $e^{i\theta} g$ is circular. Thus it will be sufficient to prove our assertion in case $\lambda_j \ge 0$ $(j \in J)$. Indeed if $\sum \lambda_j g_j = 2^{-1/2}(a+ib)$ with $a = s^*$, $b = b^*$ then $x = \sum \lambda_j x_j$, $y = \sum \lambda_j y_j$ and $\{\mu a, \mu b\}$, where $\mu = (\sum \lambda_j^2)^{-1/2}$, is semicircular in view of the preceding proposition, so that $\mu \sum \lambda_j g_j$ is circular, which implies our assertion.

The fact we just proved clearly implies a). To prove b) note that it is sufficient to consider the case where the orthonormal system is a basis.

Let $U : \mathcal{G} \to \mathcal{G}$ be the unitary mapping $(g_j)_{j\in J}$ to the given basis. It will be sufficient to note that there is an orthogonal transformation V of the real Hilbert space H spanned by $\{x_j \mid j \in J\} \cup \{y_j \mid j \in J\}$ such that on the complexification $H + iH \supset \mathcal{G}$ the complexification of V extends U. Indeed the orthogonal transformation V of H extends to a trace-preserving automorphism of the C^*-algebra spanned by H and hence maps circular system to a circular system.

To conclude the proof let g be circular. In view of 1.10 and 1.11 we may assume we are in the context of 1.11. Thus we may assume $g = 2^{-3/2}(\ell(e_1) + \ell(e_1)^* + i\ell(e_2) + i\ell(e_2)^*)$. Let $h_1 = 2^{-1/2}(e_1 + ie_2)$, $h_1 = 2^{-1/2}(e_1 - ie_2)$, which is an orthonormal system so that $2g = \ell(h_1) + \ell(h_2)^*$. Clearly $\|g\| \le 1$. Conversely, let

$$\xi_n = n^{-1/2}(h_2 \otimes h_1 + h_2 \otimes h_1 \otimes h_2 \otimes h_1 + \cdots + h_2 \otimes h_1 \otimes \cdots \otimes h_2 \otimes h_1)$$

so that $\|\xi_n\| = 1$, and $\|2g\xi_n - 2\xi_n\| \to 0$ showing that $\|g\| \ge 1$. □

2.3. PROPOSITION. *Let (A,φ) be a C^*-probability space and let $1 \in D \subset A$ be a commutative $*$-subalgebra and let $(f(s))_{s\in S} \subset A$ be a semicircular system. Assume D and $\{f(s) \mid s \in S\}$ form a free pair of sets and suppose $p \in D$ is a self-adjoint idempotent such that $\varphi(p) \ne 0$. Then $(\varphi(p)^{-1/2}pf(s)p)_{s\in S}$ is a semicircular system in $(pAp, \varphi(p)^{-1}\varphi \mid pAp)$ and the pair of sets $\{\varphi(p)^{-1}pf(s)p \mid s \in S\}$ and pDp is free.*

PROOF: It is clearly sufficient to prove the proposition in case S is finite and D is generated as an algebra by a finite set $\{p, d_1, \ldots, d_m\}$. Passing now to the context of 1.13 let $Y(s,n)$ $(s \in S)$ be independent self-adjoint Gaussian matrices as in 1.13 and let $\{p(n), d_1(n), \ldots, d_m(n)\} \subset \Delta_n$ be such that $p(n) = p(n)^* = p(n)^2$ and the joint distribution of $\{p(n), d_1(n), \ldots, d_m(n)\}$ converges to the joint distribution of $\{p, d_1, \ldots, d_m\}$. In particular $p(n)$ is a diagonal matrix with entries 0 and 1 and if k_n is the number of 1's then $\lim_{n\to\infty} k_n n^{-1} = \varphi(p)$. We may clearly assume that the non-zero entries of $p(n)$ procede the zero entries. Then, leaving aside the zeros which border the non-zero $k_n \times k_n$ matrix, the matrices $(nk_n^{-1})^{1/2}p(n)Y(s,n)p(n)$ are random $k_n \times k_n$ matrices of the same type as $Y(s,n)$ and $p(n)d_j(n)$ correspond to diagonal matrices in Δ_{k_n}. Therefore, since $k_n \to \infty$ as $n \to \infty$, 1.13 applies to these matrices. Hence the limit distribution of $((nk_n^{-1})^{1/2}p(n)Y(s,n)p(n))_{s\in S}$ is that of a semicircular family and the sets $\{(nk_n^{-1})^{1/2}p(n)Y(s,n)p(n) \mid s \in \mathbb{N}\}$ and $\{p(n)d_j(n) \mid j = 1,\ldots,m\}$ are asymptotically free. Since the limit distribution of

$$((nk_n^{-1})^{1/2}p(n)Y(s,n)p(n))_{s\in S} \cup (p(n)d_j(n))_{1\le j\le m}$$

is the distribution of

$$(\varphi(p)^{-1/2}pf(s)p)_{s\in S} \cup (pd_j)_{1\le j\le m}$$

in $(pAp, \varphi(p)^{-1}\varphi \mid pAp)$ the assertion follows. □

We pass now to the study of the polar decomposition of a circular random variable. This requires some preparation.

2.4. LEMMA. *Let (A,φ) be a non-commutative probability space. Let $u, v \in A$ be invertible elements and let $1 \in B \subset A$ be a subalgebra such that the following conditions hold:*

a) *$(\{u,u^{-1}\}, \{v,v^{-1}\} \cup B)$ is a free pair of sets in (A,φ).*
b) *$\varphi(u) = \varphi(u^{-1}) = \varphi(v) = \varphi(v^{-1}) = 0$*
c) *$\varphi(vb) = \varphi(bv^{-1}) = \varphi(vbv^{-1}) = 0$ whenever $b \in B$ and $\varphi(b) = 0$.*

Then $(\{uv, (uv)^{-1}\}, B)$ is a free pair of sets in (A,φ).

PROOF: Let $t = uv$. In view of a) and b) we have that $\varphi(t^k) = 0$ if $k \ne 0$. Thus we must show that

$$\varphi(b_0 t^{k_1} b_1 t^{k_2} b_2, \ldots, t^{k_n} b_n) = 0$$

where $k_j \ne 0$ for $1 \le j \le n$ and $\varphi(b_p) = 0$ for $1 \le p \le n-1$. Clearly we may assume that either $\varphi(b_0) = 0$ or $b_0 = 1$ and similarly $\varphi(b_n) = 0$ or $b_n = 1$. Note that the monomial $b_0 t^{k_1} b_1 t^{k_2} b_2, \ldots, t^{k_n} b_n$ is actually a product of terms of the form u, u^{-1} alternating with terms of one of the following types $v, v^{-1}b_j, vb_jv^{-1}, vb_j, b_jv^{-1}$ (for those j such that $\varphi(b_j) = 0$). Since all these elements have expectation equal to zero, we conclude in view of a) that the expectation of the product is zero. □

2.5. COROLLARY. *Let (A,φ) be a non-commutative probability space and suppose φ is a trace. Let α be an automorphism of A such that $\varphi \circ \alpha = \varphi$ and let $1 \in B \subset A$ be a subalgebra such that $\alpha(b) = b$ for $b \in B$. Let further $u, v \in A$ be invertible elements such that the following conditions are satisfied:*

a) *$(\{u,u^{-1}\},\ B \cup \{v,v^{-1}\})$ is a free pair of sets in (A,φ).*
b) *$\varphi(u) = \varphi(u^{-1}) = 0$*
c) *$\alpha(v) = cv$ where $c \in \mathbb{C}$, $c \ne 1$.*

Then $(\{uv, (uv)^{-1}\}, B)$ is a free pair of sets in (A,φ).

PROOF: If $b \in B$, we have

$$\varphi(vb) = \varphi(\alpha(vb)) = c\varphi(vb)$$

and hence $\varphi(vb) = 0$. Similarly $\varphi(bv^{-1}) = 0$ because

$$\varphi(bv^{-1}) = \varphi(\alpha(bv^{-1})) = c^{-1}\varphi(bv^{-1}).$$

Also, $\varphi(vbv^{-1}) = \varphi(b)$ since φ is a trace. Thus, the conditions in Lemma 2.4 are satisfied and the desired conclusion follows. □

2.6. PROPOSITION. *Let (M, τ) be a W^*-probability space such that τ is a faithful trace state. Let $x \in M$ be a circular element and let $x = vb$ be its polar decomposition. Then we have:*

a) *The pair (v, b) is $*$-free in (M, τ).*
b) *v is unitary and $\tau(v^k) = 0$ if $k \in \mathbf{Z}\backslash\{0\}$*
c) *$\tau(b^k) = \frac{4}{\pi}\int_0^1 t^k(1-t^2)^{1/2}dt$ if $k \geq 0$.*

PROOF: a) It will be useful to pass to a larger W^*-probability space. Consider the W^*-probability space $(L(\mathbf{Z}), \lambda)$, where λ is the canonical trace, and let $(A, \varphi) = (L(\mathbf{Z}), \lambda) * (M, \tau)$. We shall identify M and $L(\mathbf{Z})$ with W^*-subalgebras in A. Let $u \in L(\mathbf{Z}) \subset A$ be the unitary corresponding to translation by $1 \in \mathbf{Z}$. There is also no loss of generality to assume that M is generated as a W^*-algebra by x. Then there is an unique automorphism α of (A, φ) such that $\alpha(u) = u$ and $\alpha(x) = -x$. Let $B \subset A$ be the $*$-algebra generated by $b = (x^*x)^{1/2}$. It is easily seen that $(A, \varphi), \alpha, u, v, B$ satisfy the assumptions of Corollary 2.5 (the invertibility of v follows by examining the circular element $1/2(\ell(h_1) + \ell(h_2)^*)$ considered in the proof of Proposition 2.2). It follows that the pair (uv, b) is $*$-free.

Let us show that in order to conclude the proof of a) it suffices to prove that x and ux have the same $*$-distribution. Indeed, this implies the existence of an injective weakly continuous homomorphism $\mu : M \to A$ such that $\varphi \circ \mu = \tau$ and $\mu(x) = ux$. This, in turn, would imply that $\mu(v) = uv$ and $\mu(b) = b$ and the $*$-freeness of (v, b) would follow from that of (uv, b).

To show that x and ux have the same $*$-distribution we use our results on random matrices (Theorem 1.13). Let $x_n = 2^{-3/2}(Y(1,n) + iY(2,n)) \in M_n$ and let $u_n = \sum_{1\leq j\leq n} \exp(2\pi ij/n)e(j,j;n) \in \Delta_n$. It follows from Theorem 1.13 that the $*$-distribution of (x_n, u_n) converges to the $*$-distribution of (x, u). Thus, the equality of the $*$-distribution of x and ux will follow if we show that the $*$-distributions of x_n and $u_n x_n$. The random matrix x_n is characterized by the fact that the $2n^2$ random variables which are the real and imaginary parts of its entries are Gaussian, independent, have first moments equal to 0 and second moments $(8n)^{-1}$. It is clear that $u_n x_n$ is a matrix with the same properties and hence that the $*$-distributions of x_n and $u_n x_n$ are equal.

b) As we already mentioned in the proof of a) the unitarity of v follows by examining a concrete circular element $1/2(\ell(h_1) + \ell(h_2)^*)$. The fact that $\tau(v^k) = 0$ if $k \in \mathbf{Z}\backslash\{0\}$ is a consequence of the fact that $e^{2\pi i\theta}x$ is also circular and hence $\tau(v^k) = \tau((e^{2\pi i\theta}v)^k)$.

c) Here again we use the concrete circular element $x = 1/2(\ell(h_1) + \ell(h_2)^*)$. In 3.5 of [5] we have shown that the moments of x^*x are the same as those of the square of a semicircular element, i.e.

$$\tau((x^*x)^k) = \frac{2}{\pi}\int_{-1}^{1} t^{2k}(1-t^2)^{1/2}dt.$$

This gives

$$\tau(b^{2k}) = \frac{4}{\pi}\int_0^1 t^{2k}(1-t^2)^{1/2}dt$$

and in view of the Stone–Weierstrass theorem this equality extends also to the odd moments. □

2.7. REMARK: It is not hard to derive from Proposition 2.6a) the following fact about the polar decomposition of random matrices. If $X(n) = \sum_{1\leq i,j\leq n} a(i,j;n)e(i,j;n) \in M_n$ is such that the $2n^2$ random variables $\operatorname{Re} a(i,j;n)$, $\operatorname{Im} a(i,j;n)$ are independent, Gaussian and have first moments equal zero and second moments equal n^{-1}, let $u_n b_n = X(n)$ be the polar decomposition, then the pair (u_n, b_n) is asymptotically $*$-free. Similarly, consider a real matrix, i.e. $\operatorname{Im} a(i,j;n) = 0$ and the n^2 random variables $a(i,j;n)$ are independent, Gaussian and have their first two mements equal to 0 and respectively n^{-1}. Then again the two components of the polar decomposition are asymptotically $*$-free as $n \to \infty$.

We pass to another kind of results which can be derived from the results on random matrices.

2.8. PROPOSITION. *Let (A, φ) be a C^*-probability space and let $D \subset A$ be a commutative $*$-subalgebra. Let*

$$\Omega = \{h(p,q;s) \mid 1 \leq p \leq q \leq N,\ s \in S\} \cup \{f(p,q;s) \mid 1 \leq p < q \leq N,\ s \in S\}$$

be a semicircular family such that the pair of sets (Ω, D) is free. Further, let $f(p,p;s) = 0$ and if $p > q$ let $h(p,q;s) = h(q,p;s)$, $f(p,q;s) = -f(q,p;s)$ and consider

$$k(s) = N^{-1/2} \sum_{1\leq p,q\leq N} (2^{-1/2} + \delta_{p,q}(1 - 2^{-1/2}))(h(p,q;s) + if(p,q;s)) \otimes e_{pq}$$

$$d(t) = \sum_{1\leq p\leq N} d(p,t) \otimes e_{pp}$$

in $(A \otimes \mathfrak{M}_N,\ \varphi \otimes \tau_N)$ where $\{d(p,t) \mid 1 \le p \le N,\ t \in T\} \subset D$. Then $(k(s))_{s \in S}$ is a semicircular family and the pair of sets $(\{k(s) \mid s \in S\}, \{d(t) \mid t \in T\})$ is free.

PROOF: The proof is along the same lines as the other proofs based on 1.13. The $*$-distribution of (Ω, D) is approximated by that of matrices, the matrices for Ω being Gaussian random matrices like in 1.13 and the ones for D constant diagonal matrices. Then the matrices built out of the $n \times n$ approximants, are $Nn \times Nn$ matrices to which 1.13 applies yielding the desired conclusion. □

3. Factors

This section contains some applications of circular and semicircular systems to factors.

3.1. LEMMA. *Let A be a $*$-algebra with unit and let $(w_{ij})_{1 \le i,j \le n}$ be a system of matrix units in A with $\Sigma\, w_{ii} = 1$. Let further $\Omega \subset A$ be a set which generates A as a $*$-algebra. Then the set $\bigcup_{1 \le j,k \le n} w_{1j}\Omega w_{k1}$ generates $w_{11}Aw_{11}$ as a unital $*$-algebra. The same assertion holds for $*$-algebra replaced by C^*-algebra or W^*-algebra.*

PROOF: Define $\Theta(\Sigma) = \bigcup_{1 \le j,k \le n} w_{1j}\Omega w_{k1}$ for any set $\Sigma \subset A$. It is easily seen that

$$\Theta(\Sigma^*) = (\Theta(\Sigma))^*$$

and

$$\Theta(\Sigma_1)\Theta(\Sigma_2) \supset \Theta(\Sigma_1\Sigma_2).$$

Hence if Σ is the semigroup generated by $\Omega \cup \Omega^*$ and if S is the semigroup generated by $\Theta(\Omega \cup \Omega^*)$ we have $S \supset \Theta(\Sigma)$.

Hence the $*$-algebra with unit generated in $w_{11}Aw_{11}$ by $\Theta(\Omega)$ contains $\Theta(S) + \mathbb{C}w_{11}$ which spans $\Theta(A) = w_{11}Aw_{11}$.

The assertions for C^*-algebras and W^*-algebras are immediate corollaries. □

If $\mathcal{M}$ is a II_1-factor and $\alpha \in \mathbf{R}_{>0}$ we denote by $\mathcal{M}_\alpha$ a II_1-factor isomorphic to $e(\mathcal{M} \otimes B(H))e$ where $e \in \mathcal{M} \otimes B(H)$ is a self-adjoint idempotent in the II_∞-factor with trace α. If $\alpha \le 1$ we may define $\mathcal{M}_\alpha$ to be isomorphic to $e\mathcal{M}e$ with $e \in \mathcal{M}$, $e = e^* = e^2$ and trace of e equal α.

Besides writing a free group as a free product, it will also be convenient to denote by $F(S)$ the free group with generators indexed by a set S. We shall also sometimes write $F(|S|)$ where $|S|$ is the cardinal of S.

3.2. DEFINITION: A family of unitary elements $(u_\iota)_{\iota \in I}$ in a $*$-probability space is said to have *a free $*$-distribution* if $\varphi(\pi(g)) = \delta_{g,e}$ for $g \in F(I)$, with $\pi : F(I) \to U(A)$ denoting the homomorphism such that $\pi(g_\iota) = u_\iota$ $(\iota \in I)$.

3.3. THEOREM. *Let $N \in \mathbf{N}$ and let S be a set with at least 2 elements. We have*

a) $(L(F(S) * \mathbf{Z}/N\mathbf{Z}))_{1/N} \simeq L(F(|S|N^2 - N + 1)$

b) $(L(F(S)))_{1/N} \simeq L(F(|S|N^2 - N^2 + 1))$

PROOF: a) We shall realize $L(F(S) * \mathbf{Z}/N\mathbf{Z})$ as a W^*-algebra of $A \otimes \mathfrak{M}_N$ where (A, φ) is a W^*-probability space with φ a faithful trace and containing a sufficiently large semicircular system. We shall use to this end Proposition 2.8. Let $\{f(p;s) \mid 1 \le p \le N,\ s \in S\} = \omega_1 \subset A$ be a semicircular system and let $\{g(p,q;s) \mid 1 \le p < q \le N,\ s \in S\} = \omega_2 \subset A$ be a circular system such that the pair of sets (ω_1, ω_2) is $*$-free in (A, φ). Then $L(F(S) * \mathbf{Z}/N\mathbf{Z})$ can be identified with the W^*-algebra $\mathcal{X}$ generated by

$$k(s) = \sum_{1 \le p \le N} f(p;s) \otimes e_{pp} + \sum_{1 \le p < q \le N} (g(p,q;s) \otimes e_{p,q} + g(p,q;s)^* \otimes e_{qp}) \quad (s \in S)$$

and

$$d = \sum_{1 \le j \le N} \exp(2\pi ij/N) 1 \otimes e_{jj}.$$

It is easily seen that this set of generators is equivalent to

$$\begin{array}{ll} f(p;s) \otimes e_{pp} & 1 \le p \le N,\ s \in S \\ g(p,q;s) \otimes e_{pq} & 1 \le p < q \le N,\ s \in S. \end{array}$$

If $g(p,q;s) = v(p,q;s)b(p,q;s)$ is the polar decomposition, then we get another set of generators

$$\begin{array}{ll} f(p;s) \otimes e_{pp} & 1 \le p \le N,\ s \in S \\ v(p,q;s) \otimes e_{pq} & 1 \le p < q \le N,\ s \in S \\ b(p,q;s) \otimes e_{qq} & 1 \le p < q \le N,\ s \in S. \end{array}$$

Since the distribution of $f(p;s)$ and $b(p,q;s)$ are measures without atoms (see Proposition 2.6) it follows that there are unitary elements $F(p;s)$ and $B(p,q;s)$ in A, such that they generate the same W^*-subalgebras as $f(p;s)$ and respectively $b(p,q;s)$ and $\varphi((F(p;s))^k) = \varphi((B(p,q;s)^k) = 0$ if $h \in \mathbf{Z}\backslash\{0\}$. Thus there is also the following set of generators of $\mathcal{X}$:

$$\begin{array}{ll} F(p;s)\otimes e_{pp} & 1\le p\le N,\ s\in S \\ v(p,q;s)\otimes e_{pq} & 1\le p<q\le N,\ s\in S \\ B(p,q;s)\otimes e_{qq} & 1\le p<q\le N,\ s\in S. \end{array}$$

Note that in view of Proposition 2.6 the family of unitary elements

$$\Gamma = \{F(p;s) \mid 1\le p\le N,\ s\in S\}\cup\{v(p,q;s)\mid 1\le p<q\le N,\ s\in S\}\cup$$
$$\cup\{B(p,q;s)\mid 1\le p<q\le N,\ s\in S\}$$

has a free $*$-distribution.

We have $(L(F(S)*\mathbf{Z}/N\mathbf{Z}))_{1/N} \simeq (1\otimes e_{11})\mathcal{X}(1\otimes e_{11})$. Let $w_{11} = 1\otimes e_{11}$ and $w_{1p} = v(1,p;\sigma)\otimes e_{1p}$ (here $2\le p\le N$ and $\sigma\in S$ is a fixed element) and consider the system of matrix units $w_{pq} = w_{1p}^* w_{1q}$ in $\mathcal{X}$. In view of Lemma 3.1 we have that $(1\otimes e_{11})\mathcal{X}(1\otimes e_{11}) = \mathcal{Y}\otimes e_{11}$ where $\mathcal{Y}$ is generated by

$$\begin{aligned}\gamma = {} & \{v(1,p;\sigma)v(p,q;s)v(1,q;\sigma)^{-1} \mid 2\le p<q\le N,\ s\in S\backslash\{0\}\}\cup \\ & \cup\{v(1,q;s)v(1,q;\sigma)^{-1}\mid 1<q\le N,\ s\in S\backslash\{0\}\}\cup \\ & \cup\{v(1,p;\sigma)v(p,q;\sigma)v(1,q;\sigma)^{-1}\mid 2\le p<q\le N\}\cup \\ & \cup\{v(1,q;\sigma)B(p,q;s)v(1,q;\sigma)^{-1}\mid 1\le p<q\le N,\ s\in S\}\cup \\ & \cup\{v(1,p;\sigma)F(p,s)v(1,p;\sigma)^{-1}\mid 2\le p\le N,\ s\in S\}\cup \\ & \cup\{F(1;s)\mid s\in S\}.\end{aligned}$$

It is easily seen that in $F(\Gamma)$ the set γ is a free set of generators for a subgroup. Hence since Γ has a free $*$-distribution it follows that $\mathcal{Y}\simeq L(F(\gamma))$. Counting elements, we have $|\gamma| = N^2|S| - N + 1$, which gives the desired conclusion.

b) The proof of b) is a slight modification of the proof of a). Thus $L(F(S))$ can be identified with the W^*-algebra $\mathcal{R}$ generated by

$$k(s) = \sum_{1\le p\le N} f(p;s)\otimes e_{pp} + \sum_{1\le p<q\le N}(g(p,q;s)\otimes e_{pq} + g(p,q;s)^*\otimes e_{qp})\ \ (s\in S')$$

and

$$h = \sum_{1\le j\le N} ja\otimes e_{jj}$$

where $S' = S\backslash\{\sigma_1\}$, $f(p;s)$ and $g(p,q;s)$ are as in the proof of a) and a is a unitary element with a free $*$-distribution and such that $(\{a\},\{\omega_1\cup\omega_2\})$ is a free pair.

An equivalent set of generators is

$$\begin{array}{ll} f(p;s)\otimes e_{pp} & 1\le p\le N,\ s\in S' \\ g(p,q;s)\otimes e_{pq} & 1\le p<q\le N,\ s\in S' \\ a\otimes e_{pp} & 1\le p\le N. \end{array}$$

We introduce again the polar decomposition $g(p,q;s) = v(p,q;s)b(p,q;s)$ and the unitary elements $B(p,q;s)$, $F(p,s)$. Thus we get another set of generators

$$\begin{array}{ll} F(p;s)\otimes e_{pp} & 1\le p\le N,\ s\in S' \\ v(p,q;s)\otimes e_{pq} & 1\le p<q\le N,\ s\in S' \\ B(p,q;s)\otimes e_{qq} & 1\le p<q\le N,\ s\in S' \\ a\otimes e_{pp} & 1\le p\le N. \end{array}$$

The family of unitary elements $F(p;s)$, $v(p,q;s)$, $B(p,q;s)$, a has a free $*$-distribution. Let further $\sigma_2\in S'$ and $S'' = S'\backslash\{\sigma_2\}$ and let $w_{11} = 1\otimes e_{11}$, $w_{1p} = v(1,p;\sigma_2)\otimes e_{1p}$ $(2\le p\le N)$, $w_{pq} = w_{1p}^* w_{1q}$. Then $(1\otimes e_{11})\mathcal{R}(1\otimes e_{11}) = \mathcal{P}\otimes e_{11}$ where $\mathcal{P}$ is generated by

$$\begin{aligned}\chi = {} & \{v(1,p;\sigma_2)av(1,p;\sigma_2)^{-1}\mid 2\le p\le N\}\cup \\ & \cup\{a\}\cup \\ & \cup\{v(1,p;\sigma_2)v(p,q;s)v(1,q;\sigma_2)^{-1}\mid 2\le p<q\le N,\ s\in S''\}\cup \\ & \cup\{v(1,q;s)v(1,q;\sigma_2)^{-1}\mid 1<q\le N,\ s\in S''\}\cup \\ & \cup\{v(1,p;\sigma_2)v(p,q;\sigma_2)v(1,q;\sigma_2)^{-1}\mid 2\le p<q\le N\}\cup \\ & \cup\{v(1,q;\sigma_2)B(p,q;s)v(1,q;\sigma_2)^{-1}\mid 1\le p<q\le N,\ s\in S'\}\cup \\ & \cup\{v(1,p;\sigma_2)F(p,s)v(1,p;\sigma_2)^{-1}\mid 2\le p\le N,\ s\in S'\}\cup \\ & \cup\{F(1;s)\mid s\in S'\}.\end{aligned}$$

Again, it is easy to check that these elements form a set of free generators for a subgroup of the free group generated by the $F(p;s)$, $v(p,q;s)$, $B(p,q;s)$ and a. The assertion follows by counting the elements of χ. □

We have as an immediate consequence of Theorem 3.3b) the following corollary.

3.4. Corollary. *The fundamental group of $L(F(\mathbb{N}))$ contains the multiplicative gorup of positive rational numbers.*

3.5. Corollary. *Let $G = \underset{\iota \in I}{*} \mathbb{Z}/n_\iota\mathbb{Z}$, where I is an at most countable set and $n_\iota \geq 2$ for all $\iota \in I$. Then we have*

$$L(F(\mathbb{N})) \simeq L(G * F(\mathbb{N})).$$

Proof: It follows from Theorem 3.3a) and b) that:

$$(L(F(\mathbb{N}) * \mathbb{Z}/m\mathbb{Z}))_{1/m} \simeq L(F(\mathbb{N})) \simeq (L(F(\mathbb{N})))_{1/m}.$$

This in turn implies

$$L(F(\mathbb{N}) * \mathbb{Z}/m\mathbb{Z}) \simeq L(F(\mathbb{N})).$$

Taking free products of II_1-factors we have:

$$L(F(\mathbb{N})) \simeq (L(F(\mathbb{N})))^{*I} \simeq \underset{\iota \in I}{*} L(F(\mathbb{N}) * \mathbb{Z}/n_\iota\mathbb{Z}) \simeq L(F(\mathbb{N}) * G).$$

□

REFERENCES

1. Voiculescu, D., *Symmetries of some reduced free product C^*-algebras*, in "Operator Algebras and Their Connections with Topology and Ergodic Theory", Lecture Notes in Math., Vol. II, **32**, Springer–Verlag (1985), 556–588.
2. ______, *Additon of certain non-commuting random variables*, J. Functional Analysis, Vol. 66, No. 3 (1986), 323–346.
3. ______, *Multiplication of certain non-commuting random variables*, J. Operator Theory **18** (1987), 223–235.
4. ______, *Non-commutative random variables and spectral problems in free product C^*-algebras*, Preprint, Berkeley (1988), to appear in Rocky Mountain Journal.
5. ______, *Limit laws for random matrices and free products*, Preprint, IHES (1989).
6. Wigner, E., *Characteristic vectors of bordered matrices with infinite dimensions*, Ann. of Math. **62** (1955), 548–564.
7. ______, *On the distribution of the roots of certain symmetric matrices*, Ann. of Math. **67** (1958), 325–327.

Received October 27, 1989

Department of Mathematics
University of California, Berkeley
Berkeley, CA 94720