The Bi-Free Extension of Free Probability

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Abstract. Free probability is a non-commutative probability theory adapted to variables with the highest degree of non-commutativity. The theory has connections with random matrices, combinatorics and operator algebras. Recently we realized that the theory has an extension to systems with left and right variables, based on a notion of bi-freeness. We provide a look at the development of this new direction. The paper is an expanded version of the plenary lecture at the 10th ISAAC Congress in Macau.

1. Introduction

Free probability is now in its early thirties. After such a long time I became aware that the theory has a natural extension to a theory with two kinds of variables: left and right. This is not the same as passing from a theory of modules to a theory of bi-modules, since our left and right variables will not commute in general, a non-commutation which will appear already when we shall take a look at what the Gaussian variables of the theory are.

We call the theory with left and right variables bi-free probability and the independence relation that underlies it is called bi-freeness. This new type of independence does not contradict the theorems of Muraki [**15**] and Speicher [**20**] about the possible types of independence in non-commutative probability with all the nice properties ("classical", free, Boolean and if we give up symmetry also monotonic and anti-monotonic), the reason being that we play a new game here, by replacing the usual sets of random variables by sets with two types of variables.

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The observation about the possibility of left and right variables could have been made at the very beginning of free probability. At present it becomes necessary to look back and think about the problems which would have appeared earlier had we been aware of the possibility. Several advances on this road have been made. Developments are happening faster since the lines along which free probability developed are serving often as a guide.

2. Free probability background

Free probability is a non-commutative probability framework adapted for variables with the highest degree of non-commutativity. By "highest" non-commutativity we mean the kind of non-commutativity one encounters for instance in free groups, free semigroups or in the creation and destruction operators on a full Fock space. At this heuristic level, Bosonic and Fermionic creation and destruction operators are "less non-commutative" because the commutation or anti-commutation relations they satisfy represent restrictions on the non-commutativity. These are the reasons for the adjective "free" in the name of free probability. It should also be noticed that heuristically, when non-commutativity is at the highest, a certain homogeneity appears which simplifies matters.

The distinguishing feature of free probability among non-commutative probability theories is the independence relation, called free independence or freeness, on which it is based.

Thus the notions of random variables and of expectation values are the usual ones in non-commutative probability that is quantum mechanical observables, and their expectation values or some purely algebraic version of these. So our random variables will be operators T on some complex Hilbert space $\mathcal{H}$ and there will be a unit vector $\xi \in \mathcal{H}$ so that the expectation of T is $\langle T\xi, \xi\rangle$ (we use the mathematician's scalar product which is linear in the first and conjugate linear in the second variable). The purely algebraic version is a "non-commutative probability space" which is a unital algebra $\mathcal{A}$ over $\mathbb{C}$ with a linear expectation functional $\varphi : \mathcal{A} \to \mathbb{C}$, $\varphi(1) = 1$ and the elements $a \in \mathcal{A}$ are the non-commutative random variables (in the Hilbert space setting $\mathcal{A}$ is $\mathcal{L}(\mathcal{H})$ the linear operators on $\mathcal{H}$ and $\varphi(a) = \langle a\xi, \xi\rangle$).

The distribution μ_α of a family $\alpha = (a_i)_{i\in I}$ of non-commutative random variables in a non-commutative probability space $(\mathcal{A}, \varphi)$ is the information provided by the collection of non-commutative moments $\varphi(a_{i_1} \dots a_{i_n})$ when $n \in \mathbb{N}$ and $i_1, \dots, i_n \in I$. This information can be structured in better ways. For instance in the case of just one variable a, which is a bounded hermitian operator on a Hilbert space $\mathcal{H}$ then $\varphi(a^n) = \langle a^n\xi \mid \xi\rangle$ are precisely the moments of a probability measure, which we shall also denote by μ_a and which is given by $\mu_a(\omega) = \langle E(a;\omega)\xi, \xi\rangle$ where $E(a;\omega)$ is the spectral project of a for the Borel set $\omega \subset \mathbb{R}$.

The definition of freeness, which is the notion of independence, for a family of subalgebras $(\mathcal{A}_i)_{i\in I}$ which contain the unit $1 \in \mathcal{A}_i$ $(i \in I)$ of $(\mathcal{A}, \varphi)$ is that:

$$\varphi(a_1 \dots a_n) = 0$$

whenever $\varphi(a_j) = 0$, $a_j \in \mathcal{A}_{i_j}$ $1 \le j \le n$ and $i_j \ne i_{j+1}$ if $1 \le j < n$. A family of sets $\alpha_i \subset (\mathcal{A}, \varphi)_{i\in I}$ is free if the algebras $\mathcal{A}_i$ generated by $\{1\} \cup \alpha_i$ are freely independent. When compared with the usual notion of independence (often called “classical” or “tensor product” independence) in quantum mechanics, a key difference is the non-commutation of independent variables in the free case. In the usual definition, at least in distribution, the independent variables commute. So if $a, b \in (\mathcal{A}, \varphi)$ are classically independent and $\varphi(a) = \varphi(b) = 0$ then $\varphi(abab) = \varphi(a^2)\varphi(b^2)$ which may be $\ne 0$ while $\varphi(abab) = 0$ when a, b are freely independent.

Once random variables, distributions and independence of random variables in a non-commutative probability theory have been defined, one can imitate classical probability theory and look at the limit processes which give rise to basic types of variables: Gaussian, Poisson, etc. For instance, a free central limit process will consider a sequence a_k, $k \in \mathbb{N}$ of freely independent variables in some $(\mathcal{A}, \varphi)$ which are identically distributed, i.e., $\varphi(a_k^p) = m_p$ for all $k \in \mathbb{N}$, and assuming the variables are centered $\varphi(a_k) = m_1 = 0$ one looks at the limits as $n \to +\infty$ of the distributions of $n^{-1/2}(a_1 + \cdots + a_n)$. One finds that the limit distribution if $m_2 > 0$ is a semi-circle law, that is if we normalize $m_2 = 1$ the moments are those of a probability measure on $\mathbb{R}$ with support $[-2, 2]$ and density $\frac{1}{2\pi}\sqrt{4 - t^2}$ with respect to Lebesgue measure.

Similarly, to find a free Poisson distribution one takes for each n freely independent variables $a_k^{(n)}$, $1 \le k \le n$ which have identical distributions corresponding to Bernoulli measures $\left(1 - \frac{\alpha}{n}\right)\delta_0 + \frac{\alpha}{n}\delta_1$ and then looks at the limit distribution of $a_1^{(n)} + \cdots + a_n^{(n)}$ as $n \to \infty$. The distribution one finds (in the case of $\alpha > 0$) looks like a tilted shifted semicircle with the possibility of an additional atom at 0, that is the probability measure $(1 - a)\delta_0 + \nu$ if $0 \le a \le 1$ and only ν if $a \ge 1$ where ν has support $[(1-\sqrt{a})^2, (1+\sqrt{a})^2]$ and density $(2\pi t)^{-1}\sqrt{4a - (t - (1 + a))^2}$.

This law is quite different from the classical Poisson law which is

$$\sum_{n \geq 0} e^{-a} \frac{a^n}{n!} \delta_n$$

a measure concentrated on the natural numbers.

Actually the free Gaussian and free Poisson laws are the same as well-known limit eigenvalue distribution laws in random matrix theory: the Wigner semicircle law (the limit distribution for eigenvalues of a suitably normalized hermitian Gaussian random matrix with i.i.d. entries) and the Marchenko–Pastur law. Clearly the fact that the free Gauss law and the free Poisson law appear in random matrix theory provided a strong indication of a connection between free probability and random matrices.

The explanation for the connection between random matrices and free probability I found in [**25**] is that free independence appears asymptotically among large random matrices under suitable conditions. The algebra of $N \times N$ random matrices with entries p-integrable for all $1 \leq p < \infty$ over a probability space (Ω, Σ, μ) can be endowed with an expectation functional φ_N, where $\varphi_N(Y) = N^{-1} E(\mathrm{Tr}_N Y)$. Thus random matrices become non-commutative random variables and by doing this their entries are forgotten and only non-commutative moments are remembered. The simple occurrence of asymptotic freeness is that an m-tuple $(Y_1^{(N)}, \ldots, Y_m^{(N)})$ of Hermitian Gaussian random matrices with i.i.d. suitably normalized entries is asymptotically free as $N \to \infty$. This generalizes to sets of independent random matrices with distributions invariant under unitary conjugation and even, using a combination of concentration and operator algebra techniques to a kind of generic asymptotic freeness result. The fact that free probability can be modeled by random matrices in the limit $N \to \infty$, was the source of the applications of free probability to the operator algebras of free groups (see [**26**], [**28**]), a subject we will not discuss here.

The computations of non-commutative distributions in free probability has evolved in two directions. On one hand my initial analytic approach using complex analysis and a bit of operator algebras evolved toward an analytic approach and connections with non-commutative analysis. On the other hand the streamlining of the computation of moments led to Roland Speicher's combinatorial approach to free probability. In essence free probability from the point of view of moments and commulants can be viewed as replacing the lattice of partitions of $\{1, \ldots, n\}$ which underlies classical probability by the lattice of non-crossing partitions. A partition of $\{1, \ldots, n\}$ is non-crossing if there are no $1 \leq a < b < c < d \leq n$ so that $\{a, c\}$ and $\{b, d\}$ lie in different blocks of the partition.

All this seems to be connected to the discovery of t'Hooft [**12**] about the large N-limit of gauge theory: that in the large N-limit of the gauge group, ($U(N)$ with $N \to \infty$) the contribution to the expectation values concentrates on planar diagrams. Given a partition of $\{1, \dots, n\}$ if we draw limits connecting the elements in the same block, the resulting diagram is planar precisely when the partition is non-crossing. On the other hand the large N limit of random matrix models, was early on recognized by physicists as a kind of simplified large N-limit of group theories situation. Perhaps the connection to the large N limit of gauge theories is also the answer to the question: if free probability is a successful non-commutative probability theory, why is the non-commutative probability which underlies quantum mechanics the one based on "classical" independence? The answer seems to be that free probability is related to another region of quantum theory, to the large N limit of gauge theories.

Imitation of basic classical probability for the corresponding notions of free probability has developed in many directions. The free parallel to classical probability goes quite far and after more than thirty years one should wonder about the extent of this parallelism. A partial list of items which appear in the free/classical parallel includes: limit laws, stochastic processes with independent increments, convolution operations corresponding to addition or multiplication of independent random variables, combinatorics of cumulants, continuous entropy, extreme values, exchangeability.

Some comments are here in order. This is only a rough parallel, to be taken with a grain of salt sometimes. For instance, there is a free entropy theory [**30**], resembling more classical entropy than von Neumann's quantum entropy of states. On the other hand the free entropy is a "continuous" entropy, an analogue of Shannon's differential entropy, in contrast with the fact that in classical probability the fundamental entropy notion is the discrete one.

The parallelism for infinitely divisible and stable laws (see [**2**]) is quite close, but there are a few surprises. The $\mathcal{X}^2$-law, the distribution of the square of a Gaussian variable in the free setting coincides unexpectedly with a free Poisson law (i.e., a Marchenko–Pastur law). Another interesting detail is perhaps that free and classical Cauchy laws are the same.

A quite unexpected rather recent addition to the parallel was the discovery by Koestler and Speicher ([**13**]) that there is a free analogue to de Finneti's exchangeability theorem. At first sight such a result is quite unlikely, since invariance under permutations is too weak a condition for joint distributions of non-commutative variables, the number of non-commutative moments of monomials grows exponentially with the degree compared with the polynomial growth for commuting variables. The discovery was that instead of classical permutations the appropriate symmetry is provided by Wang's C^*-algebraic universal quantum permutation groups.

Other rather unexpected items with free analogues are for instance extreme values [**1**] and optimal transportation ([**3**], [**10**]).

An important feature of free probability is that conditional probabilities have a quite natural free counterpart: a "base change" from the complex field $\mathbb{C}$ to some unital algebra $\mathcal{B}$ over $\mathbb{C}$. This works especially nice in the setting of von Neumann algebras with faithful trace states, where there are unique state-preserving conditional expectations onto von Neumann subalgebras. We should note that because of the non-commutativity, "conditional free" is a much more complex matter than in the classical commutative setting. For instance conditional independence when dealing with group examples amounts to free products of groups with amalgamation over a subgroup. Also, in the setting of von Neumann algebras of type II_1, one may have to face the complexity of a subfactor inclusion $\mathcal{B} \subset \mathcal{A}$, to describe the position of $\mathcal{B}$ in $\mathcal{A}$.

3. Bi-free independence

The framework for dealing with left and right variables is that of *a pair of faces in a non-commutative probability space* $(\mathcal{A}, \varphi)$, which is a pair $1 \in \mathcal{B} \subset \mathcal{A}$, $1 \in \mathcal{C} \subset \mathcal{A}$ of subalgebras in $\mathcal{A}$, the first one $\mathcal{B}$ being the left face and the second $\mathcal{C}$ the right face. Often such a structure arises from $((z_i)_{i\in I}, (z_j)_{j\in J})$ a two-faced set of non-commutative random variables in $(\mathcal{A}, \varphi)$, where $(z_i)_{i\in I}$ are the left and $(z_j)_{j\in J}$ the right variables, the algebras $\mathcal{B}$ and $\mathcal{C}$ being then the algebras generated by $\{1\} \cup \{(z_i)_{i\in I}\}$ and $\{1\} \cup \{(z_j)_{j\imath J}\}$ respectively. The distribution of such a system is that of the left and right variables taken together $(z_i)_{i\in I} \cup (z_j)_{j\in J}$ in $(\mathcal{A}, \varphi)$, that is expectation values of monomials in left and right variables.

To explain why left and right variables are natural in free probability we shall revisit how classical independence is defined from the tensor product of Hilbert spaces and then try to imitate this in the free setting.

Let $(T_i)_{i\in I}$ be operators on a Hilbert space $\mathcal{H}$ with the state vector $\xi \in \mathcal{H}$, $\|\xi\| = 1$ and $(S_j)_{j\in J}$ operators on $\mathcal{K}$ with state vector η. Two sets of non-commutative random variables are classically independent if they have the same joint distribution as two sets of the form $(T_i \otimes I_{\mathcal{K}})_{i\in I}$, $(I_{\mathcal{H}} \otimes S_j)_{j\in J}$ on $\mathcal{H} \otimes \mathcal{K}$ with the state vector $\xi \otimes \eta$ (to make this quite general the operators are not bounded and the tensor product is not completed).

The free analogue of the tensor product of the Hilbert spaces with state vector, for a family $(\mathcal{H}_i, \xi_i)_{i\in I}$ of such spaces is $(\mathcal{H}, \xi) = \underset{i\in I}{*}(\mathcal{H}_i, \xi_i)$ where

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \bigoplus_{i_1\neq i_2\neq i_3\neq\cdots\neq i_n} \overset{\circ}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{i_n}$$

and $\overset{\circ}{\mathcal{H}}_i = \mathcal{H}_i \ominus \mathbb{C}\xi_i$. The moment left and right make their appearance is when we want to lift operators acting on the spaces $\mathcal{H}_i$ to operators acting on $\mathcal{H}$: this can be done in two ways, as left and as right operators respectively. Indeed there are left and right factorizations, identifications via isomorphisms

$$V_i : \mathcal{H}_i \otimes \left(\mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \bigoplus_{i\neq i_1\neq\cdots\neq i_n} \overset{\circ}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{i_n}\right) \to \mathcal{H}$$

$$W_i : \left(\mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \bigoplus_{i_1\neq\cdots\neq i_n\neq i} \overset{\circ}{\mathcal{H}}_{i} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{i_n}\right) \otimes \mathcal{H}_i \to \mathcal{H},$$

where we view $\mathcal{H}_i$ as $\mathbb{C}\xi_i \oplus \overset{\circ}{\mathcal{H}}_i$ and $\xi_i \otimes$ or $\otimes \xi_i$ acting as a blank. If T is an operator on $\mathcal{H}_i$ we define $\lambda_i(T) = V_i(T \otimes I)V_i^{-1}$ and $\rho_i(T) = W_i(I \otimes T)W_i^{-1}$ the left and right operators respectively.

When defining an independence based on the free products of Hilbert spaces we have a choice between left and right. If we choose all operators to be left operators or all operators to be right operators we get the usual free independence. Bi-freeness arises when we combine left and right.

Two two-faced systems in $(\mathcal{A}, \varphi)$, $((b'_i)_{i\in I'}, (c'_j)_{j\in J'})$ *and* $((b''_i)_{i\in I''}, (c''_j)_{j\in J''})$ *are bi-free* if there are $(\mathcal{H}_1, \xi_1)$ and $(\mathcal{H}_2, \xi_2)$ Hilbert spaces with state vectors and operators $((T'_i)_{i\in I'}, (S'_j)_{j\in J'})$ on $\mathcal{H}_1$ and $((T''_i)_{i\in I'}, (S''_j)_{j\in J''})$ on $\mathcal{H}_2$ so that the distribution of $((b'_i)_{i\in I'}, (c'_j)_{j\in J'}, (b''_i)_{i\in I''}, (c''_j)_{j\in J''})$ is the same as that of

$$((\lambda_1(T'_i))_{i\in I'}, (\rho_1(S'_j))_{j\in J'}, (\lambda_2(T''_i)_{i\in I''}, \rho_2(S''_j)_{j\in J''}))$$

on $\mathcal{H}$ w.r.t. the state vector ξ where $(\mathcal{H}, \xi) = (\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$. (To achieve full generality in this the operators will not be bounded and all completions left out.)

Bi-freeness defined in this way has the right properties to serve as a non-commutative independence relation for a new type of systems of non-commutative random variables with two faces (two kinds of variables, left and right variables).

Among the consequences of this definition, if $((b'_i)_{i\in I'}, (c'_j)_{j\in J'})$ and $((b''_i)_{i\in I''}, (c''_j)_{j\in J''})$ are bi-free then $(b'_i)_{i\in I'}$ and $(b''_i)_{i\in I''}$ are free and similarly $(c'_j)_{j\in J'}$ and $(c''_j)_{j\in J''}$ are free. On the other hand $(b'_i)_{i\in I'}$ and $(c''_j)_{j\in J''}$ are classically independent and also $(c'_j)_{j\in J'}$ and $(b''_i)_{i\in I''}$ are classically independent. This bi-freeness involves some freeness (for the same kind of variables and classical independence for different kinds). However bi-freeness does not reduce just to this combination of free and classical independences.

We should also point out that on a free product of Hilbert spaces $(\mathcal{H}_i, \xi_i)$ if T is an operator on $\mathcal{H}_i$ and S an operator on $\mathcal{H}_j$ then $\lambda_i(T)$ and $\rho_j(S)$ commute when $i \neq j$ but if $i = j$ then $[\lambda_i(T), \rho_i(S)] = [T, S] \oplus O$ where $\mathcal{H}_i$ is identified with the subspace $\mathbb{C}\xi \oplus \overset{\circ}{\mathcal{H}}_i$ of the free product space.

Two-faced systems of variables where the left and right variables commute will be called bi-partite.

We have chosen to call the left and right variables "faces" of a system in reference to Janus. In roman mythology the two faces of Janus were used to look into the past and into the future and combining these two kinds of observations to deal with the transition. It would be interesting if models involving some past/future interface could be developed in the non-commutative probability setting which we discuss here.

To conclude this section we will mention a few basic examples of bi-freeness.

If $(\mathcal{X}_i, \xi_i)$ are Hilbert spaces with state-vectors and $\mathcal{L}(\mathcal{X}_i)$ denotes the algebra of linear operators on $\mathcal{X}_i$ and $\varphi_i(\cdot) = \langle \cdot \xi_i, \xi_i \rangle$ is the expectation functional on $\mathcal{L}(\mathcal{H}_i)$ let $(\mathcal{X}, \xi)$ be the free product of the $(\mathcal{X}_i, \xi_i)$ and $\varphi(\cdot) = \langle \cdot \xi, \xi \rangle$ the expectation functional on $\mathcal{L}(\mathcal{X})$. It is almost tautological then that $(\lambda_i(\mathcal{L}(\mathcal{X}_i)), \rho_i(\mathcal{L}(\mathcal{X}_i)))_{i\in I}$ are bi-free in $(\mathcal{L}(\mathcal{X}), \varphi)$.

If $\mathcal{H}$ is a complex Hilbert space and $\mathcal{T}(\mathcal{H}) = \bigoplus\limits_{n\geq 1} \mathcal{H}^{\otimes n} \oplus \mathbb{C}1$ is the full Fock space, let $l(h)$ be the left and $r(h)$ be the right creation operator: $l(h)\xi = h \otimes \xi$ and $r(h)\xi = \xi \otimes h$. If $\omega_i \subset \mathcal{H}$ are subsets, $i \in I$ which are pairwise orthogonal $i \neq j \Rightarrow \omega_i \perp \omega_j$ then in $(\mathcal{L}(\mathcal{T}(\mathcal{H})), \langle \cdot 1, 1 \rangle)$ the family of two-faced sets of non-commutative random variables $((l(\omega_i) \cup l^*(\omega_i)),\ (r(\omega_i) \cup r^*(\omega_i)))_{i\in I}$ is bi-free.

Similarly let $(\mathcal{G}_i)_{i\in I}$ be groups and $\mathcal{G} = \underset{i\in I}{*}\mathcal{G}$ their free product. If $g \in \mathcal{G}$ we shall denote by $L(g)$ the left shift by g on $l^2(\mathcal{G})$ and by $R(g)$ the right shift on $l^2(\mathcal{G})$ by g and let τ on $\mathcal{L}(l^2(\mathcal{G}))$ by the functional $\varphi(\cdot) = \langle \cdot\delta_e, \delta_e\rangle$ where δ_g where $g \in \mathcal{G}$ is the canonical basis of $l^2(\mathcal{G})$. Then the family $(L(g))_{g\in\mathcal{G}_i}$, $(R(g))_{g\in\mathcal{G}_i})_{i\in I}$ of two-faced sets in $(\mathcal{L}(l^2(\mathcal{G})), \varphi)$ is bi-free.

4. Generalities on operations on bi-free systems of variables

Like for other types of independences operations on bi-free systems of variables give rise to corresponding convolution operations on the distributions. For instance if $z' = ((z'_i)_{i\in I}, (z'_j)_{j\in J})$, $z'' = ((z''_i)_{i\in I}, (z''_j)_{j\in J})$ are bi-free two-faced systems of non-commutative random variables in $(\mathcal{A}, \varphi)$ and if $z' + z'' = ((z'_i + z''_i)_{i\in I}, (z'_j + z''_j)_{j\in J})$ then the distribution $\mu_{z'+z''}$ depends only on the distributions $\mu_{z'}, \mu_{z''}$. This yields an operation on the distributions of systems of variables with these index sets, so that

$$\mu_{z'} \boxplus \boxplus \mu_{z''} = \mu_{z'+z''}.$$

The operation $\boxplus\boxplus$ will be called additive bi-free convolution, in analogy with the additive free convolution $\boxplus$ in free probability. Clearly many kinds of such convolution operations can be defined, like in the case of free probability and it is also possible to combine different operations on left and right variables. For instance passing from z', z'' to $((z'_i + z''_i)_{i\in I}, (z'_j z''_j)_{j\in J})$ defines an additive-multiplicative bi-free convolution which we shall denote by $\boxplus\boxtimes$.

In classical probability the linearizing map for the additive convolution is provided by the logarithm of the Fourier transform, in particular for probability measures with compact support the sequence of derivatives of all orders of the logarithm of the Fourier transform at zero, is a sequence of polynomials in the moments of the measure which add when the probability measures are convolved. Roughly up to normalization these polynomials are the classical cumulants of the probability measure.

The bi-free cumulants can be described as follows. Let $z = ((z_i)_{i\in I}, (z_j)_{j\in J})$ be a two-faced system of non-commutative random variables with index see $\mathcal{I}, \mathcal{J}$. Given a map $\alpha : \{1, \dots, n\} \to \mathcal{I} \coprod \mathcal{J}$ the bi-free cumulant R_α is a polynomial in variables $X_{\alpha(k_1),\dots,\alpha(k_r)}$ where $\{k_1 < \dots < k_r\} \subset \{1, \dots, n\}$ so that the quantity $R_\alpha(\mu_z)$ obtained under the substitution

$$X_{\alpha(k_1),\dots,\alpha(k_r)} \to \varphi(z_{\alpha(k_1)} \dots z_{\alpha(k_r)})$$

has the property that $R_\alpha(\mu_{z'} \boxplus\boxplus \mu_{z''}) = R_\alpha(\mu_{z'}) + R_\alpha(\mu_{z''})$ and moreover R_α is homogeneous of degree n when $X_{\alpha(k_1),\dots,\alpha(k_r)}$ is assigned degree r

and the coefficient of $X_{\alpha(1),\dots,\alpha(n)}$ is 1. *The simplest result about cumulants is their existence and uniqueness* which we proved in [**31**]. This is a consequence of quite general considerations about the bi-free convolution being an operation which can be described by polynomials at the level of moments and which then yields an inverse limit of simply connected abelian complex Lie groups. Such Lie groups are isomorphic to their Lie algebras via the exponential maps. Roughly the bi-free cumulants are the result of using the inverse of these isomorphisms.

Such general considerations were also used in free probability in [**22**], at the very beginning of the theory, to prove existence and uniqueness of cumulants, before the effective results about cumulants which were the result of later developments. This very primitive result about existence and uniqueness was sufficient to prove an algebraic central limit theorem [**22**] and find that the semicircle law plays the role of the Gauss law in free probability. In the bi-free setting a development along similar lines has taken place and will be the subject of the next section.

5. Bi-free central limit and bi-free Gaussian distributions ([31])

From the existence and uniqueness of bi-free cumulants one immediately finds that the bi-free cumulants of degrees 1 and 2, when $z = ((z_i)_{i\in I}, (z_j)_{j\in J})$ are $R_\alpha(\mu_z) = \varphi(z_{\alpha(1)})$, $\alpha : \{1\} \to I \coprod J$ and respectively $R_\alpha(\mu_z) = \varphi(z_{\alpha(1)}z_{\alpha(2)}) - \varphi(z_{\alpha(1)})\varphi(z_{\alpha(2)}), \alpha : \{1,2\} \to I \coprod J$. With this at hand one then easily proves an *algebraic bi-free central limit theorem* ([**31**]) if $z^{(k)} = ((z_i^{(k)})_{i\in I}, (z_j^{(k)})_{k\in J})$, $k \in \mathbb{N}$, *are bi-free and their distributions coincide* $\mu_z(k) = \mu$, $k \in \mathbb{N}$ *then assuming* $\varphi(z_p) = 0$ for all $p \in I \coprod J$, *the central limit process*

$$S^{(N)} = \left(\left(N^{-1/2}\sum_{1\le k\le N} z_i^{(k)}\right)_{i\in I}, \left(N^{-1/2}\sum_{1\le k\le N} z_j^{(k)}\right)_{j\in J}\right)$$

has a limit distribution γ.

Indeed, the cumulants of $S^{(N)}$ are easily seen to converge and this is equivalent with the convergence of the moments, that is the convergence of the distributions. Moreover one easily sees that the limit distributions one finds, which are the centered bi-free Gaussian distributions, are precisely those for which $R_\alpha(\mu_z) = 0$, where $\alpha : \{1,\dots,n\} \to I \coprod J$ and $n \neq 2$. To find all these centered bi-free Gaussian distributions is equivalent to finding for each covariance matrix $(C_{pq})_{p,q\in I\coprod J}$ a distribution μ_z so that $\varphi(z_p)=0$, $\varphi(z_pz_q)=C_{pq}$ and μ_z is equal to the limit distribution of a central limit process.

Using these simple remarks we found ([**31**]) that in case I and J are finite sets, the bi-free Gaussian distributions are the distributions of $((l(h'_i)+l^*(h''_i))_{i\in I},(r(h'_j)+r^*(h''_j))_{j\in J}$ in $(\mathcal{L}(\mathcal{T}(\mathcal{H})),\langle\cdot 1\mid 1\rangle)$ where $h'_i,h''_i,h'_j,h''_j\in\mathcal{H}$. Here l,l^*,r,r^* are left and respectively right creation and destruction operators on the full Fock space $\mathcal{T}(\mathcal{H})$. The covariance matrix which determines the distribution depends only on the scalar products of the vectors h'_i,h''_i,h'_j,h''_j or $i\in I$, $j\in J$.

It should be noted that the left and right operators which realize the bi-free Gaussian distribution do not commute in general, indeed:

$$[l(h'_i)+l^*(h''_i),r(h'_j)+r^*(h''_j)]=(\langle h'_j,h''_i\rangle-\langle h'_i,h''_j\rangle)\mathcal{P}$$

where $\mathcal{P}$ is the rank one projector operator $\mathcal{P}=\langle\cdot 1\mid 1\rangle 1$.

6. The combinatorics of bi-freeness

Knowing that bi-free cumulants, which linearize the additive bi-free convolution exist, immediately leads to the question of extending to the bi-free setting what is known in free probability about computing free convolutions in free probability. This meant looking for bi-free extensions on one hand of the analytic machinery and on the other hand of the combinatorial machinery of free probability. This section will briefly deal with the extension of Speicher's non-crossing partitions approach [**21**] and I will turn later to my initial analytic approach.

A first step in the combinatorial approach to bi-freeness was the paper of M. Mastnak and A. Nica [**14**], who found a connection to the combinatorics of double-ended queues and identified some of the basic objects. This beginning was then brought to fruition in a work of I. Charlesworth, B. Nelson and P. Skoufranis [**4**] and also carried further to go beyond the field of scalars $\mathbb{C}$ to a general algebra $\mathcal{B}$ in [**5**].

Instead of non-crossing partitions one considers so-called bi-non-crossing partitions ([**4**]), that is for each $n\in\mathbb{N}$ and each map $\chi:\{1,\dots,n\}\to\{L,R\}$ the set $BNC(\chi)$ is the set of partitions π of $\{1,\dots,n\}$ so that $s_\chi^{-1}\pi$ is non-crossing, where s_χ is a permutation of $\{1,\dots,n\}$ defined as follows. If $\chi^{-1}(L)=\{i_1<\dots<i_p\}$ and $\chi^{-1}(R)=\{j_1<\dots<j_q\}$ then

$$s_\chi(k)=\begin{cases} i_k & \text{if } 1\le k\le p\\ j_{n+1}-k & \text{if } p<k\le n.\end{cases}$$

The role of the map χ in formulae connecting non-commutative moments and cumulants is to indicate in an ordered product which are the factors which are left and respectively right variables. With this change of lattices of partitions, the connection between moments and cumulants is of the same kind as in the free setting, that is based on an incidence algebra and the corresponding Möbius function. Note, that since $BNC(\chi)$ and the lattice of non-crossing partitions $NC(n)$ of $\{1, \dots, 2\}$ are isomorphic via the permutation s_χ, the Möbius functions are also related via s_χ. However, since the role of s_χ is to change the order of factors in a product and all possible χ are to be considered, the resulting cumulant formulae are a quite non-trivial generalization.

Like in the free setting, also in the bi-free generalization the independence relation corresponds to the vanishing of mixed cumulants.

To illustrate these results in one of the simplest cases: the formulae expressing moments of Gaussian variables in terms of covariances using pair partitions is the relation between moments and cumulants (in the Gaussian case the only non-zero ones are covariances). If $((z_i)_{i\in I}, (z_j)_{j\in J})$ is a bi-free Gaussian two-faced system in $(\mathcal{A}, \varphi)$ and $BNC_2(\chi)$ denotes the bi-non-crossing pair-partitions for a given $\chi : \{1, \dots, n\} \to \{L, R\}$ then:

$$\varphi(z_{\alpha(1)}, \dots, z_{\alpha(n)}) = \sum_{(\{a_k,b_k\})_{1\le k\le m} \in BNC_2(\chi)} \prod_{1\le k\le m} \varphi(z_{\alpha(a_k)} z_{\alpha(b_k)})$$

where $\chi(\alpha^{-1}(I)) \subset \{L\}$, $\chi(\alpha^{-1}(J)) \subset \{R\}$.

7. One-variable free convolutions of free probability

Before we discuss the simplest bi-free convolutions, we need to briefly recall their free probability precursors.

If $a, b \in (\mathcal{A}, \varphi)$ are freely independent non-commutative random variables, then $\mu_{a+b} = \mu_a \boxplus \mu_b$ which is the definition of the additive free convolution $\boxplus$. The transform which linearizes additive free convolution and which can be used for its computation is the R-transform ([**23**]), $R_a(z)$ defined by the formulae

$$G_z(z) = \sum_{n\ge 0} z^{-n-1} \varphi(a^n) = \varphi((z1 - a)^{-1})$$

$$G_a(K_a(z)) = z,\ R_a(z) = K_a(z) - z^{-1}$$

and which satisfies

$$R_{a+b} = R_a + R_b.$$

Here R_a is either a formal power series if we work purely algebraically or the germ of a holomorphic function at 0 in the case of a Banach algebra $\mathcal{A}$. Note that in the case of a hermitian operator a, μ is a compactly supported probability measure on $\mathbb{R}$ and G_a is its Cauchy–Stieltjes transform. The inversion used to define K_a is for z near 0. Note also that since the sum of hermitian operators is a hermitian operator, $\boxplus$ is an operation on probability measures on $\mathbb{R}$. In this case one uses the above result from ([**23**]) to get R_{a+b} and then by the same formulae used backwards one finds G_{a+b} and gets μ_{a+b} by the solution of a moment problem which boils down to finding the distributional boundary values of $-\mathrm{Im}\ G_{a+b}(x+i\epsilon)$ as $\epsilon \downarrow 0$.

In [**24**] we found also another transform which computes the multiplicative convolution. If $a, b \in (\mathcal{A}, \varphi)$ are free then $\mu_{ab} = \mu_a \boxtimes \mu_b$. Assuming $\varphi(a) \neq 0$ one considers the moment-generating series

$$\psi_a(z) = \sum_{n\geq 1} z^n \varphi(a^n) = \varphi((1-za)^{-1}) - 1$$

and defines $\chi_a(\psi_a(z)) = z$, $S_a(z) = \frac{z+1}{z}\chi_a(z)$ which then satisfies

$$S_{\mu_a \boxtimes \mu_b}(z) = S_{\mu_a}(z) S_{\mu_b}(z).$$

The map $\mu \to S_\mu$ is a free analogue of the Mellin transform. Surprisingly if μ_a, μ_b are compactly supported probability measures on $(0, \infty)$, then so is $\mu_a \boxtimes \mu_b$.

The proofs of these results in [**23**] and [**24**] were analytic, using operator theory and complex analysis. Later alternative combinatorial proofs were found. For more references about free convolution see [**29**].

8. Partial bi-free transforms and the computation of the simplest bi-free convolutions

The simplest bi-free convolutions arise from operations on two bi-free two-faced pairs (a, b) and (c, d) in some non-commutative probability space $(\mathcal{A}, \varphi)$. The three operations which we can consider combining addition and multiplication give rise to bi-additive, additive-multiplicative and bi-multiplicative convolutions

$$\begin{aligned}
\mu_{a+c,b+d} &= \mu_{a,b} \boxplus\boxplus \mu_{c,d} \\
\mu_{a+c,bd} &= \mu_{a,b} \boxplus\boxtimes \mu_{c,d} \\
\mu_{ac,bd} &= \mu_{a,b} \boxtimes\boxtimes \mu_{c,d}.
\end{aligned}$$

Looking for the simplest situations, we may restrict our attention to "two-bands moments, starting left" that is to moments of the form $\varphi(a^p b^q)$ for a two-faced pair (a, b). Note that in case the pairs (a, b), (c, d) satisfy the additional simplifying assumption $[a, b] = 0$, $[c, d] = 0$ and since we may also find realizations of the bi-freeness so that

$[a,d] = [b,c] = 0$ we will get in all three cases pairs where left and right commute (i.e., bipartite pairs):

$$[a+c, b+d] = [a+c, bd] = [ac, bd] = 0.$$

Thus the convolution operation at the level of two-bands moments actually completely describes the bi-free convolution operations in the case of bi-partite pairs.

We found in [**32**] and [**33**] three transforms which together with the one-variable transforms which we discussed in Section 7 provide the solution to computing these simplest bi-free convolutions.

If (a,b) is a two-faced pair in $(\mathcal{A}, \varphi)$, in the Banach-algebra setting, the moment-generating functions we use can be written:

$$\begin{aligned} G_{a,b}(z,w) &= \varphi((z1-a)^{-1}(w1-b)^{-1}) \\ H_{a,b}(z,w) &= \varphi((1-za)^{-1}(1-wb)^{-1}) \\ F_{a,b}(z,w) &= \varphi((z1-a)^{-1}(1-wb)^{-1}) \end{aligned}$$

which of course have also formal power-series versions. In case a, b are commuting hermitian operators and φ is given by a probability measure on $\mathbb{R}^2$, $G_{a,b}(z,w)$ is a double Cauchy–Stieltjes transform.

The reduced partial transforms are defined by the formulae

$$\begin{aligned} \tilde{R}_{a,b}(z,w) &= 1 - \frac{zw}{G_{a,b}(K_a(z), K_b(w))} \\ \tilde{S}_{a,b}(z,w) &= \frac{z+1}{z}\frac{w+1}{w}\left(1 - \frac{1+z+w}{H_{a,b}(\chi_a(z), \chi_b(w))}\right) \\ \tilde{T}_{a,b}(z,w) &= \frac{w+1}{w}\left(1 - \frac{z}{F_{a,b}(K_a(z), \chi_b(w))}\right) \end{aligned}$$

where K_a, χ_a are according to the notation used in 7. We called these transforms, "reduced" because in case $\varphi(a^p b^q) = \varphi(a^p)\varphi(b^q)$ for all $p \geq 0$, $q \geq 0$ we have $\tilde{R}_{a,b} = 0$, $\tilde{S}_{a,b} = 1$, $\tilde{T}_{a,b} = 1$.

The key properties of these transforms are that if (a,b) and (c,d) are bi-free in $(\mathcal{A}, \varphi)$ we have

$$\begin{aligned} \tilde{R}_{a+c,b+d} &= \tilde{R}_{a,b} + \tilde{R}_{c,d} \\ \tilde{S}_{ac,bd} &= \tilde{S}_{a,b}\tilde{S}_{c,d} \\ \tilde{T}_{a+c,bd} &= \tilde{T}_{a,b}\tilde{T}_{c,d}. \end{aligned}$$

To compute the bi-free convolutions at the level of two-bands moments the reduced transforms are used in conjunction with the one-variable free transforms applied to the marginals. For instance to compute $\boxplus\boxtimes$ one uses $(R_a, S_b, \tilde{T}_{a,b})$ and $(R_c, S_d, \tilde{T}_{c,d})$ to get $R_{a+c} = R_a + R_c$,

$S_{bd} = S_b S_d$ and $\tilde{T}_{a+c,bd} = \tilde{T}_{a,b}\tilde{T}_{c,d}$. Note that the computation of $\tilde{T}_{a+c,bd}$ requires the knowledge of K_{a+c} and χ_{bd} which are obtained from R_{a+c} and S_{bd}. Then from $(R_{a+c}, S_{bd}, \tilde{T}_{a+c,bd})$ one finds G_{a+c}, ψ_{bd} and then one can recover from $\tilde{T}_{a+c,bd}$ the moment generating function $F_{a+c,bd}$.

Our work [**32**], [**33**] about the $\tilde{R}$, $\tilde{S}$ and $\tilde{T}$ transforms is analytic. It takes as starting point our one-variable results in free probability about the R and S-transforms, but instead of our original proofs, the alternative proofs of Uffe Haagerup [**11**] turned out to be better suited for approaching the bi-free generalization. Soon after these results were obtained analytically, Paul Skoufranis ([**16**], [**17**]) was able to find also alternative combinatorial proofs.

9. Bi-free extreme values

In classical probability theory the max of two independent random variables has as distribution function the product of the distribution functions of the random variables. The realization that there is a free probability analogue of this basic observation was the starting point of our joint work with Gerard Ben Arous [**1**] on free extreme values. We showed in [**34**] how to extend this basic fact also to the bi-free setting, which opens the way to study basic bi-free extreme value questions. We will explain the free "dictionary" and go on to explain the bi-free "dictionary" for extreme values.

The non-commutative probability framework $(\mathcal{A}, \varphi)$ will be that of a von Neumann algebra $\mathcal{A}$ and φ a normal state. If $P, Q \in \mathcal{A}$ are hermitian projections, then $P \wedge Q$, $P \vee Q \in \mathcal{A}$ denote the projections unto $P\mathcal{H} \cap Q\mathcal{H}$ and $\overline{P\mathcal{H} + Q\mathcal{H}}$ where $\mathcal{H}$ is the Hilbert space on which $\mathcal{A}$ acts. If $X = X^*$, $Y = Y^*$ are hermitian operators then $X \vee Y$ is defined with respect to Ando's spectral order, that is

$$E(X \vee Y; (-\infty, a]) = E(X; (-\infty, a]) \wedge E(Y; (-\infty, a]),\ a \in \mathbb{R}$$

where $E(X; \omega)$ is the spectral projection of X for the Borel set $\omega \subset \mathbb{R}$. There is a similar definition of $X \wedge Y$.

If $(X_i)_{i \in I}, (Y_i)_{i \in I}$ are families of hermitian elements in $(\mathcal{A}, \varphi)$, one defines a free max-convolution for distributions of such families, so that

$$\mu_{(X_i)_{i\in I}} \boxvee \mu_{(Y_i)_{i\in I}} = \mu_{(X_i \vee Y_i)_{i\in I}}.$$

In the one-variable case the distribution corresponds to a probability measure μ on $\mathbb{R}$ with compact support. To compute the free max-convolution it is convenient to pass to the distribution function

$$F_\mu(a) = \mu((-\infty, a])$$

and to consider the corresponding operation, denoted also by $\Box\!\!\!\!\!\vee$, on distribution functions. Then one has

$$(F \,\Box\!\!\!\!\!\vee\, G)(t) = (F(t) + G(t) - 1)_+ \,.$$

This is what replaces in free probability in this case the multiplication of the distribution functions.

For the bi-free extension one considers a bi-free pair of faces of hermitian elements in $(\mathcal{A}, \varphi)$

$$z' = ((z_i')_{i \in I}, (z_j')_{j \in J})$$
$$z'' = ((z_i'')_{i \in I}, (z_j'')_{j \in J})$$

and

$$z' \vee z'' = ((z_i' \vee z_i'')_{i \in I}, (z_j'' \vee z_j'')_{j \in J}).$$

This gives then rise to a bi-free max-convolution operation on the distributions

$$\mu_{z'} \,\Box\!\!\!\!\!\vee\,\Box\!\!\!\!\!\vee\, \mu_{z''} = \mu_{z' \vee z''}.$$

The simplest case of bi-free max-convolutions to consider is that of the distributions of two-faced pairs of hermitian operators which commute (i.e., the bipartite case). The distribution of such a pair is described by a probability measure μ with compact support on $\mathbb{R}^2$. The bivariate distribution function is then

$$F_\mu(s, t) = \mu((-\infty, s] \times (-\infty, t]).$$

The marginals of a bivariate distribution function $F(s, t)$ will be denoted by $F_1(s)$ and $F_2(t)$. The operation on the bivariate distribution functions is also denoted by $\Box\!\!\!\!\!\vee\Box\!\!\!\!\!\vee$. The result of [**34**] is that if F, G are bivariate distribution functions and $H = F \,\Box\!\!\!\!\!\vee\Box\!\!\!\!\!\vee\, G$ the $H_j = (F_j + G_j - 1)_+$, $j = 1, 2$ and

$$\frac{H_1(s)H_2(t)}{H(s,t)} - 1 = \left(\frac{F_1(s)F_2(t)}{F(s,t)} - 1\right) + \left(\frac{G_1(s)G_2(t)}{G(s,t)} - 1\right)$$

if $F(s, t) > 0$, $G(s, t) > 0$, $H_1(s) > 0$, $H_2(t) > 0$ and $H(s, t) = 0$ otherwise.

This opens the way to finding the bi-freely max-stable and max-infinitely-divisible laws, which is reduced to a classical analysis problem. Note that the determination of the univariate free max-stable laws in [**1**] showed that these are generalized Pareto laws, related to "peaks over thresholds" in classical extreme values theory. One may wonder whether the bi-free max-stable and max-infinitely divisible laws will also turn out to be similarly related to classical bivariate extreme values questions.

10. Concluding remarks

The replacement of the complex field $\mathbb{C}$ by a general algebra B in bi-freeness was briefly sketched in [**31**] and then developed in detail together with the corresponding combinatorics of cumulants in [**5**]. In free probability B-valued R-transforms were initially found analytically in [**27**], the bi-free B-valued transforms have now been developed using combinatorics in [**19**].

General infinite divisibility results for the simplest cases have already been obtained [**8**], [**9**]. See also [**6**] about the operator theory side of bi-free Gaussian pairs.

In free probability random matrix realizations in the large N limit play a key role. The question about random matrix realizations for bi-free probability has not yet been clarified. On one hand realizations for certain bipartite situations, that is when left and right commute, are easy to construct but do not seem to add much to what we already know from free probability. Going beyond the bipartite case the best results at present are in [**18**].

The question whether there are de Finetti type theorems for bi-freeness was considered in [**7**]. It is not clear at present whether the Köstler–Speicher theorem [**13**] has a complete bi-free analogue.

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