RESEARCH SUMMARY

PATRICK CORN

1. Introduction

Algebraic geometry is, at heart, the study of polynomials and their solutions. Modern algebraic geometers use a powerful array of technical tools in order to understand and generalize these very natural objects. At the same time, explicit computations are often necessary to provide evidence for or against conjectures and solve theoretical problems.

My research is in computational arithmetic geometry; I am concerned with surfaces defined by the vanishing of polynomials over fields k of arithmetic interest (for example, the field $\mathbb{Q}$ of rational numbers and its finite extensions), and the problem of deciding whether these surfaces have any points whose coordinates are in k.

Consider a variety X over the field of rational numbers $\mathbb{Q}$ (what we will say applies with obvious generalizations over any number field k). Suppose we wish to determine whether or not $X(\mathbb{Q})$ is nonempty. An obvious necessary condition is the following:

$$X(\mathbb{Q}) \neq \emptyset \Rightarrow X(\mathbb{R}) \neq \emptyset \text{ and } X(\mathbb{Q}_p) \neq \emptyset \text{ for all primes } p > 0. \tag{1}$$

But we are interested in a sufficient condition as well, which leads to the question of whether the converse holds. The converse to (1) for an arbitrary $\mathbb{Q}$-variety X is known as the "Hasse principle," which suggests that it is true. Unfortunately, it is not at all true for many classes of varieties X, including the class of smooth cubic hypersurfaces in $\mathbb{P}^3$; examples of such surfaces which fail the Hasse principle have been known for over forty years, cf. [SD62] and [CG66].

In [Man71], Manin proposed a method for explaining the failure of the Hasse principle: all the existing counterexamples to the Hasse principle failed another, deeper necessary condition for having a rational point, based on the Brauer group $\mathrm{Br}(X)$, whose definition is analogous to, and shares many of the properties of, the Brauer group $\mathrm{Br}(k)$ of a field. For a smooth projective variety X over a number field k, let $X(\mathbb{A}_k) = \prod_v X(k_v)$ (this can be interpreted in general as a fact rather than a definition), and set

$$X(\mathbb{A}_k)^{\mathrm{Br}} = \{(P_v)_v \in X(\mathbb{A}_k) : \sum_v \mathrm{inv}_v \mathcal{A}(P_v) = 0 \ \forall \ \mathcal{A} \in \mathrm{Br}(X)\},$$

where $\mathrm{inv}_v \colon \mathrm{Br}\, k_v \to \mathbb{Q}/\mathbb{Z}$ comes from class field theory, and $\mathcal{A}(P_v)$ is the image of $\mathcal{A}$ under the map $\mathrm{Br}(X) \to \mathrm{Br}(k_v)$ coming from functoriality and the map $P_v \colon \mathrm{Spec}\, k_v \to X$. Class field theory implies that

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k).$$

The above inclusions lead us to a more refined way to show that there are no rational points; even if $X(\mathbb{A}_k)$ is nonempty, it might still be true that

$$X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset. \tag{2}$$

If X satisfies (2), we say that X has a Brauer-Manin obstruction to the existence of rational points. (If X does not satisfy (2), we say that the Brauer-Manin obstruction for X is empty.)

2. Del Pezzo surfaces; results

Del Pezzo surfaces are smooth, projective, geometrically connected surfaces with ample anticanonical sheaf. The self-intersection of the anticanonical sheaf is a positive integer ≤ 9, called the degree. Del Pezzo surfaces of degree $d \geq 5$ over number fields satisfy the Hasse principle, but counterexamples to the Hasse principle are known with $d = 2, 3, 4$. (Surfaces with $d = 1$ have a canonical k-point.)

We choose to study Brauer-Manin obstructions on Del Pezzo surfaces because they are relatively easy to study; for instance, the Picard group of a Del Pezzo surface is free of rank $10 - d$, with generators that can be described in terms of "exceptional curves" on the surface. Exceptional curves are curves with negative self-intersection; on each Del Pezzo surface there are finitely many. For example, the exceptional curves on a Del Pezzo surface of degree 3, which is a nonsingular cubic surface in $\mathbb{P}^3$, are simply the famous 27 lines on the cubic surface.

There is an isomorphism

$$H^1(k, \operatorname{Pic}(\overline{X})) \cong \operatorname{Br}(X)/\operatorname{Br}(k) \tag{3}$$

where $\overline{X} = X \times_k \overline{k}$, coming from a spectral sequence in cohomology. Class field theory implies that to compute $X(\mathbb{A}_k)^{\operatorname{Br}}$, we need only to describe generators for the right side of (3), which is a finite abelian group because the left side of (3) is. So one natural question to ask is: what is the structure of this group? This question is answered by Swinnerton-Dyer for degrees $d \geq 3$ in [SD93]; the only possibilities are $1, \mathbb{Z}/2, (\mathbb{Z}/2)^2, \mathbb{Z}/3$, and $(\mathbb{Z}/3)^2$.

In my thesis, I carry out this computation for the more difficult cases $d = 2$ and $d = 1$, with the help of the computer algebra system MAGMA, and obtain a complete list of the possible groups $H^1(k, \operatorname{Pic}\overline{X})$ for all Del Pezzo surfaces X. It follows from results in [Sko01] that this completes the classification of $H^1(k, \operatorname{Pic}\overline{X})$ for all rational surfaces X (surfaces which are birational to $\mathbb{P}^2$ over $\overline{k}$).

This is interesting in its own right, as a problem in the cohomology of finite groups. But unfortunately, the isomorphism (3) is very hard to describe explicitly; however one defines it, one has to use the fact that $H^3(k, \overline{k}^\times) = 0$ for number fields k. So given a nontrivial element of $H^1(k, \operatorname{Pic}\overline{X})$, it is difficult to construct an element in $\operatorname{Br}(X)$ whose class it maps to. However, there are other options.

For instance, following Swinnerton-Dyer's article [SD99], one can construct certain cyclic Azumaya algebras (elements of $\operatorname{Br}(X)$ defined analogously to central simple algebras) explicitly in the case $d = 3$, and one can actually show that all nontrivial elements of $H^1(k, \operatorname{Pic}\overline{X})$ are of this form. These cyclic algebras have the added advantage that their invariants are easy to write down–that is, given a local point P_v, the quantity $\operatorname{inv}_v \mathcal{A}(P_v)$ can be computed simply and quickly. The difficulty lies in the fact that there are infinitely many k_v-points if there is one, so one has to prove that the invariants of $\mathcal{A}$ at some suitable finite set of points give the entire list of possibilities for $\operatorname{inv}_v \mathcal{A}(P_v)$. (Often there is only one possibility, so it does not matter which point we use, and we only have to use one point.)

In [CTKS87] the authors study diagonal cubic surfaces, surfaces of the form $ax^3 + by^3 + cz^3 + dt^3 = 0 \subset \mathbb{P}^3_{\mathbb{Q}}$, and give an algorithm for computing the Brauer-Manin obstruction on them; their approach does not use cyclic algebras, and so an alternative algorithm using Swinnerton-Dyer's methods suggests itself. I have implemented this alternative algorithm using MAGMA, and used it to construct a list of diagonal cubic surfaces with positive coefficients ≤ 200 with a nonempty Brauer-Manin obstruction. I also find k-points on the surfaces with coefficients in the same range with an empty Brauer-Manin obstruction, lending credence to the conjecture that the Brauer-Manin obstruction is the only one (see below). This extends work originally done in [CTKS87].

In my thesis, I also attempt to generalize this algorithm to general cubic surfaces X, and also to Del Pezzo surfaces of degree 2 using the classification of $H^1(k, \operatorname{Pic} \overline{X})$ described above. The paper [KT04] describes an algorithm for "diagonal" degree-2 Del Pezzo surfaces; the general case treated in my thesis is more difficult, but the ideas are similar, with a notable exception where the cyclic algebra construction breaks down and it is not immediately clear how to proceed.

3. Conclusions and future research directions

The results in my thesis lead to other questions and improvements. For instance, the ultimate programming goal would be an algorithm that, when given just an arbitrary cubic surface in $\mathbb{P}^3$ in terms of the coefficients of the defining polynomial, outputs a computation of the Brauer-Manin obstruction. This is precisely what Adam Logan's MAGMA package [Log04] does for Del Pezzo surfaces of degree 4, given explicitly as intersections of two quadrics in $\mathbb{P}^4$. While my algorithms go a long way toward this goal, there is still work to be done–even simply writing down the equations of the exceptional curves on a cubic surface, given only the defining equation of the surface, is difficult. Indeed, finding the smallest normal extension L over which the exceptional curves are all defined is equivalent to finding the Galois group of a polynomial of degree 27.

Another interesting theoretical area of research I intend to explore is the important question: is the Brauer-Manin obstruction the only one? That is, if a smooth projective k-variety X has local points everywhere and empty Brauer-Manin obstruction, must it have a k-point? In general, the answer to this question is no; but the answer is yes for many classes of varieties. For diagonal cubic surfaces, it is conjectured, but not yet proven, that the Brauer-Manin obstruction is the only one. (The book [Sko01] is a good reference for the various results and conjectures in this area.)

There are also other ideas for attacking the problem of finding rational points on surfaces; writing down explicit equations for torsors is another possible approach in which I am interested. And there are other interesting classes of surfaces on which one can study these questions, including K3 surfaces.

Much less is known about rational points on higher-dimensional varieties than about rational points on curves. There are many basic problems left to be solved. Computational methods such as the ones in my thesis will be essential in forming and testing new conjectures and theorems. I look forward to solving problems such as the ones described above and studying new research topics in number theory and algebraic geometry.

References

[CG66] J. W. S. Cassels and M. J. T. Guy. On the Hasse principle for cubic surfaces. *Mathematika*, 13:111–120, 1966.

[CTKS87] Jean-Louis Colliot-Thélène, Dimitri Kanevsky, and Jean-Jacques Sansuc. Arithmétique des surfaces cubiques diagonales. In *Diophantine Approximation and Transcendence Theory: Seminar, Bonn 1985*, pages 1–108. Springer-Verlag, Berlin, 1987. Lecture notes in mathematics, 1290.

[KT04] Andrew Kresch and Yuri Tschinkel. On the arithmetic of del Pezzo surfaces of degree 2. 2004. To appear in Proc. London Math Soc.

[Log04] Adam Logan. MAGMA algorithm for computing the Brauer-Manin obstruction on a Del Pezzo surface of degree 4. Available at `http://www.liv.ac.uk/~adaml/math/index.html`, 2004.

[Man71] Yuri I. Manin. Le groupe de Brauer-Grothendieck en géométrie diophantienne. In *Actes due Congrés International des Mathématiciens (Nice, 1970), Tome 1*, pages 401–411. Gauthier-Villars, Paris, 1971.

[SD62] Peter Swinnerton-Dyer. Two special cubic surfaces. *Mathematika*, 9:54–56, 1962.

[SD93] Peter Swinnerton-Dyer. The Brauer group of cubic surfaces. *Math. Proc. Cambridge Philos. Soc.*, 113(3):449–460, 1993.

[SD99] Peter Swinnerton-Dyer. Brauer-Manin obstructions on some Del Pezzo surfaces. *Math. Proc. Cambridge Philos. Soc.*, 125(2):193–198, 1999.

[Sko01] Alexei Skorobogatov. *Torsors and rational points*. Cambridge University Press, Cambridge, 2001. Cambridge tracts in mathematics, 144.