RESEARCH STATEMENT

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1. Introduction

Algebraic geometry is, at heart, the study of polynomials and their solutions. Modern algebraic geometers use a powerful array of technical tools in order to understand and generalize these very natural objects. At the same time, explicit computations are often necessary to provide evidence for or against conjectures and solve theoretical problems.

My research is in computational arithmetic geometry; I am concerned with surfaces defined by the vanishing of polynomials over fields k of arithmetic interest (for example, the field $\mathbb{Q}$ of rational numbers and its finite extensions), and the problem of deciding whether these surfaces have any points whose coordinates are in k.

2. Background

Consider a variety X over the field of rational numbers $\mathbb{Q}$. Suppose we wish to determine whether or not $X(\mathbb{Q})$ is nonempty. First, we can use the trivial fact that a necessary condition for $X(\mathbb{Q})$ to be nonempty is that the set $X(K)$, where K is any field containing $\mathbb{Q}$, must also be nonempty. The strategy is to take K to be an extension of $\mathbb{Q}$ such that whether or not $X(K)$ is empty can be easily computed. Two key examples:

- $K = \mathbb{R}$, the field of real numbers
- $K = \mathbb{Q}_p$, the field of p-adic numbers, $p > 0$ some prime

So we have the following fact:

$$X(\mathbb{Q}) \neq \emptyset \Rightarrow X(\mathbb{R}) \neq \emptyset \text{ and } X(\mathbb{Q}_p) \neq \emptyset \text{ for all primes } p > 0. \tag{1}$$

But we are interested in a sufficient condition as well, which leads to the question of whether the converse holds. The converse to (1) for an arbitrary $\mathbb{Q}$-variety X is known as the "Hasse principle," which suggests that it is true. Unfortunately, it is not at all true for many classes of varieties X, including the class of smooth cubic hypersurfaces in $\mathbb{P}^3$; Swinnerton-Dyer ([SD62]) found a smooth cubic surface which was a counterexample to the Hasse principle in 1962, and Cassels and Guy ([CG66]) found a "diagonal" cubic surface which was a counterexample in 1966:

$$5x^3 + 9y^3 + 10z^3 + 12t^3 = 0 \tag{2}$$

(Diagonal cubic surfaces are cubic hypersurfaces in $\mathbb{P}^3$ defined by the vanishing of the polynomial $ax^3 + by^3 + cz^3 + dt^3$, where $a, b, c, d \in \mathbb{Q}^*$.)

The original proofs of the failure of the Hasse principle in these and other examples were ad-hoc and very specific. In [Man71], Manin proposed a method for unifying them: they all failed another, deeper necessary condition for having a rational point, based on the Brauer group.

3. The Brauer-Manin obstruction

For any number field k, class field theory provides an exact sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus_v \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{3}$$

where the left map comes from the natural map $\mathrm{Br}(k) \to \mathrm{Br}(k_v)$, the sum runs over all places v of k, and the right map i is defined via the invariant maps $\mathrm{inv}_v \colon \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, by $i((A_v)_v) = \sum_v \mathrm{inv}_v(A_v)$.

So $\mathrm{Br}(k)$ can be viewed as the subset of $\bigoplus \mathrm{Br}(k_v)$ consisting of elements $(A_v)_v$ whose invariants sum to 0. This is precisely what we will use to construct the Brauer-Manin obstruction.

Let X be a scheme. Then we can associate to it a group $\mathrm{Br}(X)$ which is analogous to the notion of Brauer group for a field. There are two natural ways to define this; one is to use *Azumaya algebras*, which are analogous to the central simple algebras used to define the Brauer group of a field. The other way is to use étale cohomology and define $\mathrm{Br}_{\mathrm{et}}(X)$ to be $H^2_{\mathrm{et}}(X, \mathbb{G}_m)$.

Though both definitions give canonical isomorphisms $\mathrm{Br}(\mathrm{Spec}\ k) \cong \mathrm{Br}(k)$, they do not define the same groups in general. However, for smooth projective surfaces X, there is a canonical isomorphism $\mathrm{Br}_{\mathrm{Az}}(X) \cong \mathrm{Br}_{\mathrm{et}}(X)$.

Now suppose X is a smooth projective surface defined over k such that $X(k_v) \neq \emptyset$ for all places v of k. Rational points $P_v \in X(k_v)$ correspond to maps $X \to \mathrm{Spec}\ k_v$, and functoriality gives a corresponding map

$$\begin{array}{rcl} \mathrm{ev}_{P_v} \colon \mathrm{Br}(X) & \to & \mathrm{Br}(k_v) \\ \mathcal{A} & \mapsto & \mathcal{A}(P_v) \end{array}$$

For any element $(P_v)_v \in X(\mathbb{A}_k) = \prod_v X(k_v)$, define a map $f_{P_v} \colon \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$ by

$$f_{P_v}(\mathcal{A}) = ((\oplus \mathrm{ev}_{P_v})_v) \circ i)(\mathcal{A}) = \sum_v \mathrm{inv}_v \mathcal{A}(P_v).$$

Let $X(\mathbb{A}_k)^{\mathrm{Br}}$ be the set of points $(P_v)_v \in X(\mathbb{A}_k)$ such that f_{P_v} is the zero map. Then if we consider $X(k)$ naturally as a subset of $X(\mathbb{A}_k)$, the exactness of the sequence (3) implies:

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k).$$

Failing to have local points everywhere is the same as saying that $X(\mathbb{A}_k) = \emptyset$. The above inclusions lead us to a more refined way to show that there are no rational points; even if $X(\mathbb{A}_k)$ is nonempty, it might still be true that

$$X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset. \tag{4}$$

If X satisfies (4), we say that X has a Brauer-Manin obstruction to the existence of rational points. (If X does not satisfy (4), we say that the Brauer-Manin obstruction for X is empty.)

Finally, there is one remark that we will need: Clearly f_{P_v} is zero on the image of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X)$. For number fields k, this is an injection if $X(\mathbb{A}_k) \neq \emptyset$, and in fact there is an isomorphism

$$H^1(k, \mathrm{Pic}\,\overline{X}) \cong \mathrm{Br}(X)/\mathrm{Br}(k), \tag{5}$$

where $\overline{X} := X \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,\overline{k}$.

4. Del Pezzo surfaces; my research

Del Pezzo surfaces are smooth, projective, geometrically connected surfaces with ample anticanonical sheaf. Curves on Del Pezzo surfaces with negative self-intersection are called *exceptional curves*. Here are some basic facts about Del Pezzo surfaces:

Proposition 4.1. *(a) For a Del Pezzo surface X over a field k, let d be the self-intersection of the canonical class. Then $1 \leq d \leq 9$, and either*

- $\overline{X} \cong \mathbb{P}^2$ *blown up at $9-d$ points, or*
- $d = 8$ *and* $\overline{X} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

The number d is called the degree *of X.*

(b) $\operatorname{Pic}(\overline{X}) \cong \mathbb{Z}^{10-d}$*, generated by the classes of the (finitely many) exceptional curves on X.*

(c) Del Pezzo surfaces of degree ≥ 5 over number fields satisfy the Hasse principle; Del Pezzo surfaces of degree 1, 5, or 7 have a canonical k-point.

For proofs, see [Man74] and [CT03].

The explicit description of the geometric Picard group of X given in the proposition allows us to compute $H^1(k, \operatorname{Pic} \overline{X})$ explicitly. In general, by inflation-restriction, $H^1(k, \operatorname{Pic} \overline{X}) \cong H^1(\operatorname{Gal}(L/k), \operatorname{Pic} X_L)$, where L is a field over which all the exceptional curves on $\overline{X}$ are individually defined. So we can reduce to the problem of computing the cohomology of a finite subgroup of the automorphism group of the configuration of the exceptional curves preserving incidences, with coefficients in $\mathbb{Z}^{10-d}$ and an explicitly specified action. For instance, if X is a nonsingular cubic hypersurface in $\mathbb{P}^3$, which is precisely the same thing as a Del Pezzo surface of degree 3, then the exceptional curves on X turn out to be the famous 27 lines on X, and the automorphism group is a well-known group $W(E_6)$ of order 51840.

In the cases $d = 4$ and $d = 3$, results of Swinnerton-Dyer in [SD93] allow us to carry out the computation of $H^1(k, \operatorname{Pic} \overline{X})$ in complete generality, and in fact some geometry filters in: there is a correspondence between nontrivial elements of the cohomology group and certain subsets of the exceptional curves which must be fixed by the action of Galois.

In my thesis, I carry out this computation for the more difficult cases $d = 2$ and $d = 1$, with the help of the computer algebra system MAGMA, and obtain a complete list of the possible groups $H^1(k, \operatorname{Pic} \overline{X})$ for all Del Pezzo surfaces X. It follows from results in [Sko01] that this completes the classification of $H^1(k, \operatorname{Pic} \overline{X})$ for all rational surfaces X (surfaces which are birational to $\mathbb{P}^2$ over $\overline{k}$).

This is interesting in its own right, as a problem in the cohomology of finite groups. But unfortunately, the isomorphism (5) is very hard to describe explicitly; however one defines it, one has to use the fact that $H^3(k, \overline{k}^\times) = 0$ for number fields k. So given a nontrivial element of $H^1(k, \operatorname{Pic} \overline{X})$, it is difficult to construct the Azumaya algebra to which it corresponds. However, there are other options.

For instance, following Swinnerton-Dyer's article [SD99], one can construct certain cyclic Azumaya algebras explicitly in the case $d = 3$, and one can actually show that all nontrivial elements of $H^1(k, \operatorname{Pic} \overline{X})$ are of this form. These cyclic algebras have the added advantage that their invariants are easy to write down–that is, given a local point P_v, the quantity $\operatorname{inv}_v \mathcal{A}(P_v)$ can be computed simply and quickly. The difficulty lies in the fact that there are infinitely many k_v-points if there is one, so one has to prove that the invariants of $\mathcal{A}$ at some suitable finite set of points give the entire list of possibilities for $\mathrm{inv}_v \mathcal{A}(P_v)$. (Often there is only one possibility, so it does not matter which point we use, and we only have to use one point.)

In [CTKS87] the authors study diagonal cubic surfaces, and give an algorithm for computing the Brauer-Manin obstruction on them; their approach does not use cyclic algebras, and so an alternative algorithm using Swinnerton-Dyer's methods suggests itself. I have implemented this alternative algorithm using MAGMA, and used it to construct a list of diagonal cubic surfaces with positive coefficients ≤ 200 with a nonempty Brauer-Manin obstruction. I also find k-points on the surfaces with coefficients in the same range with an empty Brauer-Manin obstruction, lending credence to the conjecture that the Brauer-Manin obstruction is the only one (see below). This extends work originally done in [CTKS87].

In my thesis, I also attempt to generalize this algorithm to general cubic surfaces X, and also to Del Pezzo surfaces of degree 2. The paper [KT04] describes an algorithm for "diagonal" degree-2 Del Pezzo surfaces; the general case treated in my thesis is more difficult, but the ideas are similar, with a notable exception where the cyclic algebra construction breaks down and it is not immediately clear how to proceed.

5. Conclusions; future research directions

The results in my thesis lead to other questions and improvements. For instance, the ultimate programming goal would be an algorithm that, when given just an arbitrary cubic surface in $\mathbb{P}^3$ in terms of the coefficients of the defining polynomial, outputs a computation of the Brauer-Manin obstruction. This is precisely what Adam Logan's MAGMA package [Log04] does for Del Pezzo surfaces of degree 4, given explicitly as intersections of two quadrics in $\mathbb{P}^4$. While my algorithms go a long way toward this goal, there is still work to be done–even simply writing down the equations of the exceptional curves on a cubic surface, given only the defining equation of the surface, is difficult. Indeed, finding the smallest normal extension L over which the exceptional curves are all defined is equivalent to finding the Galois group of a polynomial of degree 27.

Del Pezzo surfaces have mainly been chosen because they are fairly easy to describe and to work with; I am also interested in rational points on other special surfaces, such as K3 surfaces.

Another interesting theoretical area of research I intend to explore is the important question: is the Brauer-Manin obstruction the only one? That is, if a smooth projective k-variety X has local points everywhere and empty Brauer-Manin obstruction, must it have a k-point? In general, the answer to this question is no; but the answer is yes for many classes of varieties. For diagonal cubic surfaces, it is conjectured, but not yet proven, that the Brauer-Manin obstruction is the only one. (The book [Sko01] is a good reference for the various results and conjectures in this area.)

Much less is known about rational points on higher-dimensional varieties than about rational points on curves. There are many basic problems left to be solved. Computational methods such as the ones in my thesis will be essential in forming and testing new conjectures and theorems. I look forward to solving problems such as the ones described above and studying new research topics in number theory and algebraic geometry.

References

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