The Brauer-Manin obstruction for diagonal cubic surfaces: an outline of the method

Let V be a diagonal cubic surface over $\mathbb{Q}$. Let $K = \mathbb{Q}(\theta)$, θ a primitive third root of unity. We will look at $X = V_K$ and compute the Brauer- Manin obstruction to its rational points.

Any obstruction comes from the group $\mathrm{Br}(X)/\mathrm{Br}(K)$, which is examined in [1] by Swinnerton-Dyer and in [2] by Colliot-Thélene. In the case of a diagonal cubic surface, this group is either trivial (in which case there is no obstruction and, for diagonal surfaces, a rational point), order-nine (in which case there are obvious rational points), or order-three. Henceforth we assume that $\#(\mathrm{Br}(X)/\mathrm{Br}(K)) = 3$.

The paper [1] gives a method of constructing, for any cubic surface, a generator of $\mathrm{Br}(X)/\mathrm{Br}(K)$ of the form (χ, f), where χ is a character $\mathrm{Gal}(\overline{K}/K) \to \mathbb{Z}/3\mathbb{Z}$, and f is a nontrivial element of $k(X)^*/k(X)^{*3}$. We will follow this method in the case when X is of the form $ax^3 + by^3 + cz^3 + dt^3 = 0 \subseteq \mathbb{P}^3$, $a, b, c, d \in \mathbb{Q}^*$.

First, we massage the coefficients a, b, c, d as in [2]. We divide out by cubes of integers and permute the a, b, c, d so that ad/bc is a cube in $\mathbb{Q}_3$ (we can assume that some cross-ratio is a cube in $\mathbb{Q}_3$ because of a result in [2]–if the surface has bad reduction at a prime p and none of the cross-ratios is a cube in $\mathbb{Q}_p$, then there cannot be a Brauer-Manin obstruction).

§1: The fields L and M; the lines and the generators of Pic(X_L)

We let $L = K(\gamma)$, where γ is a cube root of ad/bc, and let χ be the projection $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(L/K)$ composed with our choice of an isomorphism $\mathrm{Gal}(L/K) \to \mathbb{Z}/3\mathbb{Z}$, which is as follows: send σ to 1, where $\sigma(\gamma) = \theta\gamma$.

The first step in our computation is to compute the equations of lines defined over a cubic extension $M = L(\sqrt[3]{\frac{c}{a}})$. Let α and β be cube roots of c/a and d/b in M, respectively, and define the following:

$$\begin{aligned}
e_1 &= Z_X(x + \alpha z, y + \theta\beta t)\\
e_2 &= Z_X(x + \theta\alpha z, y + \theta^2\beta t)\\
e_3 &= Z_X(x + \theta^2\alpha z, y + \beta t)\\
f_{12} &= Z_X(x + \alpha z, y + \theta^2\beta t)\\
f_{13} &= Z_X(x + \theta^2\alpha z, y + \theta\beta t)\\
f_{23} &= Z_X(x + \theta\alpha z, y + \beta t)\\
g_1 &= Z_X(x + \theta^2\alpha z, y + \theta^2\beta t)\\
g_2 &= Z_X(x + \alpha z, y + \beta t)\\
g_3 &= Z_X(x + \theta\alpha z, y + \theta\beta t)
\end{aligned}$$

where $Z_X(p_1, p_2)$ denotes the intersection in $\mathbb{P}^3$ of the hypersurfaces defined by p_1 and p_2 with X.

Note that the notation in the above definitions corresponds to the notation in Hartshorne for the 27 lines. Also note that the above nine lines form a nine which is defined over K; indeed, these nine lines are just the intersection of X with the cubic hypersurface $ax^3 + cz^3 = 0$. Now the triple e_1, e_2, e_3 is $Z_X(xt\theta\gamma - yz, ax^3 + cz^3)$, which σ sends to $Z_X(xt\theta^2\gamma - yz, ax^3 + cz^3)$, which is just $f_{12} + f_{13} + f_{23}$.

Define μ in Pic X_M by $\mu = l_1 + l_2 + l_{12}$, where the l's are the classes of the e's and f's in Pic, and let τ be the element of $\mathrm{Gal}(M/L)$ that sends e_1 to e_2, e_2 to e_3, and e_3 to e_1; then a computation shows that μ is fixed by τ, so $\mu \in \mathrm{Pic}\ X_L$. The generators of Pic $\overline{X}$ given in Hartshorne are just $\mu, l_1, l_2, \ldots, l_6$, where l_4, l_5, and l_6 are the classes in Pic of three other skew lines defined over a different cubic extension of L.

The strategy will be to find a positive divisor $\mathfrak{a}$ defined over L whose class in Pic is μ. Now, just as above, it's easy to compute that $\sigma(f_{12} + f_{13} + f_{23}) = g_1 + g_2 + g_3$, which after mapping everything to Pic yields

$$\begin{aligned}
\sigma(3\mu - 2(l_1 + l_2 + l_3)) &= 6\mu - 2(l_1 + l_2 + l_3) - 3(l_4 + l_5 + l_6) \\
3\sigma\mu - (6\mu - 4(l_1 + l_2 + l_3)) &= 6\mu - 2(l_1 + l_2 + l_3) - 3(l_4 + l_5 + l_6) \\
\sigma\mu &= 4\mu - 2(l_1 + l_2 + l_3) - (l_4 + l_5 + l_6)
\end{aligned}$$

and similarly starting with the equation $\sigma^2(f_{12} + f_{13} + f_{23}) = e_1 + e_2 + e_3$ yields $\sigma^2\mu = 4\mu - (l_1 + l_2 + l_3) - 2(l_4 + l_5 + l_6)$. So then we see that

$$\mu + \sigma\mu + \sigma^2\mu = 9\mu - 3(l_1 + \ldots + l_6) = 3h, \tag{1}$$

where h is the class of a hyperplane section $\mathfrak{h}$.

§2: Construction of $\mathfrak{a}$

If we let

$$\phi = \frac{y + \beta t}{x + \alpha z} \in k(X_M),$$

then $(\phi) = e_3 + f_{23} - e_1 - f_{12}$. Now

$$\phi \cdot \tau\phi \cdot \tau^2\phi = -a/b;$$

and in fact we can find an element $m \in M$ such that $N_{M/L}(m) = b/a$ (let $m = -1/\phi(P)$, where $P \in X(M)$ is not on any of the lines above).

Solving for m is made easier in MAGMA by another lemma in [2]–if we have checked that all the above things are true for our surface, then in fact $-b/a$ is a norm for $\mathbb{Q}(\beta)/\mathbb{Q}$ and for $\mathbb{Q}(\sqrt[3]{\frac{ab}{cd}})/\mathbb{Q}$. (This follows from the Hasse Norm Theorem and a brief computation.) So we can solve the norm equation over whichever of these extensions suits us. In practice, we choose $m \in \mathbb{Q}(\beta)$.

We let $f_1 = 1 - m\phi + (m\tau m)\phi\tau\phi$, and $\mathfrak{a} = (f_1) + e_1 + e_2 + f_{12}$. It is easy to check that $\mathfrak{a}$ is fixed by τ, so that it is defined over L, and from the definition it is clear that it is a positive divisor whose class in Pic is μ.

§3: Construction of f and computation of the obstruction

Equation (1) implies that if we choose a hyperplane section $\mathfrak{h}$ defined over K, and $\mathfrak{a}$ is chosen as above, then the divisor

$$(\mathfrak{a} - \mathfrak{h}) + \sigma(\mathfrak{a} - \mathfrak{h}) + \sigma^2(\mathfrak{a} - \mathfrak{h})$$

is principal. So let f be a function in $k(X)$ with this divisor. A lemma cited in [1] says that (χ, f), which is an element of Br $k(X)$, comes from an element of $\mathrm{Br}(X)$, and that it does not come from an element of Br K, because $\mathfrak{a}-\mathfrak{h}$ is not principal. (Its class in Pic is nonzero.)

For an explicit computation of f, note that we can lift σ to an element of $\mathrm{Gal}(M/K)$ to see how it acts on the components of $\mathfrak{a}$. So lift to an element $s \in \mathrm{Gal}(M/K)$ such that $s(\alpha)=\theta\alpha$ and $s(\beta)=\theta^2\beta$.

Then we compute:

$$\begin{aligned}
(f) &= \mathfrak{a}+\sigma(\mathfrak{a})+\sigma^2(\mathfrak{a})-3\mathfrak{h}\\
&= (f_1)+s(f_1)+s^2(f_1)+(e_1+e_2+f_{12})+s(e_1+e_2+f_{12})+s^2(e_1+e_2+f_{12})-3\mathfrak{h}\\
&= (f_1\cdot s(f_1)\cdot s^2(f_1))+(e_1+e_2+f_{12})+(f_{23}+f_{13}+g_3)+(g_1+g_2+e_3)-3\mathfrak{h}\\
&= (f_1\cdot s(f_1)\cdot s^2(f_1))+\left(\frac{ax^3+cz^3}{G^3}\right)\\
&= \left(f_1\cdot s(f_1)\cdot s^2(f_1)\cdot\frac{ax^3+cz^3}{G^3}\right),
\end{aligned}$$

where G is the linear homogeneous polynomial we choose to define $\mathfrak{h}$. (We will see shortly that it does not matter which G we choose.) So we may as well take

$$f=f_1\cdot s(f_1)\cdot s^2(f_1)\cdot\frac{ax^3+cz^3}{G^3}.$$

We can actually explicitly write down f as the quotient of two cubic polynomials, as follows: $f=r\cdot F(x,y,z,t)/G^3$, where r is a constant and $F(x,y,z,t)=$

$$\begin{aligned}
&(3\mathrm{Tr}(m))x^2y+(3\mathrm{Tr}(\beta m))x^2t-\left(\frac{3b}{a}\mathrm{Tr}(1/m)\right)xy^2+\left(\frac{3b}{a}\mathrm{Tr}(\beta/m)\right)xyt\quad-\\
&\left(\frac{3b}{a}\mathrm{Tr}(\beta^2/m)\right)xt^2+\left(\frac{b}{a}(6+\mathrm{Tr}(m/\tau m))\right)y^3+\left(\frac{b}{a}(1-\theta)\mathrm{Tr}(m\beta/\tau m)\right)y^2t\quad-\\
&\left(\frac{b}{a}(2\theta+1)\mathrm{Tr}(m\beta^2/\tau m)\right)yt^2+\left(\frac{3c}{a}(1-\theta)\right)z^3+\left(\frac{d}{a}(6+\theta^2\mathrm{Tr}(m/\tau m))\right)t^3.
\end{aligned}$$

Here the trace Tr is computed with respect to the extension $\mathbb{Q}(\beta)/\mathbb{Q}$, in which m can be taken to lie by a result cited above.

To compute the Brauer-Manin obstruction for the algebra (χ,f), we compute

$$\sum_v [f(P_v),\gamma]_v,$$

where $(\ ,\ )_v$ is the local norm residue symbol, $(a,b)_v=\theta^{[a,b]_v}$, P_v is a v-adic point, and the sum runs over $v|3abcd$ (these are the places where the surface and possibly the algebra have bad reduction–in the treatment in [2], there may be other places where the algebra has bad reduction, but we don't have to worry about that with this method). Without doing any computation, we can already assure that the summands when $v|3$ are zero, since ad/bc is a cube in $\mathbb{Q}_3$. We can also replace f by F, since the norm residue symbol is zero on cubes.

It doesn't matter which v-adic point we take, by the result in [2] alluded to above–we check before we begin the process that at the places $v|3abcd$ above, the surface is K_v-rational, or equivalently one of the cross-ratios is a cube in K_v. If there is a place where this does not happen, the Brauer-Manin obstruction cannot happen; and if this happens at all places of bad reduction, then another lemma in [2] assures us that all points in $X(K_v)$ evaluate to the same thing in $\mathrm{Br}(K_v)$ (under the map $P_v \mapsto (\chi, f)(P_v)$).

In practice, we take P_v to be a point $(A_1 : A_2 : A_3 : A_4)$ with two of the A_i's equal to zero. We are guaranteed such a point if v lies above a prime congruent to 2 (mod 3), and if v lies above a prime congruent to 1 (mod 3), our massage of the coefficients yields exactly two out of four of the coefficients not divisible by that prime (if three or four are, we massage; if one is, none of the cross-ratios can be a cube in K_v, so this excludes this case; if none are, then the surface has good reduction at v). The ratio of these two coefficients must be a cube mod p if we are to have a rational point in K_v (equivalently, a rational point in $\mathbb{Q}_p$), so this guarantees a point of the above form.

Formulas for explicit computation of the symbol $[a, b]_v$ are given in [2]. It is not hard to compute; there are two cases, depending on whether v lies over a prime congruent to 1 or 2 mod 3.

Given a, b, c, d, it takes MAGMA a few seconds to: pre-check all the usual conditions on the surface we need (local points everywhere, no obvious rational points, etc.), permute the coefficients so that ad/bc is a cube in $\mathbb{Q}_3$, check whether the surface is K_v-rational at every v of bad reduction, solve the norm equation for m, and compute f, $\mathbb{Q}_v$-points for each bad v, and the sum of the invariants.

Hopefully we can generalize this computation to non-diagonal cubic surfaces where the lines are still reasonably easy to describe. Some of the lemmas in [2] will have to be generalized as well; this will be the hard part.

References

[1] Peter Swinnerton-Dyer, "Brauer-Manin obstructions on some Del Pezzo surfaces," Math. Proc. Camb. Phil. Soc. (1999), 125, pp. 193-198.

[2] J.-L. Colliot-Thélène, D. Kanevsky, J.-J. Sansuc, "Arithmetique des Surfaces Cubiques Diagonales," in *Diophantine Approximation and Transcendence Theory*, G. Wüstholz (ed.), 1987.