(ROUGH) THESIS OUTLINE

1. Statement of the theorem on diagonal cubic surfaces

I'll omit the preliminaries; some of them can be found in my research statement. This is going to be very informal and perhaps a bit confusing, since I wrote it up in the last two days. I hope this gives some flavor of the things I have been and will be working on.

The first thing is already written up separately as a paper, which I gave you the link for. It's about degree-6 Del Pezzo surfaces. It basically stands alone.

Here is the first result:

Theorem 1.1. *Let X be a diagonal cubic surface in $\mathbb{P}^3_{\mathbb{Q}}$, given by the equation*

$$ax^3 + by^3 + cz^3 + dt^3 = 0, \quad 1 \leq a, b, c, d \leq 200.$$

Suppose $X(\mathbb{Q}_p) \neq \emptyset$ for all primes p, but $X(\mathbb{Q}) = \emptyset$.

Then X has a nontrivial Brauer-Manin obstruction.

Proof: We search using MAGMA for such surfaces X, and we find that X is isomorphic to exactly one of the following N surfaces: (the list will probably actually be in an appendix–for coefficients up to 100, $N = 281$, and MAGMA is busily computing N right now for coefficients up to 200. Hopefully we can get the coefficients up higher than 200 as well.)

Then we compute the Brauer-Manin obstruction explicitly for each of these surfaces and find that it is nontrivial. This is the difficult part of the program (though the CPU time it takes is negligible compared to finding points on all the other surfaces). Details of this computation will be given in the next two sections. ■

2. General cohomology computations

Computing the Brauer-Manin obstruction explicitly for diagonal cubic surfaces with coefficients up to 100 has been done in [CTKS87] (though I found a few minor typos in their table). Their method is different from the one I use to obtain the above result; here is a brief description of my method and some results related to it.

The method is an application of some ideas of Swinnerton-Dyer in [SD93] and [SD99]. The first main idea uses the well-known isomorphism (valid for any smooth, projective surface X over a number field k):

$$H^1(k, \operatorname{Pic} \overline{X}) \cong \operatorname{Br}(X)/\operatorname{Br}(k). \tag{1}$$

(The standard map $\operatorname{Br}(k) \to \operatorname{Br}(X)$ is an injection if $X(k_v) \neq \emptyset$ for all places v; algebras in the image of this map are sometimes called "constant.")

If X is a Del Pezzo surface, $\operatorname{Pic} \overline{X}$ is free of rank $10 - d$, where d is the degree of X. So one natural question is: what are the possibilities for the finite abelian group $H^1(k, \operatorname{Pic} \overline{X})$?

This question is answered in [SD93] for Del Pezzo surfaces of degree $d = 4, 3$, and the following result generalizes this:

Theorem 2.1. *If X is a Del Pezzo surface of degree d over a number field k, then $H^1(k, \operatorname{Pic} \overline{X})$ is isomorphic to one of the following groups:*

1. *$d \geq 4$: $1, \mathbb{Z}/2, (\mathbb{Z}/2)^2$*
2. *$d \geq 3$: $\mathbb{Z}/3, (\mathbb{Z}/3)^2$*
3. *$d \geq 2$: $(\mathbb{Z}/2)^r$ ($3 \leq r \leq 7$), $\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^s$ ($s = 0, 1, 2$), $(\mathbb{Z}/4)^2$*
4. *$d \geq 1$: $(\mathbb{Z}/2)^8$, $\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^s$ ($s = 3, 4$), $(\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/2)^t$ ($t = 1, 2$), $\mathbb{Z}/5$, $(\mathbb{Z}/5)^2$, $\mathbb{Z}/6$, $\mathbb{Z}/6 \oplus \mathbb{Z}/2$, $\mathbb{Z}/6 \oplus \mathbb{Z}/3$, $(\mathbb{Z}/6)^2$*

Proof (sketch): Suppose L is the smallest Galois extension over which the exceptional curves on $\overline{X}$ are defined. Then $\operatorname{Gal}(L/k)$ must preserve exceptional curves, and hence embeds into the group $W(E_r)$ of automorphisms of the exceptional curves preserving incidences ($r = 9-d$). By inflation-restriction, the cohomology group we want is isomorphic to $H^1(H, \mathbb{Z}^{r+1})$, where H is a subgroup of $W(E_r)$, with an explicitly given action.

This cohomology group is computable in theory, but for $r \geq 6$, $W(E_r)$ becomes much too large for naive computations to succeed. Special cases of this theorem have been worked out by [Ura96], in the case where H is cyclic, and by [KT04], in the case of "diagonal" Del Pezzo surfaces of degree 2; in the latter case, H is a subgroup of a certain subgroup of $W(E_7)$ of order 128. But apparently no one has worked out the general case until now.

Suppose that we are looking for elements of order m in $H^1(H, \operatorname{Pic} \overline{X})$. Following [SD93], let $\overline{P} = \operatorname{Pic} \overline{X}$, $P = (\operatorname{Pic} \overline{X})^H$, and m be a positive integer. The short exact sequence

$$0 \to \overline{P} \to \overline{P} \to \overline{P}/m\overline{P} \to 0,$$

where the first map is multiplication by m, gives rise to a long exact sequence from which we obtain the isomorphism

$$H^1(H, \overline{P})[m] \cong (\overline{P}/m\overline{P})^H/(P/mP).$$

So the strategy is to search for elements $\overline{\alpha} \in \overline{P}/m\overline{P}$, such that there is a subgroup H of $W(E_r)$ which fixes α mod $m\overline{P}$, but such that H does not fix any element of α's equivalence class outright. This is a computation that MAGMA handles without undue difficulties. So we get a handle on how big the 2-part, 3-part, 5-part, and 7-part of the H^1 can be (no other primes occur because they don't divide $|W(E_r)|$), and also see that the H^1 is a p-group unless it contains an element of order 6. (We check the cases $m = 8, 9, 10, 15$ to see that there are no elements of these orders; similar computations are done in the case $d = 4, 3$ by Swinnerton-Dyer.) The rest of the computation is relatively tedious. [If you want more details, you can ask me specifically, but I'm trying not to get too bogged down in details.] ■

This result also gives rise to a conjecture which is analogous to a result of Swinnerton-Dyer; we state the results together:

Theorem 2.2. *(a) ([SD99]) If X is a Del Pezzo surface of degree 3 and $H^1(k, \operatorname{Pic} \overline{X})[2] \neq 0$, then X satisfies the Hasse principle.*

(b) (Conjecture) If X is a Del Pezzo surface of degree 2 and $H^1(k, \operatorname{Pic} \overline{X})[3] \neq 0$, then X satisfies the Hasse principle.

Proof (rough sketch): In both cases, suppose X has local points everywhere. The assumption in part (a) implies that there is a set of six skew lines defined over a quadratic extension of k. Blowing them down over the quadratic extension yields a Severi-Brauer variety S with local points everywhere, but the Hasse principle holds for S, so S has an L-point. Then X has an L-point, but general theory of cubic forms implies that if a cubic polynomial has a solution over L, it has a solution over k.

Similar things come out of the computer analysis for Del Pezzos of degree 2 (one can blow down lots of lines over a small extension). I have not yet put everything together for a proof, but I think it should be true, and the proof should not be too hard. I think I may have asserted that this was true in the email I sent you, but I can't find a proof yet; I may have it written down somewhere. For now, I'll call it a conjecture. ■

3. Back to diagonal cubic surfaces

Now that we have a good idea what the left side of (1) can be, it seems natural to use this to generate the elements of the Brauer group that we use to compute the Brauer-Manin obstruction. Unfortunately, the isomorphism is not easy to describe explicitly in general. In fact, it relies on the fact that $H^3(k, \overline{k}^\times) = 0$, so an explicit description of (1) would presumably use an explicit splitting of a 3-cocycle.

However, in some cases this isomorphism can be described. The main idea is that if there is a cyclic extension L/k such that the restriction map $H^1(k, \overline{P}) \to H^1(L, \overline{P})$ is trivial, then we can use inflation-restriction and the cohomology of cyclic groups to avoid the difficulties resulting from working with 3-cocycles. In this case, we should look at cyclic cubic algebras, since quadratic ones don't give rise to Brauer-Manin obstructions, by the above theorem. If we construct cyclic algebras in $\operatorname{Br}(k(X))$ which actually lie in the image of the standard injection $\operatorname{Br}(X) \hookrightarrow \operatorname{Br}(k(X))$, the isomorphism (1) gives rise to the following explicit criterion:

Proposition 3.1. *Suppose $L = k(\gamma)$ is a cubic cyclic extension of k, with $\gamma^3 \in k$. Let $\zeta \in k$ be a third root of unity. The cyclic algebra $A(\gamma^3, f, \zeta) = k\langle i, j\rangle$ ($i^3 = \gamma^3$, $j^3 = f$, and $ij = \zeta ji$) comes from the image of* $\operatorname{Br}(X)$ *if and only if*

$$(f) = N_{L/k} D$$

for some divisor $D \in \operatorname{Div} X_L$. It is constant (i.e. in $\operatorname{Br}(k)$*) if and only if D is principal.*

Proof: A diagram chase so standard that it is not included in any reference I have (though the result is cited in a few papers). So I'll include a proof myself in my thesis. ■

The nice thing about these algebras is that their invariants can be easily written down: if $P_v \in X(k_v)$, then $\operatorname{inv}_v A(\gamma^3, f, \zeta)(P_v) = [\gamma^3, f(P_v)]_v$, the local norm residue symbol from class field theory. (The closed brackets indicate that it is written additively, with a choice of a cube root of unity, namely ζ.)

I've implemented an algorithm for computing the Brauer-Manin obstruction completely explicitly for diagonal cubic surfaces; I'll refer you to another quick summary I wrote a while ago:

`http://math.berkeley.edu/ corn/paperdiagonal.pdf`

It explains the details of the construction in this case.

4. General cubic surfaces and Del Pezzo surfaces of degree 2

The assumption that there is a nontrivial Brauer-Manin obstruction on a Del Pezzo surface X leads to some restrictions on the equations defining X. In particular, we have:

Theorem 4.1. *(a) If X is a cubic surface with a nontrivial Brauer-Manin obstruction over a field k containing the third roots of unity, then the equation for X can be written in one of the following two forms:*

$$\begin{aligned} N_{L/K}(f) &= cN_{M/K}(g) \\ f_1 f_2 f_3 &= cN_{M/K}(g) \end{aligned}$$

where $c \in K$, L and M are cyclic cubic extensions of K, K/k is obtained by at most three quadratic extensions, and f, g, and the f_i are linear polynomials with coefficients in L, M, and K respectively.

(b) If X is a Del Pezzo surface of degree 2 with $H^1(k, \operatorname{Pic} \overline{X})[2] \neq 0$, then there are two cases. In the first case, the equation of X can be written in the form

$$w^2 = bB_2^2 - B_1 \cdot B_1^\sigma,$$

where σ is the nontrivial Galois group element of a quadratic extension L/k, B_1 and B_2 are quadratic polynomials in three variables defined over L and k respectively, and $b \in k$. The equation is to be understood as defining a surface in a weighted projective space $\mathbb{P}(1,1,1,2)$, where w is the variable of weight 2.

In the second case, the group of automorphisms of the 56 exceptional curves is a subgroup of a copy of $S_8 \subset W(E_7)$, but it is difficult to say much more about X, or even to give an example of an X which falls into this case.

Proof: Part (a) is nearly immediate from [SD99]. Part (b) follows from work of Dolgachev on symmetric determinantal equations and some Galois cohomology (I will write this up in more detail in my thesis). ■

The nice part about these descriptions of the surfaces is that the equations of the cyclic algebras described in Proposition (3.1) can be written down easily; in part (a), the rational function given in Proposition (3.1) can be written down in terms of conjugate polynomials of f and g, and in part (b), the rational function we need is simply B_2/x^2, where x is one of the three variables in which B_2 is written. This can be used to create a general algorithm for computing the Brauer-Manin obstruction on these surfaces. (Alas, the cyclic algebra idea does not work at all for the "second case" of part (b).)

I am currently working on implementing this algorithm in MAGMA. Some important details of the implementation for diagonal cubic surfaces came from lemmas with fairly ad-hoc, specific proofs in [CTKS87] which I must try to generalize; this has been the hard part so far.

References

[CTKS87] Jean-Louis Colliot-Thélène, Dimitri Kanevsky, and Jean-Jacques Sansuc. Arithmétique des surfaces cubiques diagonales. In *Diophantine Approximation and Transcendence Theory: Seminar, Bonn 1985*, pages 1–108. Springer-Verlag, Berlin, 1987. Lecture notes in mathematics, 1290.

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