On the components of $X_0(p^n)$

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Abstract: We show he ordinary locus of $X_0(p^n)(\mathbf{C}_p)$ is normally the set of $\mathbf{C}_p$-valued points on $2n$ affinoids which correspond to components of the stable model of $X_0(p^n)$. We then show the points on Edixhoven's "horizontal" components of $X_0(p^2)$ correspond to elliptic curves which are p-isogenous to curves which Buzzard calls "too supersingular."

Fix a prime p. Suppose $n \geq 1$. We first present a viewpoint of the ordinary locus of $X_0(p^n)$, slightly different from that taken in Katz-Mazur and Edixhoven ([K-M], [E]) which allows one to see the stable structure of the ordinary locus. Next, we give a moduli-theoretic interpretation of Edixhoven's (semi)stable model of $X_0(p^2)$ [E].

Edixhoven found the p-adic stable model of $X_0(p^2)$ by blowing up the Katz-Mazur-Edixhoven model ([K-M], [E]) at the supersingular points of the reduction. We interpret points on the components of this model in terms of the canonical subgroups of elliptic curves described by Katz in [K-pPMF]. In particular, (E, C) corresponds to a point "on" one of Edixhoven's "horizontal" components of $X_0(p^2)$ if and only if E/pC is "too supersingular" in the sense of Buzzard [B] i.e., has no canonical subgroup.

One can easily add tame level structure which we plan to do in a subsequent article.

These results facilitate a determination of which eigenforms on $X_0(Np^2)$ give rise to a representations potentially crystalline at p. In a future article, using the results in this paper and the description of the Fontaine monodromy operator given in [C-I], we will show that the representation attached to an eigenform on $X_0(Np^2)$, $p > 3$, which isn't old and doesn't come by twisting from a form on $X_0(Np)$ is potentially crystalline at p.

0. Semi-stable coverings.

Our approach to the semi-stable reduction of a curve is encapsulated by the following fact:

Semi-stable coverings of a curve correspond to semi-stable models of the curve. which is proven in [C-SM, Prop. 2.1] and which we now explain.

First, a wide open is a rigid space conformal to $C\backslash D$ where C is a smooth complete curve and D is a finite disjoint union of affinoid disks in C, which contains at least one in each connected component. A wide open disk is the complement of one affinoid disk in $\mathbf{P}^1$ (it is conformal to $B(0,1)$) and a wide open annulus is conformal to the complement of two disjoint such disks (it is conformal to $A(r,1)$, where $r \in |\mathbf{C}_p|$, $0 < r < 1$). One can also characterize these spaces as smooth one dimensional rigid spaces W which contains an affinoid X such that $W\backslash X$ is a finite union of annuli and $H^0(W)$ is isomorphic to $H^2(W, W-X)$ and is finite dimensional.

By the ends of W, we mean $\varinjlim_{X}$ Conn. Comp.$(W\backslash X)$ where X runs over the affinoid subdomains of W. We call an affinoid subdomain X of W **underlying** if the map from Conn. Comp.$(W\backslash X)$ to the set of ends of W is bijective. . A **semi-stable covering** of a curve C is a finite admissible covering $\mathcal{D}$ of C by connected wide opens such that

(i) if $U \neq V \in \mathcal{D}$, $U \cap V$ is a finite collection of disjoint wide open annuli,

(ii) if $T, U, V \in \mathcal{D}$ are pairwise distinct, $T \cap U \cap V = \emptyset$.

(iii) for $U \in \mathcal{D}$, if

$$Z_U := U\backslash\Big(\bigcup_{\substack{V\in\mathcal{D}\\ V\neq U}} V\Big),$$

Z_U is a non-empty affinoid whose reduction is irreducible and has at worst regular singular points. We call Z_U an **underlying affinoid** of $\mathcal{D}$.

A final object in the category of semistable coverings exists and corresponds to the stable model of C if and only if the stable model of C exists and its reduction has at least two components.

1. The vertical components of $\mathbf{X_0(p^n)}$.

For an elliptic curve over $\mathbf{C}_p$, with ordinary reduction, let $K(E)$ denote the kernel of reduction and $K_n(E)$ the cyclic subgroup of order p^n in $K(E)$. (more generally, if E is arbitrary, the canonical cyclic subgroup of order p^n, when it exists (see [B, Def. 3.4])).

Let $X_0(p^n)$ be the complete smooth rigid curve over $\mathbf{Q}_p$ associated to the course moduli problem that associates to a scheme S over $\mathbf{Q}_p$ the set of pairs (E, C) where C is a subgroup of E of order p^n. There is an (Atkin-Lehner) involuton α of $\coprod_{n\geq 0} X_0(p^n)$ which takes the point in $X_0(p^n)$ corresponding to the pair (E, C) to the point corresponding to $(E/C, E[p^n]/C)$. There are also two maps

$$\pi_f, \pi_\nu \colon \coprod_{n\geq 1} X_0(p^n) \to \coprod_{n\geq 0} X_0(p^n)$$

which take the point in $X_0(p^n)$ corresponding to the pair (E, C) to the point corresponding to (E, pC) and to $(E/p^{n-1}C, C/p^{n-1}C)$, respectively. We have,

$$\alpha \circ \pi_\nu = \pi_f \circ \alpha. \tag{1}$$

There are several affinoids to consider in $X_0(p^n)$. First, there the are the affinoids $\mathbf{X}_{a\,b}$, $a + b = n$, implicit in the work of Katz-Mazur-Edixhoven whose $\mathbf{C}_p$-valued points correspond to pairs (E, C) where E is a generalized elliptic curve over (the ring of integers in) $\mathbf{C}_p$ with ordinary (this includes multiplicative) reduction and $C \cap K(E) = K_a(E)$ (The reduction of C modulo p is what is called "(a, b)-cyclic" [K-M, 13.4.1]) ([K-M], [E]).

More precisely:

Let $\mathcal{X}_0(p)$ denote the model of $X_0(p)$ found by Deligne and Rapoport [D-R, Th. 1.16]. The formal completion of $\mathcal{X}_0(p)$ along the smooth locus of its reduction is an affinoid subdomain of $X_0(p)$ with two connected components. We take $\mathbf{X}_{1\,0}$ to be the component containing the cusp ∞ and $\mathbf{X}_{0\,1}$ the cusp 0. When $p \geq 5$, we could also obtain the more general $\mathbf{X}_{a\,b}$ from the "(a, b)-component" of Edixhoven's model, in this way. (See also [K-M, §13.4].)

Now let $\pi_{a\,b} = \pi_f^b \circ \pi_\nu^a$. Then, if $a + b = n$

$$\mathbf{X}_{a\,b} = \pi_{a-1,b}^{-1}\mathbf{X}_{1\,0} \cap \pi_{a\,b-1}^{-1}\mathbf{X}_{0\,1}$$

where we take $\pi_{-1\,n}^{-1}\mathbf{X}_{1\,0} = \pi_{n\,-1}^{-1}\mathbf{X}_{0\,1} = X_0(p^n)$. The rigid space $\mathbf{X}_{a\,b}$ is an affinoid subdomain of $X_0(p^n)$ becaause a finite cover of an affinoid is an affinoid and the intersection of two affinoid subdomains of o curve is an affinoid subdomain (see Corollary A7).

When $a \geq b$ (which we'll assume until pointed out otherwise), the points on $\mathbf{X}_{a\,b}$ also correspond to pairs $(E, \mathcal{P})$ where E is an elliptic curve with ordinary reduction and $\mathcal{P}$ is a pairing on $K_a := K_a(E)$ onto μ_{p^b}. Indeed, define a pairing $\mathcal{P}$ on K_a by setting

$$\mathcal{P}(A, B) = (R, S)_{E,n}$$

where $S \in C$, $p^b S = B$ and $R \in K_n$, $p^b R = A$. Here $(\ ,\ )_{E,i}$ denotes the Weil pairing into μ_{p^i} on $E[p^i]$. On the other hand,

$$C_{\mathcal{P}} = \{S \in E[p^n] \colon p^b S \in K_a, (R, S)_{E,n} = \mathcal{P}(p^b R, p^b S), \forall R \in K_n\},$$

is a cyclic group of order p^n such that $C_{\mathcal{P}} \cap K_n = K_a$.

We see that $\mathbf{Z}_p^*$ acts on $\mathbf{X}_{a\,b}$ via $\tau_r \colon (E, \mathcal{P}) \mapsto (E, \mathcal{P}^r)$, for $r \in \mathbf{Z}_p^*$.
$(C_{\mathcal{P}^r} = \{T \colon \exists S \in C_{\mathcal{P}}, T \equiv rS \bmod K, p^b T = p^b S\}.)$

One can use this point of view to show that, over $\mathbf{C}_p$, $\mathbf{X}_{a\,b}$ has as many irreducible components as classes mod squares in $(\mathbf{Z}/p^b\mathbf{Z})^*$ (which is 2 if $b > 0$ and $p > 2$, or $p = 2$ and $b = 2$; and 4 if $p = 2$ and $b > 2$: and 1 otherwise). Let B_b

be the set of orbits of $((\mathbf{Z}/p^b\mathbf{Z})^*)^2$ in the set of primitive p^b-th roots of unity. The set $C_{a\,b}$ of isomorphism classes of pairs $(A, \mathcal{P})$, where k is a cyclic group of order p^a and $\mathcal{P}$ is a pairing on A onto μ_{p^b}, is in 1-1 correspondence with B_b. Indeed, if $(A, \mathcal{P})$ represents an element of $C_{a\,b}$ and P generates A, the class of $\mathcal{P}(P, P)$ in B_b depends only on $(A, \mathcal{P})$. Also, $\mathbf{Z}_p^*$ acts on $C_{a\,b}$; $(A, \mathcal{P}) \mapsto (A, \mathcal{P}^r)$.

Now fix b and set $B := B_b$. All the pairs $(E, \mathcal{P})$ such that the class of $\mathcal{P}(P, P)$ for a generator P of $K_a(E)$ equals a given element of B lie on the same component of $\mathbf{X}_{a\,b}$ because the map $\mathbf{X}_{a\,b} \to B$ which takes the point corresponding to $(E, \mathcal{P})$ to the class of $\mathcal{P}$ is rigid analytic. To see this, let $\mathbf{X}^1_{a\,b}$ be the affinoid above $\mathbf{X}_{a\,b}$ in $X_1(p^n)$. Let Y be the inverse image in $\mathbf{X}^1_{a\,b}$ of a component of $\mathbf{X}_{a\,b}$. Suppose (E, Q) corresponds to a point y of Y, so p^bQ generates $K_a(E) = (Q) \cap K(E)$. Suppose $R \in K(E)$, $p^bR = Q$. Then, $y \mapsto (R, Q)_{E,n}$ is a rigid analytic map from Y to the primitive p^b-th roots of unity and its image modulo the action of $((\mathbf{Z}/p^b\mathbf{Z})^*)^2$ depends only on the image of y in $\mathbf{X}_{a\,b}$.

For $\beta \in C_{a\,b}$, call the corresponding component in $\mathbf{X}_{a\,b}$, $\mathbf{X}^\beta_{a\,b}$. It is easy to see these components are non-empty. In fact, these components are irreducible. To see this, first, the reduction of $\mathbf{X}^\beta_{b\,b}$ is isomorphic (non-canonically) to the quotient of the Igusa curve $Ig(p^b)$ by the group of automorphisms

$$H_b = \{\alpha_t : \alpha_t(E, Q) = (E, tQ), t \in (Z/p^bZ)^*, t^2 = 1\}.$$

(In fact, $\alpha_{-1} = \alpha_1$.) Indeed, let $\zeta \in \mu_{p^b}$ represent β, then there exists a $P \in K_b$ such that $\mathcal{P}(P, P) = \zeta$. Let $Q \in E[p^{2b}]$ such that $(P, Q)_{W,2b} = \zeta$. Then the point $(\bar{E}, \bar{Q})$ of $Ig(p^b)$ is well defined up to the action of H_b. If ζ is replaced with ζ^{y^2} we can replace P with yP and Q with yQ.

Now, the *coup de grace* is that $\mathbf{X}^\beta_{b\,b}$ is naturally isomorphic to $\mathbf{X}^\beta_{a\,b}$. Simply, if $(E, \mathcal{P})$ represents a point on $\mathbf{X}^\beta_{b\,b}$, $(E, \mathcal{P}')$ represents a point on $\mathbf{X}^\beta_{a\,b}$, where

$$\mathcal{P}'(R, S) = \mathcal{P}(p^{a-b}R, p^{a-b}S)$$

for $R, S \in K_a$. Denote this map from $\mathbf{X}^\beta_{b\,b}$ to $\mathbf{X}^\beta_{a\,b}$ by $r^\beta_{a\,b}$.

These components can all be defined over K_p,

where $K_p =: \begin{cases} \mathbf{Q}_p(\sqrt{(-1)^{(p-1)/2}p} & \text{if } p \text{ is odd} \\ \mathbf{Q}_2(\sqrt{-1}, \sqrt{2}) & \text{if } p = 2 \end{cases}$. To see this, define a function $f_{a\,b}$ on $\mathbf{X}^1_{a\,b}$ with values in μ_{p^b}, by: if (E, P) represents a point x of $\mathbf{X}^1_{a\,b}$,

$$f_{a\,b}(x) = (Q, P)_{E,n}$$

where $Q \in K_n(E)$ and $p^b Q = p^b P$. Clearly, $f_{a\,b}$ is a rigid analytic function, and using it, we see $\mathbf{X}^1_{a\,b}$ has at least $\phi(p^b)$ irreducible components. Now, if $\beta \in B$, let χ_β be the function on the primitive p^b-th roots of one such that $\chi_\beta(x) = \begin{cases} 1 & \text{if } x \in \beta \\ 0 & \text{otherwise} \end{cases}$. Then $g_\beta =: \chi_\beta \circ f_{a\,b}$ is a rigid analytic function on $\mathbf{X}^1_{a\,b}$ and if $\tau \in$ Gal$(\bar{\mathbf{Q}}_p/K_p)$.

$$\begin{aligned} g_\beta\big((E,P)^\tau\big) &= \chi_\beta \circ f_{a\,b}(E^\tau, P^\tau) \\ &= \chi_\beta\big((Q^\tau, P^\tau)_{E,n}\big) \\ &= \chi_\beta\big((Q,P)^\tau_{E,n}\big) \\ &= \chi_\beta(Q,P)_{E,n} \\ &= g_\beta(E,P) \end{aligned}$$

It follows that g_β is defined over K_p. But also, if x and y, in $\mathbf{X}^1_{a\,b}$, have the same image in $X_0(p^n)$, $g_\beta(x) = g_\beta(y)$, Thus g_β is the pullback of a function on $\mathbf{X}_{a\,b}$ defined over K_p. We thus see the components $\mathbf{X}^\beta_{a\,b}$ are defined over K_p.

Also, if $\gamma \in C_{a\,b}$, $\mathbf{X}^\beta_{a\,b} \cong \mathbf{X}^\gamma_{a\,b}$ because, if $r \in \mathbf{Z}^*_p$ such that $r\beta = \gamma$, τ_r resticts to an isomorphism from $\mathbf{X}^\beta_{a\,b}$ onto $\mathbf{X}^\gamma_{a\,b}$.

There is also a natural map $t^\beta_{a\,b}\colon \mathbf{X}^\beta_{a\,b} \to \mathbf{X}^\beta_{b\,b}$,

$$(E, \mathcal{P}) \mapsto (E/K_{a-b}, \mathcal{P}|_{K_a/K_{a-b}}).$$

One can show $t_{b+1,b}$ is the restriction of π_ν to $\mathbf{X}_{b+1,b}$). Also, $t^\beta_{b+1,b} \circ r^\beta_{b+1,b}$ is a lift of Frobenius. Indeed, suppose

$$t^\beta_{a\,b} \circ r^\beta_{a\,b}(E, \mathcal{P}) = (E/K_{a-b}, \mathcal{P}')$$

and $\phi_c: E \to E/K_c$ is the natural isogeny. Suppose $P, Q \in K_b$, $p^{a-b}R = P$ and $p^{a-b}S = Q$. Then,

$$\mathcal{P}'(\phi_{a-b}(R), \phi_{a-b}(S)) = \mathcal{P}(P, Q).$$

Proposition 1.1. *Suppose $(E, \mathcal{P})$ represents a point x in $\mathbf{X}_{b\,b}$. Then $\alpha(x)$ is represented by $(E', \mathcal{P}')$, where $E' = E/C_{\mathcal{P}}$ and if $\rho: E \to E'$ is the natural isogeny*

$$\mathcal{P}'(\rho(u), \rho(v)) = \mathcal{P}(p^b u, -p^b v)$$

if $u, v \in K_{2b}(E)$.

Proof. First, $\alpha(x)$ is represented by $(E/C_{\mathcal{P}}, E[p^{2b}]/C_{\mathcal{P}})$. Suppose $A, B \in K_b(E)$. Let $P, Q \in K_{2b}(E)$ such that $p^b P = A$ and $p^b Q = B$. Suppose $R \in E[p^{2b}]$, $p^b R \equiv P \bmod C_{\mathcal{P}}$ and $S \in K_{3b}(E)$, $p^b S = Q$. Write $p^b R = P + T$. Then, $p^b T = -p^b P = -A$.

$$\begin{aligned}\mathcal{P}'(\rho(Q), \rho(P)) &= (\rho(S), \rho(R))_{E',2b} \\ &= (S, R)^{p^b}_{E,3b} \\ &= (p^b S, p^b R)_{E,2b} \\ &= (Q, T)_{E,2b} = \mathcal{P}(B, -A). \quad \blacksquare\end{aligned}$$

Corollary 1.1.1. *If p is odd, $\alpha\big(\mathbf{X}^{\beta}_{b\,b}\big) = \mathbf{X}^{(\frac{-1}{p})\beta}_{b\,b}$.*

If $a < b$, we set $\mathbf{X}^{\beta}_{a\,b} = \alpha(\mathbf{X}^{\beta}_{b\,a})$.

Suppose C a curve over $\mathbf{C}_p$ which has a stable model $\mathcal{C}$. We call the rigid spaces red^{-1}x, where x is a point on $\overline{\mathcal{C}}$, **residue classes** of C. if the reduction of $\mathcal{C}$ has at least two components, we call the affinoids red^{-1}Z in C where Z is the smooth locus of a component of $\overline{\mathcal{C}}$ **underlying affinoids** of C. We define residue classes of affinoids similarly.

Let $\mathcal{X}_0(p^n)$ denote the stable model of $X_0(p^n)$, when it exists.

Theorem 1.2. *If $p \geq 23$, $c+d=n$ and $\beta \in B_{\min\{c,d\}}$ then $\mathcal{X} := \mathcal{X}_0(p^n)$ exists, its reduction has at least two irreducible components and $\mathbf{X}^{\beta}_{c\,d}$ is an underlying affinoid of $X := X_0(p^n)$.*

Proof. Since $p \geq 23$, $X_0(p)$ and hence $X_0(p^n)$ has genus at least 2, and since the reduction of $\mathcal{X}_0(p)$ has two components it follows from Proposition 1.5 of [C-SM] that the reduction of $\mathcal{X}$ has at least two irreducible components. We remark that the theorem is true for $n=1$. In fact, it follows from [D-R] that $\mathbf{X}_{1\,0}$ and $\mathbf{X}_{0\,1}$ are the underlying affinoids of $X_0(p)$, in this case

Now since $\mathbf{X}^{\beta}_{c\,d}$ is an affinoid with good reduction, it is either contained in a residue class of X or is the complement of finitely many residue classes in an underlying affinoid of X. Suppose $\mathbf{X}^{\beta}_{c\,d}$ is contained in a residue class U. Then since $\mathbf{X}^{\beta}_{c\,d}$ maps finitely onto $\mathbf{X}_{1\,0}$ or $\mathbf{X}_{0\,1}$ via π_f and the reductions of these affinoids are smooth affines with at least three points at infinity, the same is true for the reduction of $\mathbf{X}^{\beta}_{c\,d}$. Since U is a wide open disk or annulus it follows that one of the connected components of $U\backslash\mathbf{X}^{\beta}_{c\,d}$ is a wide open disk D. The image of D under π can not be contained in $\mathbf{X}_{1\,0}$ or $\mathbf{X}_{0\,1}$ since some the points in D correspond to supersingular elliptic curves. It follows that $\pi(D)$ is disconnected from $\mathbf{X}_{1\,0}$ or $\mathbf{X}_{0\,1}$ which is impossible since D is connected to $\mathbf{X}^{\beta}_{c\,d}$. Thus, $\mathbf{X}^{\beta}_{c\,d}$ is the complement of finitely many residue classes in an underlying affinoid Z of X. Since these residue classes must be disks, we see they can't exist using the same argument as above. ∎

This result should be true for all p, for sufficiently large n.

If $a \geq b > 0$ there is a natural map $\rho\colon C_{a\,b} \to C_{a\,b-1}$; $(A,\mathcal{P}) \mapsto (A,\mathcal{P}^p)$. If $a > b > 0$, there is another natural map $\sigma\colon C_{a\,b} \to C_{a-1\,b}$; $(A,\mathcal{P}) \mapsto (A/p^{a-1}A,\mathcal{P})$.

Lemma 1.3. *Suppose $a+b=n>1$ and $\beta \in C_{a\,b}$. Then π_f restricts to a finite map $\mathbf{X}^{\beta}_{a\,b} \to \mathbf{Y}^{\beta}_{a\,b}$ where*

$$\mathbf{Y}^{\beta}_{a\,b} = \begin{cases} \mathbf{X}^{\rho\beta}_{a\,b-1} & \text{if } a \geq b \geq 1 \\ \mathbf{X}^{\sigma\beta}_{a\,b-1} & \text{if } a < b \geq 1 \\ \mathbf{X}_{a-1\,0} & \text{if } a \geq 1 = b+1 \end{cases}$$

The degree of this restriction is

$$\begin{cases} p & \text{if } p > 2 \text{ and } b > 1 \text{ or } p = 2 \text{ and } b \geq 4 \\ (p-1)/2 & \text{if } p > 2 \text{ and } b = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. First note that $\pi_f \colon X_0(p^n) \to X_0(p^{n-1})$ is finite of degree p. Also if $a+b=n$ and $b > 1$, $\pi_f^{-1}\mathbf{X}_{a\,b-1} = \mathbf{X}_{a\,b}$. We have seen that the irreducible components of $\mathbf{X}_{a\,b-1}$ are the $\mathbf{X}^{\gamma}_{a\,b-1}$ for $\gamma \in C_{a,b-1}$. It is easy to see, using (1), that $\mathbf{X}^{\beta}_{a\,b}$ maps to $\mathbf{Y}^{\beta}_{a\,b}$.

Suppose now $p > 2$ and $b > 1$ or $p = 2$ and $b > 3$. Since $|C_{a\,b-1}| = |C_{a\,b}|$ This implies that each irreducible component of $\mathbf{X}_{a\,b}$ maps with degree p onto an irreducible component of $\mathbf{X}_{a\,b-1}$. This proves the lemma in this case.

Now suppose $p = 2$ and $2 \leq b \leq 3$. The lemma follows in this case because two irreducible components of $\mathbf{X}_{a\,b}$ map to each irreducible component of $\mathbf{X}_{a\,b-1}$.

Now note that $\pi_f^{-1}\mathbf{X}_{a-1\,0} = \mathbf{X}_{a\,0} \coprod \mathbf{X}_{a\,1}$ and $\pi_f|_{\mathbf{X}_{a\,0}}$ has degree 1. This completes the proof the lemma when $p = 2$ and when $b = 0$. So suppose $p > 2$ and $b = 1$. Then π_f restrict to a finite degree $p-1$ map from $\mathbf{X}^{\beta}_{a\,1} \coprod \mathbf{X}^{r\beta}_{a\,1}$ onto $\mathbf{X}_{a\,0}$. Where $\beta \in C_{a\,1}$ and $r \in (\mathbf{Z}/p\mathbf{Z})^*$ is a quadratic non-residue. The lemma folllows from the fact that $\pi_f \circ \tau_r = \tau_r \circ \pi_f$ on $\mathbf{X}_{a\,b}$, $a \geq b$. ∎

2. Annuli

Suppose $r \leq s \in \mathbf{R}$. By the **width** of an annulus isomorphic to $A(r,s)$, we mean $\log_p(s/r)$. If $\mathcal{A}$ is a wide open annulus, denote its width by $w(\mathcal{A})$.

Lemma 2.1. *Suppose $f \colon \mathcal{A} \to \mathcal{B}$ is a morphism of wide open annuli such that $C := f(\mathcal{A})$ is not contained in any affinoid subdomain. Then C is an end of $\mathcal{B}$, $f \colon \mathcal{A} \to C$ is finite and $w(C) = \deg^{\mathcal{A}}_{C}(f) w(\mathcal{A})$.*

Proof. We can suppose $\mathcal{A} = A(r,1)$, $\mathcal{B} = A(s,1)$ and

$$\lim_{|x| \to 1} |f(x)| = 1$$

. Then $h(T) := f^*(T)$ is a unit in $A(\mathcal{A})$. We can write

$$h(T) = cT^e g(T),$$

where $c \in K^*$, $e \in \mathbf{Z}$ and $|g(x) - 1| < 1$ for $x \in \mathcal{A}$. It follows that $|c| = 1$, $C = A(r^e, 1)$ and $e = \deg_{\mathcal{C}}^{\mathcal{A}}(f)$. Thus,

$$w(\mathcal{C}) = -\log_p(r^e) = -\deg(f)\log_p(r) = \deg(f)w(\mathcal{A}). \quad \blacksquare$$

If $\mathcal{A}$ is an annulus over K, there are two natural maps $\Omega^1_{\mathcal{A}/K} \to K$. Indeed, if T is a parameter on $\mathcal{A}$ we can write every element ω of $\Omega^1_{\mathcal{A}/K} \to K$ in the form

$$(\sum_{i=-\infty}^{\infty} c_i(\omega, T)T^i)\frac{\mathrm{d}T}{T},$$

where $c_i(\omega, T) \in K$. The map $\omega \mapsto c_i(\omega, T)$ is a linear map and if T' is another parameter, there is an $\epsilon \in \{\pm 1\}$ such that

$$c_1(\omega, T) = \epsilon c_1(\omega, T')$$

for all $\omega \in \Omega^1_{\mathcal{A}/K}$. (See §II of [RLC].) We call a choice of one of these two homomorphisms an **orientation** of $\mathcal{A}$ and if $\mathcal{A}$ is oriented, we denote the chosen homomorphism by $\mathrm{Res}_{\mathcal{A}}$.

The following lemma will be used in a future article where we will discuss the crystaline nature of the representation attached to an eigenform.

Lemma 2.2. *Suppose $h: \mathcal{A} \to \mathcal{B}$ is a finite surjective morphism of annuli and suppose $\mathcal{A}$ and $\mathcal{B}$ are oriented. Then, if ω is a differential on $\mathcal{B}$ and ν is a differential on $\mathcal{A}$*

$$Res_{\mathcal{A}} h^*\omega = \epsilon d\ Res_{\mathcal{B}}\omega \quad \text{ and } \quad Res_{\mathcal{B}}\ \mathrm{Tr}_h \nu = \epsilon\ Res_{\mathcal{A}}\nu,$$

where d is the degree of h and $\epsilon = -1$ if h is orientation reversing and 1 otherwise. If h is an inclusion such that $\mathcal{B}\backslash h(\mathcal{A})$ is a union of annuli, the first formula is still true with $d = 1$.

Lemma 2.3. *Suppose $h: \mathcal{A} \to \mathcal{B}$ is morphism of annuli, $\mathcal{C} \subseteq \mathcal{B}$ is a subannulus at an end of $\mathcal{B}$ and $h: h^{-1}\mathcal{C} \to \mathcal{C}$ is finite of degree d. Then $h(\mathcal{A})$ is an annulus and*

$h: \mathcal{A} \to h(\mathcal{A})$ is finite, étale of degree d. If $(d,p) = 1$, $h: \mathcal{A} \to h(\mathcal{A})$ is Galois and $\mathcal{A} \times_{h(\mathcal{A})} \mathcal{A}$ is a disjoint union of d annuli each projecting isomorphically onto $\mathcal{A}$.

Proof. Suppose $\mathcal{A} = A(r,1)$ and $\mathcal{B} = A(s,1)$ and $\mathcal{C} = A(t,1)$, $t \geq s$. We can write

$$h(T) = cT^n g(T),$$

where $|g(T) - 1| < 1$, $s \leq |c| \leq 1$ and $s \leq |c|r^n \leq 1$. We can suppose $n \geq 0$. The hypothesis about $\mathcal{C}$ implies $n = d$ and $|c| = 1$. It follows that h is finite onto $A(r^n, 1)$ of degree d.

To prove the last part observe that $g(T)^{1/d}$ makes sense ∎

By a **circle** we mean an affinoid isomorphic to $\max \mathbf{C}_p\langle T, T^{-1}\rangle$, i.e., an annulus conformal to $A[1,1]$. We call a subannulus U of an annulus A concentric if the connected components of $A \backslash U$ are annuli.

3. Horizontal Components

Suppose $p > 3$ is prime. The reduction of Edixhoven's semi-stable model $\mathcal{X}_2$ of $X_0(p^2)$ (which may be obtained by blowing up the Katz-Mazur-Edixhoven regular model over $\mathbf{Z}_p$ of $X_0(p^2)$ at the supersingular points on its reduction over the extension of $\mathbf{Q}_p^{nr}$ of degree $(p^2-1)/2$) has four vertical components $X_{2\,0}$, $X_{1\,1}^+$, $X_{1\,1}^-$ and $X_{0\,2}$ (as described above (we also let $X_{2\,0}^{\pm} = X_{2\,0}$ and $X_{0\,2}^{\pm} = X_{0\,2}$)) and $|SS|$ horizontal components (we'll frequently use SS to denote $|SS|$), $Z_2(s)$ for $s \in SS$, where SS is the set of supersingular j-invariants [E]. (It is stable if $|SS| > 1$.) Moreover, the reductions of any two of these components intersect when and only when one is vertical an one is horizontal, in which case, they intersect in one point.

Remark. *$\mathcal{X}_2$ is stable if there are at least two supersingular points mod p. In general, it may be characterized as follows. Recall, $\pi_f: X_0(p^2) \to X(1)$ is the forgetful map. Let $D \subset X(1)$ be the disk around ∞ corresponding to elliptic curves with multiplicative reduction. Then $\mathcal{X}_2$ is the minimal semistable model $\mathcal{X}$ of $X_0(p^2)$ such that the sections of π_f over D factor through embeddings $Spf(A^0(\pi_f^{-1}D)) \to \mathcal{X}$.*

An elliptic curve over $\mathbf{C}_p$ is called **too supersingular** if it has no canonical subgroup and **nearly too supersingular** if it is p-isogenous to a too supersingular curve. Nearly too supersingular curves do have canonical subgroups. (Canonical subgroups of elliptic curves are introduced by Katz [K]. These are subgroup schemes of order p. Buzzard defined canonical subgroups of order p^n [B, Def. 3.4].) If (E, C) is a pair consisting of an elliptic curve E over $\mathbf{C}_p$ with a model with good supersingular reduction and a subgroup C of order p, the **Buzzard invariant** of (E, C) is the positive real number $b(E, C)$ which is characterized by the properties, $b(E, C)$ is the valuation of the Hasse invariant of the reduction modulo p of a model for E with good reduction when E has a canonical subgroup and it is C and in general

$$b(E, C) + b(E/C, E[p]/C) = 1.$$

(It is always true that either C is the canonical subgroup of E or $E[p]/C$ is the canonical subgroup of E/C.) In particular, E is too supersingular if and only if $b(E, C) = p/(p + 1)$ for one and hence all subgroups C of E of order p. If E has a canonical subgroup K_2 of order p^2, $b(E/K_1, K_2/K_1) = pb(E, K_1)$ and if $K_1 \neq C$, $b(E/C, E[p]/C) = b(E, C)/p$.

In general, if H is a supersingular elliptic curve over $\overline{\mathbf{F}}_p$ corresponding to $s \in SS$ and $b \in \mathbf{Q}$, the pairs (E, C) such that E reduces to a curve isomorphic to H and $b(E, C) = b$ correspond to the $\mathbf{C}_p$-valued points on a concentric circle, $C_b(s)$, in the wide open annulus $A_1(s)$ in $X_0(p)$ above the singular point of the reduction of $X_0(p)$ corresponding to s. Moreover, $A_1(s)\backslash C_b(s) = W_\infty \coprod W_0$ where W_∞ is a wide open annlus connected to $\mathbf{X}_{1\,0}$ of width b.

Let $\mathbf{Z}_2(s) = \mathrm{red}^{-1}(\mathrm{Z(s)} - \bigcup_{\mathrm{i+j=2}} \mathrm{X}^{\pm}_{\mathrm{i\,j}})$ be the underlying affinoid of $\mathrm{red}^{-1}\mathrm{Z(s)}$.

Theorem 3.1. *The $\mathbf{C}_p$-valued points of $\mathbf{Z}_2(s)$ correspond to pairs (E, C) where E is a nearly too supersingular elliptic curve and C is a cyclic subgroup of order p^2 and pC is its canonical subgroup or equivalently $\mathbf{Z}_2(s)$ is the inverse image under the forgetful map to $X_0(p)$ of $C_{1/(p+1)}(s)$.*

This will be proven below.

Proposition 3.2. *Suppose $f: W \to V$ is a finite map of basic wide opens and V is not a disk. Then if X is a minimal underlying affinoid of W, $Z =: f(X)$ is a minimal underlying affinoid of V and $X = f^{-1}(Z)$.*

Remark. *A basic wide open which is neither a disk nor an annulus has a unique minimal underlying affinoid. The minimal underlying affinoids in a wide open annulus are the concentric circles. The image of a disk or an annulus under a finite map is a disk or an annulus.*

Proof. If V is not an annulus let Y be its minimal underlying affinoid. If V is an annulus, the image of X is an affinoid with irreducible reduction so must be contained in a concentric circle of V which we will call Y in this case. Let E be a connected component of $W \backslash X$. Then, because f is finite, $U =: f(E)$ is a disk or annulus containing an annulus at an end of V so in the corresponding connected component D of $V \backslash Y$. Claim, it must be contained in D. First, suppose V is not an annulus. Then by [RLC] (see also, [BL]) U must be contained in Y or in a component of $V \backslash Y$. Since the former is precluded, we have our claim, in this case. If V is an annulus, the claim follows from the fact that if h is a unit on $A(r, 1)$, $h = cT^n g$, where $c \in K^*$, $n \in \mathbf{Z}$ and $|g - 1| < 1$ so, in particular if $n \neq 0$, determines a finite map onto $A(|c|r^n, |c|)$. This implies either $U \cap Y$ equals Y or $\emptyset$. Suppose $Y \subset U$. Then there is a a proper concentric wide open subannulus A of E such that $f^{-1}(D \cup Y) \subset A$. It follows that $U - f(A)$ and Y are disconnected but this is impossible since $E - A$ and X are connected and so $f(E) \subseteq D$.

We conclude $f^{-1}Y = X$ and finiteness implies $Y = f(X) = Z$ which concludes the proof. ∎

It follows that $\pi_f(\mathbf{Z}_2(s))$ is a circle.

One thing we may conclude from Lemma 2.1 is that in the situation of Proposition 3.2, $W \backslash X$ maps finitely onto $V \backslash Y$.

There exists a semi-stable model $\mathcal{X}_1$ of $X_0(p)$ such that π_f extends to a finite morphism $\mathcal{X}_2 \to \mathcal{X}_1$ (cf. [C-SM]). This amounts to adding components corresponding to the circles $\pi_f(\mathbf{Z}_2(s))$, $s \in SS$. Let $G_0(p^n)$ be the oriented graph of the reduction of $\mathcal{X}_n$ (one vertex for every irreducible component and one edge for every singular point).

The component $X_{i\,j}^{\pm}$ of $\overline{\mathcal{X}}_n$ is the irreducible component of the reduction of $\mathcal{X}_n$ containing the reduction of $\mathbf{X}_{i\,j}^{\pm}$.

For every supersingular point s on the reduction of $X(1)$ we have one component $Z_n(s)$ of the reduction of $\mathcal{X}_n$ lying over it. Let

$$\mathbf{Z}_1(s) = \mathrm{red}^{-1}\big(\mathrm{Z_1(s)} - (\mathrm{X}_{1\,0} \cup \mathrm{X}_{0\,1})\big) = \pi_{\mathrm{f}}(\mathbf{Z}_2(\mathrm{s}))$$

be the underlying circle in $A_1(s) = \mathrm{red}^{-1}\mathrm{Z}_1(\mathrm{s})$ and let $A_{i\,j}^{\pm}(s)$ be the annulus which is the reduction inverse of the intersection of $Z_n(s)$ and $X_{i\,j}^{\pm}$. This annulus has a natural orientation corresponding to the ordered pair $(\mathbf{X}_{i\,j}^{\pm}, \mathbf{Z}_n(s))$. We also put

$$A_{i\,j}^{\pm} = \bigcup_s A_{i\,j}^{\pm}(s).$$

and $A_{1\,1}(s) = A_{1\,1}(s)^+ \cup A_{1\,1}(s)^-$.

Theorem 3.1 is equivalent to the assertion that $\mathbf{Z}_1(s)$ is the nearly too supersingular circle $C_{1/(p+`1)}$, which we will now prove.

Note that π_f has degree p and

$$\pi_f^{-1} A_1(s) \backslash \mathbf{Z}_2(s) = A_{2\,0}(s) \cup A_{1\,1}^{+}(s) \cup A_{1\,1}^{-}(s) \cup A_{0\,2}(s)$$

.

Lemma 3.3. *$w(A_{2\,0}(s)) = w(A_{0\,2}(s))$, and*

$$w(A_{1\,0}(s)) + w(A_{0\,1}(s)) = 1.$$

Proof. As the Atkin-Lehner involution α acts on $\mathcal{X}_2$ as $\mathcal{X}_2$ is canonical and $\alpha(X_{2\,0}) = X_{0\,2}$ we must have $\alpha(A_{2\,0}(s)) = A_{0\,2}(s)$ so $w(A_{2\,0}(s)) = w(A_{0\,2}(s))$. The last assertion is follows from the fact that the annulus $A_1(s)$ has width 1 and is the disjoint union of the annuli $A_{1\,0}(s)$ and $A_{0\,1}(s)$ and the circle $\mathbf{Z}_1(s)$. ∎

Lemma 3.4. *The maps induced by* π_f*:* $A_{2\,0}(s) \to A_{1\,0}(s)$*,* $A^{+}_{1\,1}(s) \to A_{1\,0}(s)$*,* $A^{+}_{1\,1}(s) \to A_{1\,0}(s)$*,* $A^{+}_{1\,1}(s) \to A_{1\,0}(s)$ *have degrees* 1*,* $(p-1)/2$*,* $(p-1)/2$ *and* p *respectively.*

Proof. This follows from Lemma 2.3 and the fact that the morphisms $X_{2\,0} \to X_{1\,0}$, $X^{\pm}_{1\,1} \to X_{1\,0}$ and $X_{2\,0} \to X_{0\,1}$ have, by Lemma 1.3, degrees 1, $(p-1)/2$ and p respectively. ∎

Lemma 3.5. $w(A_{1\,0}(s)) = w(A_{2\,0}(s))$ *and* $w(A_{0\,1}(s)) = pw(A_{0\,2}(s))$*.*

Proof. This follows from the previous lemma and Lemma 2.1. ∎

Thus $w(A_{1\,0}(s)) = 1/(p+1)$ so $\mathbf{Z}_1(s) = C_{1/(p+1)}(s)$. This concludes the proof of Theorem 3.1.

Let $W_{2\,0} = \mathrm{red}^{-1}(\mathrm{X}_{2\,0})$ etc.

Corollary 3.5.1. *The pair* (E, C) *corresponds to point in* $W =: W_{2\,0} \cup W^{-}_{1\,1} \cup W^{+}_{1\,1}$ *if and only if* E *has supersingular reduction,* $K_2(E)$ *exists and* $pC = K_1(E)$*. It corresponds to point in* $W_{2\,0}$ *if and only if* $C = K_2(E)$*, to a point in* $W^{\epsilon}_{1\,1}$ *if and only if* $C \neq K_2(E)$ *and the induced pairing* $K_1(E) \times K_1(E) \to \mu_p$ *is of type* ϵ*.*

Proof. The first sentence of the corollary is clear since we know $\pi_f^{-1} W_{1\,0} = W$. Next, there is a section s of $\pi_f \colon W_{2,0} \to W_{1\,0}$, since if (E, D) corresponds to a point in $W_{1\,0}$, $K_2(E)$ exists and $K_1(E) = D$. Then $s(P)$ will correspond to $(E, K_2(E))$. This is a section because it is when restricted to $\mathbf{X}_{1\,0}$ and $W_{2\,0}$ is irreducible. This establishes the corollary for $W_{2\,0}$. Now suppose (E, C) corresponds to a point P in $W^{\epsilon}_{1\,1}$. Then $C \neq K_2(E)$, so we get a pairing $\mathcal{P}_P$ on $K_1(E)$ onto μ_p. If c generates $K_1(E)$ and $\mathcal{P}_P(c, c) = \zeta^a$, $\chi(a)$ depends only on P and gives an analytic function on $W^{\epsilon}_{1\,1}$. It must be constant since $W^{\epsilon}_{1\,1}$ is connected. This concludes the proof. ∎

Appendix: Affinoids in Curves

We prove some well known results about curves for which we don't know a good reference.

Suppose K is a complete subfield of $\mathbf{C}_p$ with ring of integers R. Suppose C is a smooth proper curve over K and $\mathcal{C}$ is a model of C over R. If V is a subscheme of the reduction $\overline{\mathcal{C}}$ of $\mathcal{C}$, let $X(\mathcal{C},V)$ denote generic fiber of the formal completion of $\mathcal{C}$ along V. If V is a reduced open affine, $X(\mathcal{C},V)$ is an affinoid subdomain of C with reduction V. Now suppose $\mathcal{C}$ is semi-stable and S is a subset of the set of components $T := T_{\mathcal{C}}$ of $\overline{\mathcal{C}}$. Let $Y_S = \bigcup_{Z\in S} Z$, $Y_S^o = Y_S \backslash Y_{T\backslash S}$ and let $X(\mathcal{C},S) = X(\mathcal{C},Y_S^o)$. This is an affinoid subdomain if $S \neq T$ of C because if $\mathcal{C}_S$ is the blow down of $\mathcal{C}$ along $\bigcup_{\substack{Z\in S \\ Z\subset Y_S^o}} Z$, then the image Y_S' of Y_S^o in $\overline{\mathcal{C}}_S$ is a reduced open affine in $\overline{\mathcal{C}}_S$ and

$$X(\mathcal{C},S) = X(\mathcal{C}_S, Y_S').$$

Of course, $X(\mathcal{C},T) = C$. Also set $S^\infty = Y_S \backslash Y_S^o$.

If $f\colon \mathcal{T} \to \mathcal{C}$ is a morphism of semi-stable models of C, and
$S_{\mathcal{T}} = \{Z \in T_{\mathcal{T}} \colon \bar{f}(Y_{\{Z\}}^o \subseteq Y_S^o\}$

$$X(\mathcal{T}, S_{\mathcal{T}}) = X(\mathcal{C},S). \tag{1}$$

Also, if $E \subseteq \mathbf{C}_p$ is a complete extension of K,

$$X(\mathcal{C}_{R_E}, S_{\overline{R}_E}) = X(\mathcal{C},S)_E.$$

If U is another subset of T,

$$X(\mathcal{C},S)\cap X(\mathcal{C},U) = X(\mathcal{C},S\cap U) \tag{2}$$

and if $S^\infty \cap U^\infty \subseteq (S\cap U)^\infty$

$$X(\mathcal{C},S)\cup X(\mathcal{C},U) = X(\mathcal{C},S\cup U). \tag{3}$$

If $f: \mathcal{T} \to \mathcal{C}$ is a morphism of semi-stable models of C and $\bar{f}^{-1}x \in T_{\mathcal{T}}$ for $x \in S^\infty \cap U^\infty$, $S^\infty_{\mathcal{T}} \cap U^\infty_{\mathcal{T}} \subseteq (S_{\mathcal{T}} \cap U_{\mathcal{T}})^\infty$ so

$$X(\mathcal{C}, S) \cup X(\mathcal{C}, U) = X(\mathcal{T}, S_{\mathcal{T}} \cup U_{\mathcal{T}}). \tag{4}$$

If $\{W_Z: Z \in T\}$ is the semi-stable covering of C corresponding to $\mathcal{C}$,

$$X(\mathcal{C}, S) = \bigcup_{Z \in S} W_Z \backslash \bigcup_{Z \in T \backslash S} W_Z.$$

Theorem A1. *If X is an affinoid subdomain of C and $\mathcal{S}$ is a semi-stable model of C over R_K, then there exists a finite extension E of K and a semi-stable model $\mathcal{T}$ of C_E over R_E mapping to $\mathcal{S}_{R_E}$ and a subset S of $T_{\mathcal{T}}$ so that $X_E = X(\mathcal{C}, S)$.*

We first extend scalars to $\mathbf{C}_p$ (one can descend later). We will prove the translation of this theorem into the language of semi-stable coverings. That is, we will regard $\mathcal{S}$ as a semi-stable covering and find an appropriate semi-stable refinement $\mathcal{T}$ of $\mathcal{S}$. We let $\overline{\mathcal{S}}$ denote the reduction of the corresponding semi-stable model. We may and will suppose X is irreducible (equivalently, connected).

By a **residue class** U of $\mathcal{S}$, we mean the subspace of C corresponding to a point P of $\overline{\mathcal{S}}$ and we let $\overline{U} = P$. The space U is a wide open disk or wide open annulus according as P is smooth or singular and we call it either a **residue disk** or **residue annulus**. if $W \in \mathcal{S}$, let Z_W denote the underlying affinoid in W.

If $R \subseteq S$ are rigid spaces let $\mathrm{CC}(R, S)$ denote the subspace of S connected to R. If f is function on S and R is an affinoid $||f||_R$ will denote the sup-norm of the restriction of f to R. By a **circle** we mean an affinoid isomorphic to max $\mathbf{C}_p\langle T, T^{-1}\rangle$, i.e., an annulus conformal to $A[1,1]$. We call a subannulus U of an annulus A concentric if the connected components of $A \backslash U$ are annuli.

Lemma A2. *If U is a wide open disk in C, there exists a function z on C with a single pole such that $U = \{x \in C: |z(x)| > 1\}$.*

Proof. Claim: We can find a semi-stable covering $\mathcal{S}$ of C so that U is contained in a residue disk D of $\mathcal{S}$. This is clear when C has a model with good reduction.

Otherwise, there exists a semi-stable covering $\mathcal{T}$ of C such that no element of $\mathcal{T}$ is a disk (eg., the covering corresponding to the stable model if the genus of C is at least 2 and this model has at least two components). Then U must be contained in a residue class of $\mathcal{T}$. If it is contained in a residue annulus A, it must be contained in a concentric circle Z in A (see Lemma 3.2 of [RLC]). We can then take $\mathcal{S}$ to be

$$\{A\} \cup \{\ \mathrm{CC}(W\backslash A, W\backslash Z)\colon W \in \mathcal{T}\}.$$

It follows using a blowing down argument as above, applied to $\mathcal{S}$ that $Y := C\backslash D$ is an affinoid.

Now, let $P \in U$ and suppose f is a function on C with a pole only at P. Then

$$D = \{x \in C\colon |f(x)| > ||f||_Y\}$$

and there exists $r \in |\mathbf{C}_p|$, $U = \{x \in D\colon |f(x)| > r\}$. Suppose $a \in \mathbf{C}_p$, $|a| = r$. Take $z = f/a$. ∎

We will say such a z **determines** U.

Lemma A3. *If $\mathcal{D}$ is a collection of disjoint wide open disks D in C such that $D \cap X \neq \emptyset$ and $D\backslash X \neq \emptyset$, then $\mathcal{D}$ is finite.*

Proof. For $U \in \mathcal{D}$, let z_U be a function on C which determines U. Suppose $a_U \in \mathbf{C}_p$, $|a_U| = ||z_U||_X > 1$. Let $f_U = (z_U/a_U)|_X$. Then $\overline{f}_U \neq 0$ and $\overline{f}_U \overline{f}_V = 0$ if $V \neq U$ in $\mathcal{D}$. Since $\overline{X}$ is reduced of finite type over $\overline{\mathbf{F}}_p$, this implies $\mathcal{D}$ is finite. ∎

Lemma A4. *If Z is an underlying affinoid of $\mathcal{S}$, then either (i) $X \cap Z = \emptyset$, (ii) X is contained in a residue class of Z or (iii) $X \cap Z$ contains all but finitely many residue classes of Z.*

Proof. Suppose neither (i) nor (ii) is true. Let D be a residue disk in Z and z a function on C which determines D. Then since $X\backslash D = \{x \in X\colon |z(x)| \leq 1\}$, $Y := X\backslash D$ is an affinoid. Also $V := C\backslash D$ is an affinoid whose reduction is the blow down of $\overline{\mathcal{S}}$ along $T_{\mathcal{S}}\backslash\{\overline{Z}\}$ minus $\overline{D}$. Since neither conditions (i) nor (ii) hold, the image of $\overline{Y}$ in $\overline{V}$ is not a point. Since X is connected, $X \cap Z$ cannot be contained in a finite number greater than one of residue classes , so the image of Y must be a non-empty Zariski open of V. The lemma now follows from Lemma A3. ∎

Proposition A5. *If U is a residue class of $\mathcal{S}$ and $X \not\subset U$, $U \backslash X = \bigcup(\mathcal{B} \cup \mathcal{A})$ where $\mathcal{B}$ is a finite set of wide open disks and $\mathcal{A}$ is empty if U is a disk, and is either empty or a concentric wide open annulus in U if U is an annulus.*

Proof. Suppose D is a residue disk of $\mathcal{S}$ and neither $D \backslash X$ nor $(D \cap X)$ is empty. Suppose W is the element of $\mathcal{S}$ which contains D. Let z be a function on C which determines D. For each, $Q \in D \backslash X$, let $w_Q = z/(1 - z/z(Q))$ and suppose $a_Q \in \mathbf{C}_p$ such that $|a_Q| = r_Q := ||w_Q||_X$. Let $B_Q = \{x \in C\colon |w_Q(x)| > r_Q\}$ which is contained in D. Then B_Q is a wide open disk determined by $z_Q := w_Q/a_Q$. $z_Q|X \in A^0(X)$. We want to prove the collection of disks, $S := \{B_Q\colon Q \in D \backslash X\}$, is finite.

Suppose $P \in D \backslash X$. Let $A_r := A[P, r] = \{x \in D\colon |z_P(a)| = r\}$. Since X is connected and $X \not\subset D$, $X \cap A_r \neq \emptyset$ and $X \not\subset A_r$ for all $R := r_P \geq r > 1$. Let $D_r = \{x \in D\colon |z_P(a)| \geq r\}$ and $\mathcal{S}_r =$

$$\{D\} \cup \{W \backslash D_r\} \cup \mathcal{S} \backslash \{W\}.$$

Then $\mathcal{S}_r$ is a semi-stable covering of C and D_r is an underlying affinoid of $\mathcal{S}_r$. It follows that X contains all but finitely many residue classes of B_r and hence of A_r for $1 < r \leq r_P$.

Suppose $r_P \geq r_1 > \cdots > r_n > \cdots > 1 \in |\mathbf{C}_p|$ and $\{P_i\}$ is a sequence of points such that $P_i \in A_{r_i} \backslash X$. Then $\overline{z_{P_i}|_X} \neq 0$ and $\overline{z_{P_i}|_X} \cdot \overline{z_{P_j}|_X} = 0$, if $i \neq j$. Thus X must contain contain all but finitely many of the circles $A[P, r]$, $1 > r \geq r_P$ and, by Lemma A4, in each circle it contains all but finitely many residue disks.

It follows that if S is infinite there exists a sequence of points, $\{Q_i\}$, in $D \backslash X$ and a sequence of $s_j \in |\mathbf{C}_p|$ such that $1 > s_{j+1} \geq r_{Q_j}$, $Q_{j+1} \in A[Q_j, s_{j+1}]$, $s_{j+1} > |z_{Q_j}(Q_{j+1}) - z_{Q_j}(Q_{j+2})|$ (i.e., all the Q_i, $i > j$, lie in the same residue disk of a circle around B_{Q_j}). Let $f_i = (z_{Q_i} - z_{Q_i}(Q_{i+1}))$. Then $f_i|_X \in A^0(X)$, $\overline{(f_i|_X)} \neq 0$ and $\overline{(f_i|_X)}\,\overline{(f_j|_X)} = 0$, if $i \neq j$. Again, this contradicts the finite typeness of $\overline{X}$ and establishes the proposition when U is a disk.

Lemma A6. *Suppose A is a residue annulus of $\mathcal{S}$. Then if $T\colon A \cong A(R,S)$, $R, S \in |\mathbf{C}_p|$, is parameter and $X \not\subset A$, $(R,S)\backslash\{|T(x)|\colon x \in A \cap X\} = (r,s)$ for some $r, s \in |\mathbf{C}_p|$, $R \le r \le s \le S$. v Proof. First suppose $\exists W_1, W_2 \in \mathcal{S}$, $W_1 \neq W_2$, such that A is a component of $W_1 \cap W_2$. Let D_i be a residue disks in W_i for $i = 1$ or 2. Suppose z_i is a function on C which determines D_i. Then $M := C\backslash(D_1 \cup D_2)$ and $X' = X\backslash(D_1 \cup D_2)$ are affinoids. Moreover, $X \cap A = X' \cap A$ and $\overline{M}$ equals the blow down of $\overline{\mathcal{S}}$ along the components which don't correspond to the W_i. In fact, $\overline{M}$ has two components, Z_1 and Z_2, which correspond to W_1 and W_2, $z_1 \in A^0(M)$, $\overline{(z_1|_M)}$ is not constant but is on Z_2. We can and will suppose it vanishes on Z_2.*

Let g be a function on $\overline{M}$ which vanishes at and only at the singular points of $\overline{M}$ apart from the one corresponding to A (Such a function exists because $\overline{M}$ is an affine curve over a finite field.). Let $\tilde{g} \in A^0(M)$ be a lifting of g. Let $N = \{x \in M\colon |\tilde{g}(x)| = 1\}$ and $Y = \{x \in X'\colon |\tilde{g}(x)| = 1\}$. Then N and Y are affinoids, $A \subset N$, $N\backslash A$ has two connected components and $Y \cap A = X \cap A$.

Suppose the restriction of the divisor (z_1) of z_1 to $A \cup Z_{W_1}\backslash D_1$ is the effective divisor D. There exists a positive integer n and a function f on C such that

$$(f) = -nD + E$$

where E is an effective divisor supported on $C\backslash W_1$ (This is because the points on he Jacobian J of C represented by divisors supported on a non-empty open subset of $C(\mathbf{C}_p)$ form an open subgroup of $J(\mathbf{C}_p)$ and such open subgroups have torsion quotients.). We can also suppose $||f||_{Z_{W_1}} = 1$. Let $h = z_1^n f$. Then h has poles only at the pole of z_1, $||h||_M = 1$, $\overline{h|_M}(\bar{A}) = 0$ and h doesn't vanish on $A \cup Z_{W_1}$. It follows that if T is a parameter on A, $||T||_A = 1$, $|T(x)| \to 1$ as $x \to Z_{W_1}$ $h = T^n u$, $n > 0$.

Suppose $|T(A)| = (R,1)$. Then since X is connected, $X \cap A$ has at most two components and if $|T(A \cap X)| \neq (R,1)$ (which well now suppose), Y has two components Y_1 and Y_2 such that $Y_i \cap Z_{W_j} \neq \emptyset$ if and only if $i = j$. Then

$$|T(A \cap X)| = (R,1) \backslash (r,s),$$

where $r = ||h||_{Y_2}^{1/n}$ and $1/s = ||1/h||_{Y_1}^{1/n}$.

Now suppose $A \subset Z_W$. Suppose $Q \in |\mathbf{C}_p|$ and $R < Q < S$. Let $Z_Q = \{x \in u \colon |T(x)| = Q\}$. Then Z_Q is a concentric circle of A and let

$$\mathcal{S}_Q := \{W \backslash Z_Q\} \cup \{A\} \cup \mathcal{S} \backslash \{W\}$$

Then applying what we just proved to the the two components of $A \backslash Z_Q$ which are residue annuli of $\mathcal{S}_Q$ and components of $A \cap W \backslash Z_Q$, we see

$$S := (R,S) \backslash \{|T(x)| \colon x \in A \cap X\} = (r_1, s_1) \cup (r_2, s_2) \cup N,$$

where $R \le r_1 \le s_1 \le Q \le r_2 \le s_2 \le S$ and N is either empty or $\{Q\}$. Suppose $R < Q' < Q$. We also see

$$S = (r_1', s_1') \cup (r_2', s_2') \cup N',$$

where $R \le r_1' \le s_1' \le Q' \le r_2' \le s_2' \le S$ and N' is either empty or $\{Q'\}$. The only possibilities consistent with these two statements are $S = (r_1, s_2)$, (r_2, s_2) and $S = (r_1, s_1)$. ∎

It follows by an argument similar to that used in the analysis of $D \backslash X$ above that $A \backslash X = T^{-1}(r,s) \cup \bigcup \mathcal{B}$ where $\mathcal{B}$ is a finite collection of wide open disks. This completes the proof of the proposition. ∎

Now we complete the proof of the theorem. We will make several refinements of $\mathcal{S}$.

First we make sure X is not contained in any residue class. Suppose X is contained in a residue class U of $\mathcal{S}$. Let B be a closed disk contained in X. Let

$Y = B$ if U is a disk and the concentric circle in U containing B if U is an annulus. Let $V = U$ if U is a disk and the residue disk of Y containing B otherwise. Let W' be $\mathrm{CC}(W\backslash U, W\backslash Y)$. Let $\mathcal{S}'$ equal

$$\begin{cases} \{W'\colon W \in \mathcal{S}\} \cup \{U\} & \text{if } U \text{ is a disk and} \\ \{W'\colon W \in \mathcal{S}\} \cup \{U\backslash B\} \cup \{V\} & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{S}'$ is semi-stable nd X is contained in no residue class of $\mathcal{S}'$.

We will next find a semi-stable refinement $\mathcal{S}''$ of $\mathcal{S}'$ so that if $A \cap X \neq \emptyset$, $A \subset X$ for each residue annulus A of $\mathcal{S}''$. Let $\mathcal{A}$ be the set of residue annuli of $\mathcal{S}'$. For each $A \in \mathcal{A}$ let $\mathcal{C}_A$ denote the collection of concentric circles Z in A such that neither $X \cap Z$ nor $Z\backslash X$ is empty. We know from Proposition A5 that $\mathcal{C}_A$ is finite.

Let $\mathcal{S}''$ be

$$\{\ \mathrm{CC}(Z_W, W\backslash \bigcup_{\substack{A\in\mathcal{A} \\ Z\in\mathcal{C}_A}} Z)\colon W \in \mathcal{S}'\} \cup \bigcup_{A\in\mathcal{A}} \{\ \mathrm{CC}(Z, A\backslash \bigcup_{\substack{Y\in\mathcal{C}_A \\ Y\neq Z}} Y)\colon Z \in \mathcal{C}_A\}.$$

This is a semi-stable covering with the required properties. Indeed, the spaces in the collection on the left are elements of $\mathcal{S}$ with annuli cut out off the residue annuli and the spaces in the collection on the right are annuli which fill in the gaps.

Now we will make a refinement $\mathcal{S}'''$ of $\mathcal{S}''$ so that if $U \cap X \neq \emptyset$, $U \subset X$ for any residue class U of $\mathcal{S}'''$. For $W \in \mathcal{S}''$, let $\mathcal{B}_W(X)$ denote the set of residue classes of $\mathcal{S}''$ in W so that $X \cap D \neq \emptyset$ and $D\backslash X \neq \emptyset$. It follows from the construction of $\mathcal{S}''$ that the elements of $\mathcal{B}_W(X)$ are disks and from lemma A4 that $\mathcal{B}_W(X)$ is finite for each $W \in \mathcal{S}''$. For $D \in \mathcal{B}_W(X)$, let $\mathcal{D}_D$ be the set of connected components of $D\backslash X$. It follows from Proposition A5 that $\mathcal{D}_D$ is a finite collection of wide open disks.

Now suppose $D \in \mathcal{B}_W(X)$. For $S \subseteq \mathcal{D}_D$, $S \neq \emptyset$, let $B(S)$ denote the smallest closed disk in D containing $\bigcup_{E\in S} E$ and $U(S)$ the largest wide open disk containing $B(S)$ disjoint from $\bigcup_{E\in\mathcal{D}_D\backslash S} E$, if it exists and the empty set if it doesn't. Let

$$W_S = U(S)\backslash \bigcup_{\substack{T\subset S \\ B(T)\neq B(S)}} B(T)$$

Let $\mathcal{S}'''$ be

$$\{W\backslash \bigcup_{D\in\mathcal{B}_W(X)} B(\mathcal{D}_D) \colon W\in\mathcal{S}''\}\cup\{W_S\colon S\subseteq\mathcal{D}_D, W_S\neq\emptyset\}.$$

This is a semi-stable covering because

$$D=\bigcup_{S\subseteq\mathcal{D}_D} W_S$$

and if T and S are non-empty subsets of $\mathcal{D}_D$, $W_S\neq W_T$, $W_S\cap W_T\neq\emptyset$, either $W_S\cap W_T$ equals $U(T)\backslash B(T)$ or $U(S)\backslash B(S)$, so is an annulus.

Finally, $\mathcal{T}$ will be a refinement Ş$'''$ so that if Z is a residue annulus or underlying affinoid of $\mathcal{T}$ and $X\cap Z\neq\emptyset$, $Z\subseteq X$. Let $\mathcal{B}$ be the set of residue disks D of $\mathcal{S}$ such that $D\cap X=\emptyset$ but $Z\cap X\neq\emptyset$ where Z is the underlying affinoid of $\mathcal{S}'''$ containing D. For each $D\in\mathcal{B}$, let $B(D)$ be a closed disk in D. We take $\mathcal{T}$ to be

$$\{W\backslash\bigcup_{D\in\mathcal{B}} B(D)\colon W\in\mathcal{S}'''\}\cup\mathcal{B}.$$

The point is the residue annuli of $\mathcal{T}$ are the residue annuli of $\mathcal{S}'''$ and $\{D\backslash B(D)\colon D\in\mathcal{B}\}$, and the underlying affinoid with respect to $\mathcal{T}$ of $W\backslash\bigcup_{D\in\mathcal{B}} B(D)$ is $Z_W\cap X$ which is an affinoid whose reduction is a Zariski open in $\overline{Z}_W$.

Then $X=X(\mathcal{T},S)$, where S is the set of components of $\overline{\mathcal{T}}$ corresponding to the set of $W\in\mathcal{T}$ such that $W\cap X\neq\emptyset$. ∎

Corollary A7. *Suppose F is a complete subfield of $\mathbf{C}_p$ and C is a smooth proper curve defined over F. If X and Y are affinoid subdomains of C, $X\cap Y$ is an affinoid subdomain and $X\cup Y$ either equals C or is an affinoid subdomain.*

Proof. We know C has a semi-stable model over R_K where K is a finite extension of F. By Theorem A1, there exists a finite te extension E of K, a semi-stable model $\mathcal{T}$ of C over R_E and a subset S of $T_{\mathcal{T}}$ so that $X_E=X(\mathcal{C},S)$. Also by this theorem there exists a finite extension L of E, a semi-stable model $\mathcal{R}$ of C over R_L mapping to $\mathcal{S}$ and a subset U of $T_{\mathcal{R}}$ so that $Y_L = X(\mathcal{R}, U)$. It now follows from (1) and (2) that $Z = X_L \cap Y_L$ is an affinoid subdomain of C_L.

We can assume L is a Galois extension of F with Galois group G. let $\{f_\sigma : \sigma \in G\}$ and $\{g_\sigma : \sigma \in G\}$ be the natural descent data fon X_L and Y_L. That is, $f_\sigma : X_L^\sigma \to X_L$ and $g_\sigma : Y_L^\sigma \to Y_L$ are isomorphisms such that

$$f_{\sigma\tau} = f_\tau \circ f_\sigma^\tau \quad \text{and} \quad g_{\sigma\tau} = g_\tau \circ g_\sigma^\tau.$$

Now, if $\iota_X : X \to C$ and $\iota_Y : Y \to C$ are the natural inclusions, $\iota_{X_L^\sigma} = \iota_{X_L} \circ f_\sigma$ and $\iota_{Y_L^\sigma} = \iota_{Y_L} \circ g_\sigma$. Also, if N is an extension of L,

$$Z(N) = \{(x, y) \in X(N) \times Y(N) : \iota_X(x) = \iota_Y(y)\}$$

and

$$Z^\sigma(N) = \{(x, y) \in X^\sigma(N) \times Y^\sigma(N) : \iota_X^\sigma(x) = \iota_Y^\sigma(y)\}.$$

It follows that if $(x, y) \in Z^\sigma(N)$, $(f_\sigma(x), g_\sigma(y)) \in Z(N)$. Thus $\{(f_\sigma \times g_\sigma)|_{Z^\sigma} : \sigma \in G\}$ is descent data on Z. Since Z is an affinoid, it descends to an affinoid over F which represents $X \cap Y$.

The second part of the corollary follows similarly.

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