Wide Open Spaces

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Tips from Milne

Taken from **http://www.jmilne.org/math/tips.html**.

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6. If, in a moment of weakness, you do refer to a paper or book for a result, never say where in the paper or book the result can be found. In addition to making it difficult for the reader to find the result, this makes it almost impossible for anyone to prove that the result isn't actually there.

p-adic Notation

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Let $|\cdot|$ be the absolute value on $\mathbb{C}_p$ such that $|p| = p^{-1}$, and let $\mathcal{R} = p^{\mathbb{Q}}$. Let K be a complete subfield of $\mathbb{C}_p$ with ring of integers R_K and residue field $\mathbb{F}_K$.

Definition of a Wide Open

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Moreover, X is inner in another affinoid subdomain of W and it is possible to glue n open disks onto W to obtain a proper separable rigid space.

Call X an **underlying affinoid** of W and (W, X) a **wide open pair**. If X has good reduction call (W, X) basic.

Some Examples (general)

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The complement of finitely many affinoid disks over K, each in a separate residue class, in a curve of good reduction over K is a basic wide open over K.

Some Examples in Modular Curves

In $X_0(p^n)$, let $SS(p^n)$ and $O(p^n)$ be the supersingular and ordinary loci. Say an elliptic curve over K is TS if it has a model D with good or multiplicative reduction over $\mathbb{R}_p$, such that all $p+1$ of the closed subgroup schemes of $D[p]$ over $\mathbb{R}_p$ of rank p are isomorphic.

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Define:

(i) $TS(p) \subset X_0(p)$, (E, C) where E is TS.
(ii) $Y(p^2) \subset X_0(p^2)$, (E, C) where $E/C[p]$ is TS.
(iii) $Z_i(p) \subset X_0(p)$, $i = 0$ or 1, (E, C) where $\overline{C}$ is isomorphic over $\mathbb{R}_p$ to exactly p^i subgroup schemes of $\overline{E}[p]$.

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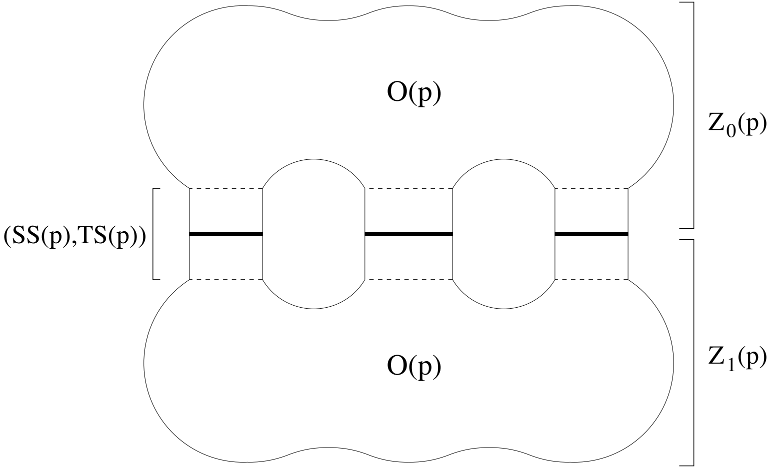
Then $(SS(p), TS(p))$, $(Z_i(p), Z_i(p) \cap O(p))$, $(SS(p^2), Y(p^2))$ are *potential* basic wide open pairs.

Wide Opens in $X_0(p)$, a Picture



Ends

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Let $W^+ = W(\mathbb{C}_p) \cup \mathcal{E}(W)$.

For each $e \in \mathcal{E}(W)$ there is a natural continuous linear map

$$\mathrm{res}_e \colon \varinjlim_A (\Omega^1_{A/K})^f \twoheadrightarrow K,$$

where A runs over all annuli around e.

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$$\mathrm{res}_e \colon \lim_{\to\, A} (\Omega^1_{A/K})^f \twoheadrightarrow K,$$

where A runs over all annuli around e. In fact,

$$\mathrm{res}_e(\omega) = 1$$

if $\omega = \mathrm{d}v/v$ for some parameter v on A_L over an extension L of K where $\{x \in A \colon |v(x)| > r\}$ is nonempty and disconnected from X for some r.

Properties of Wide Opens

Let (W, X) be a wide open pair. By glueing disks over K to the components of $W \backslash X$, we obtain a proper separable smooth one dimensional rigid space. Therefore, by the p-adic Riemann Existence Theorem, we obtain a smooth complete algebraic curve.

Properties of Wide Opens

Let (W, X) be a wide open pair. By glueing disks over K to the components of $W \backslash X$, we obtain a proper separable smooth one dimensional rigid space. Therefore, by the p-adic Riemann Existence Theorem, we obtain a smooth complete algebraic curve. Using this one can show that

$$H^1_{DR}(W/K) := \Omega^1(W/K)/\mathrm{d}A(W)$$

is finite dimensional over K. Let

$$g(W) = \frac{1}{2}\dim_K(\ker(H^1_{DR}(W/K) \to H^1_{DR}((W \backslash X)/K))).$$

Note: If W is completed to a projective curve as above, it follows from a Mayer-Vietoris argument that $g(W)$ equals the genus of that curve.

Definition of **ord** at an End

Suppose $\mathcal{A} = (A, \mathrm{res})$ is an open annulus and residue map. If f is a function and ω a differential on A, each with no zeroes or poles on A, then we define $\mathrm{ord}_{\mathcal{A}} f = \mathrm{res}(\mathrm{d}f/f)$ and

$$\mathrm{ord}_{\mathcal{A}}\omega = \mathrm{ord}_{\mathcal{A}}(\omega/\mathrm{d}z),$$

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for any $z \in A(\mathcal{A})^*$ with $\text{ord}_{\mathcal{A}}\, z = 1$.

Now suppose ν is either a meromorphic function or differential on W, with finitely many zeroes and poles. Let X be an underlying affinoid containing the support of ν. Let A be the component of $W \backslash X$ around $e \in \mathcal{E}(W)$, and let $\mathcal{A} = (A, \text{res}_e)$. We set

$$\text{ord}_e\, \nu = \text{ord}_{\mathcal{A}}\, \nu.$$

Riemann-Hurwitz for Wide Opens

Suppose $f\colon W \to V$ is a finite map. Then f maps $\mathcal{E}(W)$ to $\mathcal{E}(V)$. For $a \in W^{+} := W(\mathbb{C}_p) \cup \mathcal{E}(W)$, let $b := f(a)$ and

$$\delta_f(a) = \mathrm{ord}_a(f^{*}\mathrm{d}T)/\mathrm{ord}_b\,\mathrm{d}T$$

where T is a parameter at b. There exist annuli $\mathcal{A}$ and $\mathcal{B}$ around a and b such that f restricts to a finite étale map from $\mathcal{A}$ onto $\mathcal{B}$. Let $e_f(a)$ be the degree of this map.

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When $\deg f < p$, $\delta_f(a) = e_f(a) - 1$. On the other hand, for $p = 3$

$$f\colon z \mapsto z^3 - 3^{1/2}/z$$

gives a map from $W := A(3^{1/8}, 3^{-1/8})$ to $A(3^{3/8}, 3^{-3/8})$. If a is the end of W where $|z|$ is near $3^{1/8}$, $\delta_f(a) = -2$.

Riemann-Hurwitz for Wide Opens

Theorem: Suppose $f\colon W \to V$ is a finite map of degree d. Then

$$2g(W) - 2 = d(2g(V) - 2) + \sum_{a \in W^+} \delta_f(a).$$

Semi-stable Coverings

Let C be a wide open or smooth proper curve over K. Let $\mathcal{C}$ be a finite collection of basic wide open pairs (U, U^u) over K such that $\mathcal{C}^w := \{U, (U, U^u) \in \mathcal{C}\}$ is an admissible covering of C.

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Every curve has a semi-stable covering (after finite base extension), i.e. one can make an arbitrary curve by glueing together wide open pieces of curves with good reduction along annuli.

Theorem: Suppose $\mathcal{C}$ is a semi-stable covering of a smooth proper curve C over K. Let $\Gamma_{\mathcal{C}}$ be the unoriented graph without loops, whose vertices correspond to the elements of $\mathcal{C}$, and whose edges with endpoints corresponding to distinct $U, V \in \mathcal{C}$ correspond to the connected components of $U \cap V$. Then

$$g(C) = \sum_{U \in \mathcal{C}} g(U) + \mathit{Betti}(\Gamma_{\mathcal{C}}).$$

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Proof: Again, this follows from a Mayer-Vietoris argument, using the definition and the cohomology of disks and annuli.

Semi-stable Covering vs Semi-stable Reduction