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I'ld like to give some credit to an anonymous referee for this talk. He or she was reviewing a paper on modular curves written by Ken Mcmurdy and myself. Several suggestions were made. In particular, we were referred to some advice for writers of mathematics given by JS Milne. Milne gives a set of "tips." From what I've read of Ken Ribet, I see he rarely follows Milne's tips. Unfortunately, we and many others do. For example, Milne says

2. Refer to another obscure paper for all the basic (nonstandard) definitions you use, or never explain them at all.

6. If, in a moment of weakness, you do refer to a paper or book for a result, never say where in the paper or book the result can be found. In addition to making it difficult for the reader to find the result, this makes it almost impossible for anyone to prove that the result isn't actually there.

Alternatively, call the result "well known" without giving any reference. This may make the reader too embarassed to question it. A variant of this, which some members of the audiance may have seen, is to say, "this would be a good exercise for a graduate student."

Now I'll talk about what we wanted to use and have now proven without using Milne's tips.

Wide Open Spaces

Notation. Fix a prime p. Let $\mathbf{C}_p$ be the completion of a fixed algebraic closure of $\mathbf{Q}_p$. Let $\mathbf{R}_p \subseteq \mathbf{C}_p$ be its ring of integers, $m_{\mathbf{R}_p} \subset \mathbf{R}_p$ its maximal ideal. Let $|\cdot|$ be the absolute value on $\mathbf{C}_p$ such that $|p| = p^{-1}$ and $\mathcal{R} = p^{\mathbf{Q}}$. Let K be a complete subfield of $\mathbf{C}_p$ with ring of integers R_K and residue field $\mathbf{F}_K$.

Some years ago, I realized that rigid analytic spaces obtained from complete curves by removing a non-empty finite set of closed disks had many good properties which were helpful in understanding the arithmetic geometry of curves.

Now, I'll give an intrinsic definition of these spaces.

A **wide open** W over K is a connected one dimensional smooth rigid space W over K which contains an affinoid subdomain X such that $W\backslash X$ is the union of finitely many non-empty open annuli $A_1, \ldots, A_n$. Moreover, in each A_i there exist open subannuli $B_i^1 \subset B_i^2$ such that $A_i \backslash B_i^1$ is connected, B_i^2 is disconnected from X and if $X_i = W\backslash \bigcup B_i^j$, X_i is an affiniod and X_2 is relatively compact in X_1.

Call (W, X) a wide open pair and X an **underlying affinoid** of W.

Examples.

Open disks and open annuli over K are wide opens over K, which contain closed disks and closed annuli as underlying affinoids.

In $X_0(p^n)$, let $SS(p^n)$ and $O(p^n)$, be the supersingular and ordinary loci.

Let TS be the subspace of $X(1)$ whose points are represented by E such that all the subgroup schemes of $E[p]$ over $\mathbf{R}_p$ of rank p are isomorphic.

$TS(p) \subset X_0(p)$, points (E, C) where E represents a point of TS

$TS(p^2) \subset X_0(p^2)$, (E, C) where $E/C[p]$ represents a point of TS and

$Z_i(p) \subset X_0(p)$, $i = 0$ or 1, (E, C) where $\overline{C}$ is isomorphic over $\mathbf{R}_p$ to exactly p^i other subgroup schemes of E.

Then $(SS(p), TS(p))$, $(Z_i(p), Z_i(p) \cap O(p))$, $(SS(p^2), TS(p^2))$ are potential wide open pairs.

Let $\mathcal{E}(W) = \varprojlim_Z (CC(W\backslash Z))$, where Z ranges over subdomains of W affinoid subdomains of W, be the set of **ends** of W. Then $\mathcal{E}(W) \cong W\backslash X$ for each underlying affinoid X of W. We call the component of $W\backslash X$ corresponding to an end e an **annulus around** e.

For each $e \in \mathcal{E}(W)$ and each annulus A around e, there is a natural continuous linear map $\operatorname{res}_{e,A}$ from $\Omega^1_{\mathcal{A}/K}(W)$ onto K such that $\operatorname{res}_{e,A}(\omega) = 1$ if $\omega = dv/v$ for some parameter v on A_L over an extension L of K such that $\{x \in \mathcal{A}\colon |v(x)| > r\}$ is nonempty and disconnected from X for some r. As $\operatorname{res}_{e,B} \circ \mathit{restriction} = \operatorname{res}_{e,A}$ if $B \subseteq A$, let's call these maps res_e.

A **basic wide open pair** over K is a wide open pair (W, Z) where Z has good reduction and the components of $W\backslash Z$ are isomorphic to annuli of the form $A_K(1, s)$ and Z is disconnected from $A_K(r, s)$ for $r > 1$.

Some Properties of Wide Opens

One knows in rigid geometry affinoids subdomains are open. But they are not in Berkovich's theory. However, the Berkovich spectrum of a wide open subspace of a curve is open in the Berkovich spectrum of a curve.

Let (W, X) be a wide open pair. By glueing disks over K to the components of $W\backslash X$, we obtain a proper separable smooth one dimensional rigid space. Therefore, by the p-adic Riemann Existence Theorem, we obtain a smooth complete algebraic curve.

Using this one can show that $H^1_{DR}(W/K) := \Omega^1(W/K)/\mathrm{d}A(W)$ is finite dimensional over K. Let

$$g(W) = \frac{1}{2}\dim_K(\ker(H^1_{DR}(W/K) \to H^1_{DR}((W\backslash Z)/K))).$$

Now suppose $\mathcal{A} = (A,\ \mathrm{res})$ is an open annulus and a residue map. If f is a function and ω a differential on A, each with no zeroes or poles on A, then we define $\mathrm{ord}_{\mathcal{A}} f =\ \mathrm{res}(\mathrm{d}f/f)$ and

$$\mathrm{ord}_{\mathcal{A}}\omega =\ \mathrm{ord}_{\mathcal{A}}(\omega/\mathrm{d}z),$$

for any $z \in A(\mathcal{A})^*$ with $\mathrm{ord}_{\mathcal{A}}\, z = 1$.

Suppose ν is either a meromorphic function or differential on W, with finitely many zeroes and poles. Let X be an underlying affinoid containing the support of ν. Let A be the component of $W\backslash X$ corresponding to $e \in \mathcal{E}(W)$, and let $\mathcal{A} = (A,\ \mathrm{res}_e)$. We set $\mathrm{ord}_e\,\nu =\ \mathrm{ord}_{\mathcal{A}}\,\nu$.

Let $W^+ = W(\mathbf{C}_p) \cup \mathcal{E}(W)$ and $\mathrm{Div}(W)$ denote the set of maps from W^+ to $\mathbf{Z}$ with finite support. For any $D \in\ \mathrm{Div}(W)$ let

$$\deg D = \sum_{P \in W^+} D(P).$$

Theorem. *Let f be a rigid function and ω a differential on W, each with finitely many poles and zeroes in $W(\mathbf{C}_p)$. Then (i)* $\deg(f) = 0$, *(ii)* $\deg(\omega) = 2g(W) - 2$, *and*

$$\sum_{P \in W^+} res_P\, \omega = 0. \qquad (iii)$$

Suppose $f\colon W \to V$ is a finite map. Then f maps $\mathcal{E}(W)$ to $\mathcal{E}(V)$. For $a \in W^+$, let $b := f(a)$ and

$$\delta_f(a) = \operatorname{ord}_a(f^*\mathrm{d}T)/ \operatorname{ord}_b \mathrm{d}T$$

where T is a parameter at b. There exist annuli $\mathcal{A}$ and $\mathcal{B}$ around a and b such that f restricts to a finite étale map from $\mathcal{A}$ onto $\mathcal{B}$. Let $e_f(a)$ be the degree of this map.

When $\deg f < p$, $\delta_f(a) = e_f(a) - 1$. On the other hand, $f\colon z \mapsto z^3 - 3^{1/2}/z$ gives a map from $W := A(3^{1/8}, 3^{-1/8})$ to $A(3^{3/8}, 3^{-3/8})$. If a is the end of W where $|z|$ is near $3^{-3/8}$, $\delta_f(a) = -2$.

Theorem. *Suppose $f\colon W \to V$ is a finite map of degree d. Then*

$$2g(W) - 2 = d(2g(V) - 2) + \sum_{a \in W^+} \delta_f(a).$$

Semistable Covers

Let C be a wide open or a smooth proper curve over K. Let $\mathcal{C}$ be a finite collection of basic wide open pairs (U, U^u) over K such that $\mathcal{C}^w := \{U, (U, U^u) \in \mathcal{C}\}$ is an admissible covering of C. Then we call $\mathcal{C}$ a **semi-stable covering** over K if:

(i) if $U, V \in \mathcal{C}^w$ and $U \neq V$, $U \cap V$ is a disjoint union of connected components of $U \backslash U^u$,

(ii) if U, V and W are three distinct elements of $\mathcal{C}^w$, $U \cap V \cap W = \emptyset$.

Theorem. *Every wide open or smooth proper curve over K has a semi-stable cover over a finite extension of K.*

Suppose $\mathcal{C}$ is a semi-stable covering of a smooth proper curve C over K. Let $\Gamma_{\mathcal{C}}$ be the unoriented graph, whose vertices correspond to the elements of $\mathcal{C}$, and whose edges with endpoints are distinct and those with endpoints $U, V \in \mathcal{C}$ correspond to the connected components of $U \cap V$.

Proposition.

$$g(C) = \sum_{U \in \mathcal{C}} g(U) + Betti(\Gamma_{\mathcal{C}}).$$

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