Action of Inertia on Modular Curves

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Suppose K is the completion of the MUR of a complete local field. If A is an Abelian variety over K, by Mumford there exists a finite Galois extension L of K over which A has "semi-Abelian reduction," i.e. there is a model $\mathcal{A}_L$ of $A \otimes L$ over R_L whose reduction is an extension of an Abelian variety by an affine torus. This model is unique, if it exists. In particular, if $\sigma \in I(K) =: \mathrm{Gal}\,\bar{K}/K$ we have an isomorphism $\mathcal{A}_L^\sigma \to \mathcal{A}_L$ and so a homomorphism

$$w_A\colon I(K) \to \text{ Aut } \overline{\mathcal{A}}.$$

Theorem. *The homomorphisms w_A and $w_{\hat{A}}$ are both trivial if and only if A has semi-Abelian reduction over K.*

Theorem. *A has semi-Abelian reduction over $K(A[\ell](\bar{K}))$, $p \neq \ell > 2$.*

How do you compute w_A?

Suppose A is the Jacobian of a curve C. Then A has semi-Abelian reduction over L if and only if C has "semi-stable model" over R_L, i.e., a model over R_L whose reduction has at worst normal crossings. If the dimension of A is at least 2 the category of semi-stable models for C over L has a final object $\mathcal{C}_L$, if it is non-empty. This is called the **stable** model of C (over L). We have $Pic\mathcal{C} \cong \mathcal{A}$

$$w_C\colon I(K) \to \text{ Aut } \overline{\mathcal{C}},$$

and the obvious diagram commutes.

Modular Curves

Suppose $p > 3$. Let $X_0(p^3)$ and $J_0(p^3)$ its Jacobian. Let C_p be the completion of the curve $y^2 = x^p - x$ over $\mathbf{F} = \overline{\mathbf{F}}_p$.

Theorem. *Then the semi-Abelian reduction of $J_0(p^3)^{new}$ is isogenous to the product of $(p^2-1)/6$ copies of $Jac(C_p)$.*

Question. *Is $J_0(p^3)^{new} \cong Jac(C_p)^{(p^2-1)/6}$?*

What lies behind this is

Theorem. *The "new part" of the stable reduction of $X_0(p^3)$ is $(p^2-1)/6$ copies of C_p, each attached at ∞ to the "old part."*

Inertial Action

Let K be the MUE of $D =: \mathbf{Q}(\{\sqrt{D} : D \in \mathbf{Q}_p\})$. We get a map

$$w : I(K) \to \big(\ \mathrm{Aut}(y^2 = x^p - x)^{(p^2-1)/6}\big).$$

In fact, w factors through a map

$$I(K)^{ab} \to \big(\ \mathrm{Aut}(y^2 = x^p - x)\big)^{(p^2-1)/6}.$$

Quaternion Action

Now suppose $p = 13$. Let E be the ss elliptic curve $\bmod p$. Then, $\mathrm{End}\hat{E} \cong B := \mathbf{Z}_p[i,j,k]$, where $i^2 = -2$, $j^2 = -p$, and $ij = -ji = k$..

Then B^* acts on the residue class $U \cong A(|p|, 1)$ of $X_0(p)$ above E. Let W be the AL inv. of $X_0(p)$. Then if $\rho \in \mathbf{Z}_p[i]^*$, $w_\rho =: \rho W$ is an involution of U.

All these automorphisms preserve the central circle S in U and hence act on its reduction $\bar{S} \cong \mathbf{G}_m$.

The curves above correspond to fixed points of some $\bar{w}_\rho$. There are two kinds of ρ. If $x \in \bar{S}$ is a fixed point of w_ρ where $\rho \in \mathbf{Z}_p[i]^*$ and C_x the corresponding affine curve.

Theorem. *The map* $\mathcal{O}_D^* \to Aut(C_x)$, $b \mapsto (w \circ Art(b))|_{C_x}$ *factors through* $N_{D/F}$ *where* $F = \mathbf{Q}_p(\sqrt{-p\rho\bar{\rho}})$ *and has image of order equal to* $2p$ *and its kernel contains* $1 + p\mathcal{O}_K$.

Corollary. *Any field of semi-Abelian reduction of* $J_0(p^3)^{new}$ *contains the cyclic extension* $S(D)$ *of degree* p *of* $L =: \mathbf{Q}_p(\sqrt{-pD})^{nr}$ *for any* $D \in \mathbf{Z}_p^*$ *corresponding to the group* $\mathbf{Z}_p^* + p\mathcal{O}_L$.

Corollary. *There exists a weight 2 new form on* $X_0(p^3)$ *whose representation wildly ramified at* p.

Now

$$J_0(p^3)^{new} \sim \prod_f A_f$$

where f runs over the normalzed new forms on $X_0(p^3)$ and Jared told me that the minimal ext. $M(f)$ of K over which A_f has good reduction only contains one of the two fields $S(D)$. Given the above theorem I guessed and Jared proved

$$\sum_{M(f)\supseteq S(1)} \dim A_f = \sum_{M(f)\supseteq S(D)} \dim A_f \quad \text{for } D \in \mathbf{Z}_p^*.$$

For $p = 13$, the dimensions of the A_f are 36, 42, 42, 48.

Conjecture. *Then the semi-Abelian reduction of*

$$\prod_{M(f)\supseteq S(D)} A_f$$

is isogenous to the product of $(p^2 - 1)/12$ *copies of* $Jac(C_p)$.