Spectrum, Dimension, and Polymer Analogies in Fluid Turbulence

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The relation between fractal dimension and inertial exponents in three-dimensional incompressible fluid turbulence is analyzed. Several polymer analogies are explained, and an application to the numerical modeling of turbulence is presented.

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It has long been noted that fully developed turbulence in a three-dimensional fluid has fractal properties,[1-3] and that the stretched vortex structures that are its carriers resemble polymers.[4,5] However, the quantitative relations between vortices, polymers, and their dimensions and characteristic exponents have not been properly understood and as a result the qualitative insight has not led to significant progress in turbulence modeling and computation.

The energy spectrum $E(k)$ of homogeneous turbulence is calculated by the integration of the Fourier transform of the trace of the velocity correlation tensor over the sphere of radius $k=|\mathbf{k}|$, where $\mathbf{k}$ is the wave vector dual to the separation $\mathbf{r}$; the mean energy at a point $\frac{1}{2}\langle\mathbf{u}^2\rangle$, where the angular brackets denote an average and $\mathbf{u}$ is the velocity, equals $\frac{1}{2}\int_0^\infty E(k)dk$. Similarly, if $\xi=\mathrm{curl}\mathbf{u}$ denotes the vorticity, then $\langle\xi^2\rangle=\int_0^\infty Z(k)dk$, where $Z(k)$ is the vorticity spectrum. An elementary calculation[6] yields $Z=k^2E$ for homogeneous incompressible flow. The inertial range is the range of scales k^{-1}, small compared with the scales on which the fluid is stirred but large compared to dissipation scales. This range is the locus of an energy cascade from large to small scales, generated by vortex stretching. It is generally believed that in the inertial range $E(k)\sim k^{-\gamma}$, where γ is the universal inertial exponent. Kolmogorov obtained for γ the value $\gamma=\frac{5}{3}$; a derivation of this result on the basis of a plausible cartoon of the equations of motion can be found in Kraichnan.[7]

Vortex tubes stretch nonuniformly, and it can readily be seen that the portions that are stretched most contribute most to ξ^2. Thus a small portion of the volume available to the flow contains more than its share of vorticity. Consider a single realization of the flow, pick out a finite portion V of the volume it occupies, and define the ϵ support of the vorticity to be the smallest set Λ in V such that

$$\int_\Lambda \xi^2 dV \geq (1-\epsilon)\int_V \xi^2 dV.$$

Assume that the flow is described by Euler's equations, which are appropriate in the inertial range.[8] It is plausible and compatible with numerical results and available theory to assume that there exists a time T such that for $t>T$ the limit $D=\lim_{\epsilon\to 0}\dim\Lambda$ exists with probability 1, where $\dim(\cdots)$ denotes the Hausdorff dimension. D is the universal fractal dimension of turbulence. The limiting set $\lim_{\epsilon\to 0}\Lambda$, if it exists, is the essential support of the vorticity. The usual guess for D is $D\simeq 2.5$ (see, for example, Ref. 1). I shall be assuming in this Letter that vortex tubes remain approximately tubes as they stretch, and I shall ignore the added complexity that arises when sheetlike structures form as vortex tubes approach each other.[9] The evidence that this is a legitimate approximation is quoted, e.g., in Ref. 5.

One may well wonder whether there is a logical relation between γ and D, and further whether there is a functional relation between them. It may well be that there exists a single pair (γ,D) typical of Euler's equations, rather than a function $\gamma=\gamma(D)$. In the polymer problem I shall describe below a function $\gamma(D)$ can indeed be defined, and provides qualitative information about the hydrodynamical situation. A relation between γ and D has been derived in Refs. 1 and 2. The derivation contains the correct observation that as the energy cascades to high k, the "active" (presumably highly vortical) portion of the volume shrinks. If the "active volume" shrinks by $\beta<1$ when the scale is halved, then $\beta=2^{D-3}$. This observation, together with other assumptions, leads to the relation

$$\gamma=\tfrac{5}{3}+\tfrac{1}{3}(3-D). \tag{1}$$

Equation (1) is widely quoted, but it cannot be right: (i) One expects $d\gamma/dD>0$, contrary to what happens in Eq. (1), for as D decreases, the support of ξ decreases, ξ becomes more singular, it acquires a longer spectral tail, and γ decreases. (ii) The assumption that $\gamma=\frac{5}{3}$ when $D=3$ is false: if $D=3$, ξ is well behaved, $\langle[\xi(\mathbf{x}+\mathbf{r})-\xi(\mathbf{x})]^2\rangle$ remains bounded[10] as $r=|\mathbf{r}|\to 0$, and a Fourier transform yields $\gamma\geq 3$; conversely, we shall see below that if there is an energy cascade $D<3$, and thus $D<3$ when $\gamma\simeq\frac{5}{3}$. Note that these remarks are self-consistent: If $\gamma\geq 3$ when $D=3$, and $d\gamma/dD>0$, then it is possible that $\gamma(D=2.5)=\frac{5}{3}$. (iii) The polymer analogy will show that γ and D should depend on the integral constraints that are obeyed by the motion: conservation of volume, helicity, circulation, connectivity of vortex

lines, and energy. These do not appear in the argument that leads to Eq. (1).

The simplest polymer analogy is as follows: Consider a linear polymer in a solvent. The mean end-to-end length r of a polymer chain with N bonds behaves as $r \sim N^{1/\mu}$; μ depends on the constraints imposed on the chain.[11] For a free Gaussian chain (i.e., an arc of Brownian motion) $\mu = 2$; for a chain constrained to avoid itself $\mu = \frac{5}{3}$ (the inverse μ^{-1} of this value of μ is known as the "Flory exponent"). Once one knows how many bonds live in a sphere of radius r, one can find how many live between r and $r+dr$, and thus the two-point correlation function of the bond density can be determined; for small r it scales as $r^{\mu-3}$, and its Fourier transform for large k behaves as $k^{-\mu}$. On the other hand, the Hausdorff dimension of the chain[12] is μ. If one thinks of the chain as being analogous to a vortex tube, the analog of Z is produced by an integration of the Fourier transform over the sphere of radius k that adds a factor $\sim k^2$, and the analog of E is then obtained by a division by k^2. Thus $E(k) \sim k^{-\gamma}$ where $\gamma = D$; D equals dimensions of the chain. Clearly, $d\gamma/dD > 0$. Note that γ and D are affected by the constraint of self-avoidance.

For a vortex system things are a little more complicated since ξ, the vorticity, is a vector quantity. If $D = 3$, we have seen that $\gamma \geq 3$. If the essential support of the vorticity is a collection of arcs of Brownian motion, $D = 2$, $\langle \xi(\mathbf{x}+\mathbf{r})\xi(\mathbf{x})\rangle = \langle \xi^2 \rangle \delta(\mathbf{r})$ since the increments of Brownian motion are independent and ξ is tangent to its support. A Fourier transform then yields $\gamma = 0$. (A note of caution: We do not know, even for tube or filament-like supports, that D determines γ uniquely.) The pairs $(D = 3, \gamma \geq 3)$, $(D = 2, \gamma = 0)$ are compatible with $d\gamma/dD > 0$ and the usually accepted pair $(D \simeq 2.5, \gamma = \frac{5}{3})$ interpolates nicely between them.

The polymer analogy shows that it is reasonable to think of a functional relation $\gamma = \gamma(D)$. The shift in the graph of $\gamma(D)$ [for example, $\gamma(2) = 2$ in the polymer case, $\gamma(2) = 0$ in the vortex case] is due to the vector nature of ξ, as can best be seen in the case $D = 2$, where the vorticity correlation vanishes for $r \neq 0$ because the directions of ξ are independent, while the (scalar) density correlation in the polymer case remains nonzero. Note further that the relation $\gamma = \gamma(D)$ does not determine the values of D and γ; these depend on the constraints which will be different in the two cases.

A further relation between polymer theory and vortex theory is revealed by the random-vortex approximation.[13,14] The motion of vortex tubes and filaments is determined by a set of equations that resemble the Kirkwood equations of polymer theory.[15] The hydrodynamic interaction between distant bonds is described in the vortex case by the Biot-Savart rather than the Stokes kernel, the force between the nearby portions of a vortex is the force required to stretch the vortex, and the Brownian bombardment of the polymer by the molecules of the solvent is replaced by a Brownian motion with variance proportional to viscosity. One can view a vortex in an equilibrium range such as the inertial range as being in approximate thermal equilibrium with a potential flow, at a temperature proportional to the viscosity. Thus the comparison between the behavior of exponents relating to polymers in thermal equilibrium and the behavior of inertial exponents in turbulence is quite reasonable.

More useful and more quantitative results require a more detailed examination of the inertial range. Remember that the energy associated with a single realization of an incompressible turbulent flow is[16]

$$T = \tfrac{1}{2}\int \mathbf{u}^2 dV = \frac{1}{8\pi}\int dV \int dV' \frac{\xi(\mathbf{x})\cdot\xi(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \qquad (2)$$

where $\mathbf{x}$ is the coordinate vector. We shall need the following scaling property of (2)[17]: If all spatial dimensions are multiplied by a factor $\alpha > 0$, and the vorticity is scaled so that the circulation is invariant, then T is multiplied by α. Indeed, each of the six space dimensions is multiplied by α; each ξ is multiplied by α^{-2} by conservation of circulation, and the denumerator $|\mathbf{x}-\mathbf{x}'|$ adds a final factor α^{-1}. Assume that the vorticity is contained in a tube of some radius ρ; then one can write $T = T_1(\rho) + T_2(\rho)$, where $T_1(\rho)$ is the contribution of those parts of $\mathbb{R}^3 \times \mathbb{R}^3$ where $|\mathbf{x}-\mathbf{x}'| < \rho$, and $T_2(\rho)$ is the remainder. If ξ points in a fixed direction in the tube, then $T_1(\rho) > 0$ while $T_2(\rho)$ can change sign.

The energy cascade is generated through the stretching of vortex tubes; energy appears in ever smaller scales because the vortex radii and the radii of curvature shrink. Suppose a specific scale has been reached. Smaller scales will be generated through the stretching of some portion of the existing vortex tube by a factor L. (To simplify the discussion, we take this factor to be uniform along the stretching portion of the tube—an inessential assumption). If $L = 1$ the cascade stops; if $L > 1$ the tube fractalizes. The radius of the tube decreases by a factor $\sqrt{L}$. Compare $T_2(\rho/\sqrt{L})$ after the stretching with $T_2(\rho)$ before the stretching. $T_2(\rho/\sqrt{L})$ increases by a factor L^2/a, where a is the ratio of the old to the new average distances between pieces of the tube. Indeed, the support of ξ is L times longer, $|\mathbf{x}-\mathbf{x}'|$ is a times shorter, and for $|\mathbf{x}-\mathbf{x}'|$ not very small the contributions of the vorticity in the tube to (2) depend on its circulation but only marginally on its cross section.[16] Since presumably T_2 is dominated by nondistant interactions it is plausible to assume at first $a \sim L$ and thus T_2 scales like L. $T_1(\rho/\sqrt{L})$ is made up of $L\sqrt{L}$ pieces, each $L^{-1/2}$ times smaller than the pieces that made up $T_1(\rho)$, and thus T_1 also scales like L. By conservation of energy, $T_1 + T_2$ must remain uniformly bounded and thus T_2 must become negative to cancel the growth in T_1—this is the origin of vortex folding. However, T_2 contains a positive part—the radius of curvature of the tube decreases *a priori* like L^{-1} and thus $\sqrt{L}$ of the pieces that

make up T_1 will be aligned and make a positive contribution to T_2. To keep T_1+T_2 bounded one must assume that a is smaller than L and/or the radius of curvature decreases faster than L^{-1}, and thus the tube folds into an ever smaller portion of the available volume, characterized by a dimension D'. Clearly $D' \geq D$, and it is plausible that $D'=D$. The condition that the cascade be allowed to proceed indefinitely defines an implicit equation for D' that remains to be solved. However, in a simplified lattice model of vortex motion, where the vortex tubes are stretched at random subject to all the global constraints, a numerical calculation[18] has yielded $D' \simeq 2.4$. It is comforting that $D' > 2$, since for $D=2$ I found above $\gamma=0$ and thus an infinite energy per unit volume.

The energy expression (2) explains qualitatively why in the polymer case $D < 2$ while in the vortex case $D > 2$. In the polymer case the only constraint imposed is self-avoidance, which prevents folding, straightens out the polymer, and decreases D. In the vortex case the energy must remain bounded and thus as stretching occurs $\xi(\mathbf{x})$ and $\xi(\mathbf{x}')$ must sometimes point in opposing directions; thus folding is encouraged and D increases. In the framework of the analogy between the Kirkwood equations and the random-vortex equations, the difference resides in the difference between the Stokes and Biot-Savart kernels. The Stokes kernel imposes no constraint on the shape of the polymer, while the Biot-Savart kernel leads to the expression (2) for the energy.

The bunching of vortex lines characterized by D' is more important in practice than the mere fractalization. It suggests that energy moves across the inertial range by the progressive tightening of vortex bundles, down to the scales where viscosity can cancel tubes of opposing rotation. This model can be implemented numerically. Vortex calculations become expensive when the number of elements grows as a result of stretching and folding, and their accuracy decreases as the ratio of smallest scale to time step decreases.[8] However, the bunching and subsequent cancellation can be imposed on the calculation and results in an efficient renormalization of the calculation—the numerical details will be presented elsewhere.[19] Note that if one views vortex lines as propagators in physical space, as is done in polymer theory,[12] then this renormalization amounts to replacing a bare propagator by an effective propagator in which a summation over irreducible loops had been carried out.

I would like to end with a little speculation. One may wonder whether it is mere coincidence that the Kolmogorov value $\gamma=\frac{5}{3}$ is the inverse of the Flory exponent that corresponds to a self-avoiding random walk. One may argue that the constraints of conservation of energy and circulation create tight, renormalizable bundles of vorticity with radii near dissipation scales, and thus the more global behavior of vortex tubes is determined by the remaining constraint of conservation of helicity (the fluid analog of self-avoidance[20]). Since ξ is a vector, one still has to show that the directions of $\xi(\mathbf{x})$ and $\xi(\mathbf{x}+\mathbf{r})$ are correlated when $r=|\mathbf{r}|$ is on that more global scale and $\mu=\frac{5}{3}$. This is likely, since when $\mu=1$ the arcs are smooth and the correlation is $\simeq 1$ while the correlation is 0 when $\mu=2$, as I have shown above.

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