# ON QUANTUM GROUPOIDS

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Abstract. This note presents some recent results on the new notion of quantum groupoid from both the perspective of Poisson geometry and Operator algebras. We shall only briefly present the major results known so far on these new objects, together with some preliminary definitions on groupoids, Poisson geometry, and operator algebras.

"The knowledge at which geometry aims is the knowledge of the eternal. " - Plato, Republic, VII, 52.

## 1. Introduction

Quantum groupoids are objects which are hoped to improve our understanding of non-commutative algebras as geometric objects, like groupoids provide us with tools in the commutative case.

An important direction of study for non-commutative algebras is summarized by an allegory: a non-commutative algebra is the algebra of (complex valued) functions on some quantum space which would not be described within set theory. To an extend, this impossibility to see the quantum space is the ultimate expression of the Heisenberg's uncertainty principle: mathematically, it translates into the non-commutativity of the observable of the quantum systems, but in fact expresses our inability to have access to the quantum space itself, but only through measurements of limited precision. On the other hand, the enunciated allegory is there to state that the quantum space, if inaccessible, still exists. Thus, one tries to study it through its algebra of observable, starting by exporting to it the understood notions of the classical case.

An issue is to understand how complementary structures on a classical space could be extended to quantum space. The case of (topological) groups has been extensively studied for the last two decades, leading to the well-known quantum groups. Essentially, the multiplication, the neutral element and the inverse of the group lift to the algebra of continuous functions into a comultiplication $\Delta$, an antipode $S$ and a counit $\varepsilon$, turning this algebra into a Hopf algebra (cf [8], [1],[7]):

**Definition 1.** *A* Hopf algebra $(A,\Delta,S)$ *is a unital associative algebra $A$ together with:*

- *A unital monomorphism $\Delta : A \to A\otimes A$ which is* coassociative*:*

$$(\Delta\otimes Id)\circ\Delta = (Id\otimes\Delta)\circ\Delta$$

*Key words and phrases.* Quantum groupoid, formal deformation-quantization, symplectic manifolds, Poisson manifolds.

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*This is only the dual property of the associativity of the multiplication of the group. Note also that the tensor product, in general, has to be completed for some norm on A which is, in the case of the space of continuous functions on a locally compact group, a $C^*$–algebra. But modulo this completion, the comultiplication on $C(G)$ is given by: $\forall f \in C(G)\quad \Delta(f) = [(x,y) \in G^2 \mapsto f(xy)]$.*

- *An (involutive) unital antiautomorphism $S : A \to A$, such that:*

$$\sigma \circ \Delta \circ S = (S \otimes S) \circ \Delta$$

*where $\sigma(x \otimes y) = y \otimes x$ for all elementary tensors and is extended by linearity.*

*For a group, we just set: $S(f) = [g \longmapsto f(g^{-1})]$.*

*A* counital Hopf algebra[1] *$(A, \Delta, S, \varepsilon)$ is a Hopf algebra $(A, \Delta, S)$ together with the* counit *$\varepsilon$ satisfying:*

- *$\varepsilon$ is a unital algebra homomorphism from A into $\mathbb{C}$,*
- *$(\varepsilon \otimes Id) \circ \Delta = (Id \otimes \varepsilon) \circ \Delta = Id$,*
- *$\forall a \in A\quad [\mu \circ (S \otimes Id) \circ \Delta](a) = [\mu \circ (Id \otimes S) \circ \Delta](a) = \varepsilon(a)a$, where $\mu$ is the multiplication of A.*

*In this last case, the assumption that $\Delta$ is injective could be relaxed as it is immediately implied by the existence of a counit.*

In the case of these richer algebraic structure, the fundamental analogy we started with is needing enlargement. Indeed, a (compact) Lie group has three natural algebras associated with it: apart from the algebra of continuous functions, one has the group algebra, and the universal enveloping algebra of its Lie algebra (other algebras can be considered, as the algebra of smooth functions...). The two first algebras are dual from each other when the original group is commutative (and only locally compact), and a whole theory has been developed (cf [18]) to extend this duality to some extend to non-commutative topological locally compact groups. The first one is always commutative, but the second one is commutative if, and only if the group is. However, via the duality theory of Takesaki, there is a natural comultiplication on the group algebra, which is co-commutative, meaning that it is valued in the symmetric square-tensor product of the group algebra. Moreover, all cocommutative counital (C*−)Hopf algebras are group algebras of some locally compact groups. Thus, the extra structure of groups allow us to use other generalization of classical space into quantum space: for instance, consider a non-cocommutative algebra as "group algebra" for some quantum group. This theory is all the more satisfactory as indeed the theory in [18] applies even to these quantum spaces.

A different approach is to consider the universal enveloping algebra. The universal enveloping algebra of a Lie algebra of a Poisson group is naturally endowed with a structure of Hopf algebra (but not C* !) coding the Poisson structure - the other operations of the group having been used to defined the algebra itself. This last object is of great importance for the theory of quantum groups. In this case, one can see this algebra as coding the notion of (right invariant) differential operators on the original Poisson groups, providing thus an encoding of the differential structure of the group.

[1] Many authors actually call Hopf algebra what we named counital Hopf algebra. But in fact, other similar definitions can be found. We retain this one for the convenience of the exposition.

The quantum groupoids have appeared as a generalization of quantum groups, in the diverse meanings this notion covers. In these notes, we shall give a brief account of these developments. A first path of interest focuses on a generalization of abstract harmonic analysis to groupoids, and includes the study of the algebra of functions on a (quantum) groupoid. Another path focuses on the deformation of a universal enveloping algebra of the Lie algebroid of a Poisson groupoid: each time, the suffix "oid" refers to a generalization we will define further.

Our first part will be concerned with a brief introduction to the notion of groupoids, then include some definitions of Poisson geometry, which we shall use while presenting Poisson groupoids. Our second section will present some versions of the algebras of functions on quantum groupoids, and of the quantum groupoids algebras, in the framework of Von Neumann algebras theory. At last, our last part will be concerned with the deformation of the universal algebra of a Lie algebroid, as another approach to quantum Poisson groupoids.

## 2. Groupoids and Poisson Geometry

2.1. **The notion of groupoids.** A groupoid is a small category whose all arrows are invertible (cf [2]). More explicitly, referring to [15]:

**Definition 2.** *A groupoid is a set $\mathcal{G}$ endowed with a function $\circ : \mathcal{G}^{(2)} \to \mathcal{G}$, where $\mathcal{G}^{(2)}$ is a subset of $\mathcal{G}$, such that:*

1. *Associativity:*

$$\forall ((x,y),(y,z)) \in (\mathcal{G}^{(2)})^2 \quad ((x\circ y, z),(x, y\circ z)) \in (\mathcal{G}^{(2)})^2 \text{ and } (x\circ y)\circ z = x\circ(y\circ z)$$

2. *Existence of inverse:*
   (a) $\forall x \in \mathcal{G}\ \ \exists x^{-1} \in \mathcal{G}\ \ ((x,x^{-1}),(x^{-1},x)) \in \mathcal{G}^{(2)}$,
   (b) $\forall x \in \mathcal{G}\ \ (x^{-1})^{-1} = x$,
   (c) $\forall (x,y) \in \mathcal{G}^{(2)}\ \ x^{-1}\circ(x\circ y) = y$,
   (d) $\forall (y,x) \in \mathcal{G}^{(2)}\ \ (y\circ x)\circ x^{-1} = y$.

*The set $\mathcal{G}^{(2)}$ is called the set of composable pairs. The groupoid will be denoted $(\mathcal{G},\circ)$ or even $\mathcal{G}$ if no confusion arises.*

Let us clarify the relation between this definition and the definition in terms of category. Let us introduce the following maps:

**Definition 3.** *Given a groupoid $(\mathcal{G},\circ)$, we define the* source map $s : \mathcal{G} \to \mathcal{G}$ *and the* range map $r : \mathcal{G} \to \mathcal{G}$ *by*

$$s : x \longmapsto x^{-1}\circ x,$$
$$r : x \longmapsto x \circ x^{-1}.$$

*Denote by $\mathcal{G}^{(0)} = s(\mathcal{G})$, called the* set of units *of $\mathcal{G}$.*

We obtain now the equivalence between the definition 2 and the category interpretation of groupoids:

**Proposition 1.** • *Given a groupoid $(\mathcal{G},\circ)$, then:*
  – $\mathcal{G}^{(0)} = r(\mathcal{G})$,
  – $\mathcal{G}^{(2)} = \{(x,y) \in \mathcal{G}^2 : s(y) = r(y)\}$

  - *$\mathcal{G}$ is the set of homomorphisms of some small category whose set of objects is $\mathcal{G}^{(0)}$; more precisely:*
    * *Any element of $\mathcal{G}$ is a morphism from $s(x)$ into $r(x)$,*
    * *The product in $\mathcal{G}^{(2)}$ is the composition of the corresponding maps.*
    * *The inverse in $\mathcal{G}$ is the inverse for the corresponding maps. In particular, any morphism for the category thus defined is an isomorphism.*
- *Conversely, given a small category whose all morphisms are isomorphisms, then the set of homomorphisms of the category (where the objects are identified with their identity isomorphisms), together with the composition, is a groupoid.*

For this reason, a groupoid $(\mathcal{G},\circ)$ is sometimes said to be a groupoid *over* $\mathcal{G}^{(0)}$. More generally, a groupoid gives rise to the following hierarchy of sets:

$$\begin{aligned}
\mathcal{G}^{(0)} &= s(\mathcal{G}),\\
\mathcal{G}^{(1)} &= \mathcal{G},\\
\mathcal{G}^{(2)} &= \{(x,y)\in\mathcal{G}^2 : s(y)=r(x)\},\\
\mathcal{G}^{(3)} &= \{(x,y,z)\in\mathcal{G}^3 : (x,y)\in\mathcal{G}^{(2)}, (y,z)\in\mathcal{G}^{(2)}\},\\
&\ \vdots\\
\mathcal{G}^{(n)} &= \{(x_i)_{i=1,\ldots,n} : (x_i)_{i=1,\ldots,n-1}\in\mathcal{G}^{(n-1)}, (x_{n-1},x_n)\in\mathcal{G}^{(2)}\} \text{ for } n\geq 2.
\end{aligned}$$

We shall introduce a couple more definitions:

**Definition 4.** *Let $(\mathcal{G},\circ)$ be a groupoid. For any $x\in\mathcal{G}$, $s^{-1}(x)$ is called the* source-fiber of $x$ *and $r^{-1}(x)$ is called the* range-fiber of $x$.

**Definition 5.** *A* subgroupoid *$\mathcal{G}'$ of a groupoid $(\mathcal{G},\circ)$ is a subset of $\mathcal{G}$ closed under the multiplication and the inverse operation in $\mathcal{G}$. Moreover, if $\mathcal{G}'^{(0)}=\mathcal{G}^{(0)}$, then the subgroupoid $\mathcal{G}'$ is said to be* wide.

The class of groupoids is indeed a category, once one define a homomorphism of groupoids to be a covariant functor (in the category framework). In the set-theoretic language used throughout these notes, this translate into:

**Definition 6.** *Let $(\mathcal{G}_1,\circ_1)$ and $(\mathcal{G}_2,\circ_2)$ be two groupoids. Then a map $\varphi:\mathcal{G}_1\rightarrow\mathcal{G}_2$ is a homomorphism of groupoids when:*

$$\forall (x,y)\in\mathcal{G}_1^{(2)}\ \ (\varphi(x),\varphi(y))\in\mathcal{G}_2^{(2)} \text{ and } \varphi(x\circ_1 y)=\varphi(x)\circ_2\varphi(y).$$

Let us now see some important examples of groupoids.

**Example 1.** *A* group *is a groupoid whose set of units is reduced to one element. This unit is of course the neutral element of the group. Alternatively, a group is a non-empty groupoid $\mathcal{G}$ such that $\mathcal{G}^{(2)}=\mathcal{G}$.*

**Example 2.** *The* trivial groupoid over any set $X$ *is the groupoid $\mathcal{G}$ such that $\mathcal{G}=X$ and $\mathcal{G}^{(2)}=X^2$, together with the multiplication defined by:*

$$\forall x\in X\quad x\circ x = x.$$

**Example 3.** *Given any set $X$ endowed with an* equivalence relation $\sim$*, we define a groupoid structure over $X$ by letting $\mathcal{G}_\sim = \{(x,y) \in X^2 : x \sim y\}$ together with the multiplication:*

$$\circ : \{((x,y),(y,z)) \in \mathcal{G}_\sim^2 : (x,y,z) \in X^3\} \longrightarrow \mathcal{G}_\sim$$
$$((x,y),(y,z)) \longmapsto (x,z).$$

*This multiplication is well-defined by the transitivity of the equivalence relation, which also gives the associativity property. Now, the symmetry of $\sim$ ensures that for all $(x,y)$ in $\mathcal{G}_\sim$, $(y,x)$ is also in $\mathcal{G}_\sim$ and the latest is clearly the inverse of the former. At last, the reflexivity is not required to define the structure; the set of units of this groupoid is the set of elements $x$ of $X$ such that $x \sim x$, so when $\sim$ is reflexive then $\mathcal{G}^{(0)} = \{(x,x) : x \in X\} \simeq X$. Note that for any $(x,y) \in \mathcal{G}_\sim$, $s((x,y)) = x$ and $r((x,y)) = y$. We shall name this groupoid the* groupoid graph *of the relation $\sim$.*

In general, the set of units of a groupoid is always endowed with a canonical equivalence relation:

**Definition 7.** *Let $(\mathcal{G}, \circ)$ be a groupoid. We define on $\mathcal{G}^{(0)}$ the equivalence relation $\sim$ by:*

$$\forall (u,v) \in \mathcal{G}^{(0)} \quad u \backsim v \Longleftrightarrow s^{-1}(u) \cap r^{-1}(v) \neq \emptyset.$$

*The equivalence class of $u$ in $\mathcal{G}^{(0)}$ is called the* orbit of $u$ *in $\mathcal{G}$.*

It is easy to check that, in example 3, with the identification $X \simeq \mathcal{G}_\sim^{(0)}$, one recovers the original equivalence relation thank to the previous definition.

**Definition 8.** *A groupoid is said transitive when its unit set has only one orbit in $\mathcal{G}$.*

These notions are especially well illustrated by the following example of groupoids:

**Example 4.** Action of a group. *Let $G$ be a (multiplicative) group acting on a set $E$ on the right - the action being denoted by $\cdot$. Then, if $(u,s)$ and $(v,t)$ are elements of $E \times G$, we declare them composable if $v = u \cdot s$, and then we set:*

$$(u,s) \circ (v,t) = (u, st).$$

*We then have $(u,s)^{-1} = (u \cdot s, s^{-1})$ and the set $\mathcal{G}_{E,G} = E \times G$ together with $\circ$ is a groupoid. Moreover the set of units for this groupoid is $\mathcal{G}_{E,G}^{(0)} = \{(u,e) : u \in E\} \cong E$ where $e$ is the neutral element of $G$. Now, the orbit of an element of $E$ for the action $\cdot$ is the orbit of the corresponding unit in the groupoid $(\mathcal{G}_{E,G}, \circ)$. Thus, $\mathcal{G}_{E,G}$ is transitive if, and only if the action $\cdot$ is.*

*It is worth noticing that this could be easily extended to the action of a groupoid on a set, leading us to the construction of a groupoid from the action of another groupoid on a set. We define the action of a groupoid on a set by:*

**Definition 9.** *(cf* [17]*) Let $(\mathcal{G}, \circ)$ be a groupoid. Let $M$ be a set such that there exists a map $\mu : M \to \mathcal{G}^{(0)}$. We define the space $\mathcal{G} \diamond M = \{(\gamma, m) \in \mathcal{G} \times M : s(\gamma) = \mu(m)\}$. A left action of $\mathcal{G}$ on $M$ is a map $\cdot : \mathcal{G} \diamond M \to M$ such that:*

1. $\forall (\gamma, m) \in \mathcal{G} \diamond M \quad \mu(\gamma \cdot m) = r(\gamma)$,

2. $\forall(\gamma_1,\gamma_2,m)\in\mathcal{G}^2\times M$

$$\Big((\gamma_1,\gamma_2)\in\mathcal{G}^{(2)}\ \textit{and}\ (\gamma_2,m)\in\mathcal{G}\diamond M\Big)\Longrightarrow\left\{\begin{array}{c}(\gamma_1,\gamma_2\cdot m)\in\mathcal{G}\diamond M\ \textit{and}\\ \gamma_1\cdot(\gamma_2\cdot m)=(\gamma_1\circ\gamma_2)\cdot m\end{array}\right.,$$

3. $\forall m\in M\quad \mu(m)\cdot m=m.$

*Then notice that the same type of construction as above leads to a groupoid structure on $\mathcal{G}\diamond M$.*

We shall refer to [17] for other discussions around the notion of groupoids. Another source is [14], and an interesting presentation can also be found in [9], where the tangent groupoid to a manifold plays an important role to deal with some quantization issues.

2.1.1. *Finite dimensional Groupoid algebra.* The general theory for groupoid algebra can be found in the original text of Renault ([15]), or as well in [14], and, in a different perspective in[17]. We will limit ourselves here to the description of this object in finite dimension. Let $\mathcal{G}$ be a finite groupoid. We adopt the following notation: any element $f$ of $\mathbb{C}^{\mathcal{G}}$ is written as a formal sum $\sum_{\gamma\in\mathcal{G}}f(\gamma)\cdot\gamma$. The sum of two functions leads to the natural sum on these formal linear combination. But instead of the pointwise multiplication, we define the convolution product of two such linear combinations by:

**Definition 10.** *For $(f,g)$ in $\left(\mathbb{C}^{\mathcal{G}}\right)^2$, we define:*

$$f*g:\gamma\in\mathcal{G}\longmapsto\sum_{\substack{(\gamma_1,\gamma_2)\in\mathcal{G}^{(2)}\\ \gamma_1\gamma_2=\gamma}}f(\gamma_1)g(\gamma_2)$$

*or, equivalently,*

$$\left(\sum_{\gamma_1\in\mathcal{G}}f(\gamma_1)\cdot\gamma_1\right)*\left(\sum_{\gamma_2\in\mathcal{G}}g(\gamma_2)\cdot\gamma_2\right)=\sum_{\gamma\in\mathcal{G}}\left(\sum_{\substack{(\gamma_1,\gamma_2)\in\mathcal{G}^{(2)}\\ \gamma_1\gamma_2=\gamma}}f(\gamma_1)g(\gamma_2)\right)\cdot\gamma.$$

Then it is easy to check that $\mathbb{C}\mathcal{G}=(\ \mathbb{C}^{\mathcal{G}},+,*)$ is a unital ring (so with the obvious action of $\mathbb{C}$ on the left, it is a $\mathbb{C}$–algebra). It is called the groupoid algebra of $\mathcal{G}$. One can extend slightly this definition by having some weights in the sums above; we shall encounter later the conditions such weights should have to ensure the algebra structure. For now, let us observe the following:

**Example 5.** *Let $\mathbb{N}_n=\{1,\ldots,n\}$ for some natural integer $n>0$. Endow this set with the equivalence relation:*

$$\forall(k,k')\in\mathbb{N}_n^2\quad k\sim k'.$$

*Then, the groupoid algebra of the groupoid-graph of $\sim$ $\mathcal{G}_\sim$ (cf example 3) is the algebra of $n\times n$-complex matrices.*

2.2. **Poisson Groupoids.** A very important class of groupoids we will be concerned with in these notes is the class of *Poisson groupoids* as introduced by Pr. Weinstein. Our first step is to recall some basic definitions about Poisson geometry. Then, we introduce the notion of Poisson groupoid as a generalization of both Poisson groups and symplectic groupoids. A reference on Poisson groups and their connections with quantum groups can be found in [8].

2.2.1. *Poisson manifold.* The Poisson structure on a manifold is inspired by the Hamiltonian formalism of Newton and Einstein mechanics. We start by the notion of Poisson algebra, before developing the theory of a manifold whose algebra of function is a Poisson algebra. Throughout these notes, $\mathbb{K}$ denotes one of the field $\mathbb{R}$ or $\mathbb{C}$. All manifolds in what follows are finite dimensional.

**Definition 11.** *Let $\mathcal{A}$ be an commutative associative algebra over $\mathbb{K}$. $(\mathcal{A}, \{.,.\})$ is said to be a* Poisson algebra *when $\{.,.\} : \mathcal{A} \otimes_{\mathbb{K}} \mathcal{A} \to \mathcal{A}$ satisfies:*

1. $\{.,.\}$ *is $\mathbb{K}$-linear on $\mathcal{A} \otimes_{\mathbb{K}} \mathcal{A}$ (i.e. $\mathbb{K}$-bilinear on $\mathcal{A}$),*
2. $\forall (a,b,c) \in \mathcal{A}^3 \;\; \{ab, c\} = a\{b,c\} + \{a,c\}b$ *(Leibniz's identity),*
3. $(\mathcal{A}, \{.,.\})$ *is a Lie Algebra, i.e.*
   (a) $\forall (a,b) \in \mathcal{A}^2 \;\; \{a,b\} = -\{b,a\}$,
   (b) $\forall (a,b,c) \in \mathcal{A}^3 \;\; \{a,\{b,c\}\} + \{c,\{a,b\}\} + \{b,\{c,a\}\} = 0$ *(jacobi's identity}.*

*The bilinear map $\{.,.\}$ is called the* Poisson Bracket on $\mathcal{A}$.

We shall omit writing the field of scalar $\mathbb{K}$ wherever possible from now on.

**Remark 1.** *For any $a$ in $\mathcal{A}$, the map $\xi_a = \{a,.\}$ is a derivation called a* Hamiltonian derivation.

**Definition 12.** *An algebra homomorphism $f : \mathcal{A} \to \mathcal{B}$ between two Poisson algebras $(\mathcal{A}, \{.,.\}_{\mathcal{A}})$ and $(\mathcal{B}, \{.,.\}_{\mathcal{B}})$ is a* homomorphism of Poisson algebras*, or* Poisson homomorphism*, when:*

$$\forall (a,b) \in \mathcal{A}^2 \quad \{f(a), f(b)\}_{\mathcal{B}} = f(\{a,b\}_{\mathcal{A}}).$$

Of course, the class of Poisson algebras with Poisson homomorphisms is a category, with the obvious definition for the category of unital Poisson algebras. Let us now turn to the object of interest for Poisson geometry:

**Definition 13.** *Let $\mathcal{M}$ be a $C^\infty$–differentiable manifold, and*

$$\{.,.\} : C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}).$$

$(\mathcal{M}, \{.,.\})$ *is a $C^\infty$–*Poisson manifold *when $(C^\infty(\mathcal{M}), \{.,.\})$ is a Poisson algebra.*

**Remark 2.** *Given a $C^\infty$–Poisson manifold $(\mathcal{M}, \{.,.\})$ then $(\mathcal{M}, -\{.,.\})$ is also a Poisson manifold.*

**Definition 14.** *A map $f : \mathcal{M} \to \mathcal{N}$ between two $C^\infty$–Poisson manifolds $\mathcal{M}$ and $\mathcal{N}$ is said to be a* Poisson map *when its lift $f^* : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{M})$ is a Poisson homomorphism.*

Again, the class of $C^\infty$–Poisson manifolds together with the Poisson maps is a category. From now on we shall assume that all manifolds are $C^\infty$.

**Definition 15.** *Given $\varphi \in C^\infty(\mathcal{M})$ for $(\mathcal{M}, \{.,.\})$ a Poisson manifold, the vector field $\xi_\varphi = \{\varphi, .\}$ on $\mathcal{M}$ is said to be the* Hamiltonian vector field *of $\varphi$ (where we identify as usual derivations and vector fields).*

**Definition 16.** *A submanifold $\mathcal{N}$ of a Poisson manifold $\mathcal{M}$ is a* Poisson submanifold *if, for any point $n$ of $\mathcal{N}$ and any function $\varphi \in C^\infty(\mathcal{M})$, $\xi_\varphi(n)$ is tangent to $\mathcal{N}$.*

Denote now by $\Gamma(X)$ the algebra of smooth sections on $X$ where $X$ is some vector bundle over $\mathcal{M}$, and $T^*(\mathcal{M})$ the algebra of smooth 1-form on $\mathcal{M}$. Given a manifold $\mathcal{M}$, denote by $< .,. >$ the natural pairing between $T\mathcal{M} \wedge T\mathcal{M}$ and $T^*\mathcal{M} \wedge T^*\mathcal{M}$, and introduce the following bracket on the exterior algebra of the sections of $T\mathcal{M} \wedge T\mathcal{M}$:

**Definition 17.** *The Schouten bracket $[[.,.]]$ is defined by:*

$$\forall (u_i, v_i)_{i\in\mathbb{N}} \in \Gamma(T(\mathcal{M}))^{2n} \qquad \left[\left[\bigwedge_{i=1}^{n} u_i, \bigwedge_{j=1}^{n} v_j\right]\right] = \sum_{i,,j} (-1)^{i+j} [u_i, v_j] \wedge \bigwedge_{k\neq i} u_k \wedge \bigwedge_{h\neq j} v_h .$$

Now, we can explicit the announced connection between symplectic and Poisson geometry:

**Proposition 2.** *Let $\pi$ be a (smooth) section of $T\mathcal{M} \wedge T\mathcal{M}$; define the following bilinear map*

$$\{f, g\} =< df \wedge dg, \pi > .$$

*Then $\{.,.\}$ defines a Poisson structure on $\mathcal{M}$ if, and only if $[[\pi, \pi]] = 0$. In this case, $\pi$ is called the associated Poisson bivector. Moreover, any Poisson bracket on $\mathcal{M}$ is of such a form.*

Given $\pi$ a section of $T\mathcal{M} \wedge T\mathcal{M}$ such that $[[\pi, \pi]] = 0$, we can define a homomorphism of vector bundle $\breve{\pi} : T^*\mathcal{M} \to T\mathcal{M}$ by. Then:

**Proposition 3.** *Keeping the previous notations, a Poisson manifold for which $\breve{\pi}$ is an isomorphism is a symplectic manifold, whose symplectic form is:*

$$\omega(X, Y) = [\breve{\pi}^{-1}(X)](Y).$$

*Of course, given $(\mathcal{M}, \omega)$ is a symplectic manifold, defining $\breve{\pi}$ by the previous formula (which makes sense as $\omega$ is nondegenerate) endows $\mathcal{M}$ of a Poisson manifold structure.*

An important property of Poisson manifolds is for them to admit a decomposition into symplectic submanifolds, called *symplectic leaves*:

**Definition 18.** *Given $(\mathcal{M}, \{.,.\})$ a Poisson manifold, a* Hamiltonian curve *$\gamma : [0, 1] \to \mathcal{M}$ is a $C^\infty$–path such that there exists a function $f \in C(\mathcal{M})$ satisfying:*

$$\forall t \in [0, 1] \quad \dot{\gamma}(t) = \xi_f(\gamma(t)).$$

**Proposition 4.** *Given $(\mathcal{M}, \{.,.\})$ a Poisson manifold, define the following binary relation on $\mathcal{M}$:*

$$\forall (u, v) \in \mathcal{M}^2 \quad u \sim v \Longleftrightarrow [\exists \gamma \textit{ Hamiltonian curve } \gamma(0) = u \textit{ and } \gamma(1) = v].$$

*Then $\sim$ is an equivalence relation on $\mathcal{M}$. Its equivalence classes are Poisson submanifolds of $(\mathcal{M}, \{.,.\})$ whose induced Poisson structure is indeed symplectic - we shall call these manifolds the symplectic leaves of $\mathcal{M}$.*

A fundamental construction for us is to endow the product manifold of two Poisson manifolds with a new, natural Poisson structure.

**Definition 19.** [8]*Let* $(\mathcal{M}_1, \{.,.\}_1)$ *and* $(\mathcal{M}_2, \{.,.\}_2)$ *be two Poisson manifolds. Following the proposition 2:*

$$\exists \pi_1 \in \Gamma(T\mathcal{M}_1 \wedge T\mathcal{M}_1) \quad \forall (\varphi, \psi) \in C^\infty(\mathcal{M}_1)^2 \qquad < d\varphi \wedge d\psi, \pi_1 >= \{\varphi, \psi\}_1,$$
$$\exists \pi_2 \in \Gamma(T\mathcal{M}_2 \wedge T\mathcal{M}_2) \quad \forall (\varphi, \psi) \in C^\infty(\mathcal{M}_2)^2 \qquad < d\varphi \wedge d\psi, \pi_2 >= \{\varphi, \psi\}_2.$$

*Then define on* $C^\infty(\mathcal{M}_1 \times \mathcal{M}_2)$ *the bracket* $\{.,.\}$*:*

$$\forall (\varphi, \psi) \in C^\infty(\mathcal{M}_1 \times \mathcal{M}_2)^2 \quad < d\varphi \wedge d\psi, \pi >= \{\varphi, \psi\},$$

*where* $\forall (m,n) \in \mathcal{M}_1 \times \mathcal{M}_2 \;\; \pi(n,m) = \pi_1(n) + \pi_2(m)$*. Then* $(\mathcal{M}_1 \times \mathcal{M}_2, \{.,.\})$ *is a Poisson manifold. We shall denote this manifold by* $(\mathcal{M}_1, \{.,.\}_1) \times (\mathcal{M}_2, \{.,.\}_2)$*.*

One of the great interest of Poisson structures when concerned with quantization issues lies at least in the two following points:

- The Hamiltonian formalism has still a meaning in Quantum physics. More precisely, the Hamiltonian is the total energy of the system (sum of the kinetic and potential energy of the physical system) in both Newton and Einstein physics. Now the concept of (scalar) energy is still defined in quantum physics, and is even the foundation of the work of Von Neumann on the mathematics foundations of quantum mechanics(cf [12]).
- The Poisson structure is defined on the algebra of smooth functions of a manifold, while the symplectic structure is a priori defined on the manifold itself. Thus the Poisson structure is better suited to non-commutative generalization. However, there is a great difficulty of identifying a subset of a C*-algebra as an analog of "smooth functions algebra" (which is not norm closed in the algebra of continuous functions). There is no general solution to this question; most of the time, the algebra retained for "smooth functions" is an extension of the notion of polynomials, as, for instance, the algebraic algebra generated by the generating unitaries of a non commutative torus. This kind of algebra satisfies two of the (necessary ?) conditions for being an algebra of smooth functions: it is dense in the full C*-algebra and is closed under holomorphic calculus.

2.2.2. *Poisson groupoids.* We are now concerned by endowing a groupoid with the structure of a Poisson manifold. In the sequel, we will denote the groupoids by their underlying set and use the usual mutliplicative notation. The notion of Poisson groupoid has been introduced in [20]. We shall motivate the definition by introducing the notion of Poisson relation found in

**Definition 20.** *A groupoid* $\mathcal{G}$ *is called a* Lie groupoid *when the set* $\mathcal{G}$ *is a manifold, such that the inverse map is* $C^\infty$ *and the set* $\mathcal{G}^{(2)}$ *is a submanifold, on which the multiplication is also* $C^\infty$*.*

To motivate the notion of Poisson groupoid, we shall define the Poisson relation (cf [20]), which generalizes the notion of Poisson map:

**Definition 21.** *Let* $P_1$ *and* $P_2$ *be any sets. The graph of a transformation* $f : P_2 \longrightarrow P_1$ *will be defined (contrary to the most common convention) as* $\{(f(y), y) :$

*$y \in P_2\}$. Accordingly, if $R \subset P_1 \times P_2$ and $S \subset P_2 \times P_3$ are subsets, considered as relations $R : P_2 \longmapsto P_1$ and $S : P_3 \longrightarrow P_2$, the composite relations $R \circ S : P_3 \longrightarrow P_1$:*

$$\{(x,z) \in P_1 \times P_3 : \exists y \in P_2 \quad (x,y) \in R \text{ and } (y,z) \in S\}.$$

*If $(P_1, \{.,.\})$ and $(P_2, \{.,.\})$ are Poisson manifolds, then a* Poisson relation *$R : P_1 \longrightarrow P_2$ is a coisotropic submanifold of the product $(P_1, \{.,.\}) \times (P_2, -\{.,.\})$.*

**Proposition 5.** *A map is a Poisson map if, and only if its graph is a Poisson relation.*

In fact, one can even show:

**Proposition 6.** *Let $\phi$ be a submersion from the Poisson manifold $P$ to the manifold $Q$. Define the equivalence relation $\sim$ on $P$ by:*

$$\forall (x,y) \in P^2 \quad x \sim y \Longleftrightarrow [\phi(x) = \phi(y)].$$

*Then, there is a unique Poisson structure on $Q$ such that the graph of $\sim$ is a Poisson relation if $\sim$ is a coisotropic relation.*

We are now in position to define the structure of Poisson groupoid:

**Definition 22.** *A* Poisson groupoid $(\mathcal{G}, \{.,.\})$ *is a Lie groupoid $\mathcal{G}$, such that $(\mathcal{G}, \{.,.\})$ is a Poisson manifold, and the graph of the multiplication $\{(xy, x, y) : (x,y) \in \mathcal{G}^{(2)}\}$ is a coisotropic submanifold in $(\mathcal{G}, \{.,.\}) \times (\mathcal{G}, -\{.,.\}) \times (\mathcal{G}, -\{.,.\})$.*

The following theorem confirms that the previous definition ensures the compatibility of the Poisson structure with the main structural components of the groupoid:

**Theorem 1.** *(cf [20]) Let $(\mathcal{G}, \{.,.\})$ be a Poisson groupoid. Then:*

1. *$\mathcal{G}^{(0)}$ is a coisotropic submanifold of $\mathcal{G}$,*
2. *The inversion map is an anti-Poisson map,*
3. *There is a unique Poisson structure on $\mathcal{G}^{(0)}$ such that $s$ is an anti Poisson map and $r$ is a Poisson map.*

In preparation for the definition of the last section of this note, we shall quote this important result, which gives a condition on the lift of the source and range maps at the level of the algebras of smooth functions instead of just conditions on the groupoid itself, which, as we stated in the introduction, "disappears" in the non-commutative setting:

**Proposition 7.** *$s^*(C^\infty(\mathcal{G}^{(0)}))$ and $r^*(C^\infty(\mathcal{G}^{(0)}))$ are commuting anti-isomorphic Poisson subalgebras of $C^\infty(\mathcal{G})$.*

The corresponding result on the groupoid itself is:

**Corollary 1.** *$(s,r) : \{\mathcal{G}.\{.,.\}) \longrightarrow (\mathcal{G}, \{.,.\}) \times (\mathcal{G}, -\{.,.\})$ is a Poisson map.*

We shall refer to [20] and [21] for examples and a study of Poisson groupoids. Also, [8] and [1] contain interesting expositions on Poisson groups, in view of their quantization.

## 3. Versions of Von Neumann algebras of functions of a Quantum Groupoid and Quantum Groupoid algebras

We now turn to our main subject. As presented in the introduction, several viewpoints can be adopted to describe a quantum structure for a group, and similarly for a groupoid. This part is concerned with the search for structures encoding the properties of a quantum groupoid as a non-commutative measurable space. It is of interest to the operator algebra standpoint, and indeed the only available theories at this point are using Von Neumann algebras. We therefore will briefly recall some definitions about operator algebras, and then present a nice description of finite quantum groupoids, before presenting the more complex ideas involved in the study of infinite quantum groupoids.

3.1. **Basic definitions of Operator algebras theory.** We refer to [5],[6] for a detailed account on operator algebras. We just introduce here the fundamental notion of Von Neumann algebra, and quote some useful results we will need later in our development. All our algebra and Hilbert spaces are taken over $\mathbb{C}$.

**Definition 23.** *Let $\mathcal{H}$ be a Complex Hilbert space. A* Von Neumann algebra $\mathfrak{A}$ *acting on $\mathcal{H}$ is a subset of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ such that:*

1. $\mathfrak{A}$ *is closed under the adjoint operation, (the adjoint of $A$ is denoted $A^*$),*
2. $\mathfrak{A}$ *equals its bicommutant, namely:*
$$\mathfrak{A} = \{A \in \mathcal{B}(\mathcal{H}) : \forall B \in \mathcal{B}(\mathcal{H}) \ \forall C \in \mathfrak{A} \quad (BC = CB) \Longrightarrow (AB = BA)\}.$$

If one calls commutant of a set $\mathfrak{A}$ the set of bounded operators on $\mathcal{B}(\mathcal{H})$ which commute with all elements in $\mathfrak{A}$, then the second point of our definition 23 says that the commutant of the commutant of $\mathfrak{A}$ is $\mathfrak{A}$.

As its name suggests, a Von Neumann algebra $\mathfrak{A}$ inherits a unital subalgebra structure from $\mathcal{B}(\mathcal{H})$, and the first point of the definition 23 means it indeed inherits a $*$–subalgebra structure. The first remarkable feature of these algebras is the famous *Von Neumann bicommutant theorem*:

**Theorem 2.** (Von Neumann) *The two following assertions are equivalent:*

1. $\mathfrak{A}$ *is a Von Neumann algebra,*
2. $\mathfrak{A}$ *is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, closed for the smallest topology making continuous the maps:*
$$(\xi, \eta) \longmapsto \langle A\xi, \eta\rangle$$
*for all $(\xi, \eta)$ in $\mathcal{H}^2$, and where $\langle ., . \rangle$ is the inner product on $\mathcal{H}$.*

This last topology is called the *Weak Operator topology on $\mathcal{B}(\mathcal{H})$*. It is obvious that this topology is weaker than the norm topology of $\mathcal{B}(\mathcal{H})$, and so in particular Von Neumann algebras are normed closed. To make the class of Von Neumann algebra into a category, we introduce the following homomorphisms:

**Definition 24.** *A map $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a *-homomorphism if it is an algebra homomorphism such that $\forall T \in \mathfrak{A} \ \ \varphi(T^*) = \varphi(T)^*$. It is unital when $\varphi(1) = 1$.*

*The class of Von Neumann algebra together with unital $*$–homomorphism is a category.*

One of the very fundamental fact about commutative Von Neumann algebras is the following consequence of the spectral theorem for normal operators:

**Theorem 3.** *Given an commutative VonNeumann algebra $\mathfrak{A}$, there is a locally compact space $\Sigma$ endowed with some Borel measure $\mu$ such that $L^\infty(\Sigma,\mu)$ is (unital) $*$–isomorphic to $\mathfrak{A}$. Conversely, any space $L^\infty(\Sigma,\mu)$ where $\Sigma$ is a locally compact space and $\mu$ a Borel measure on it is a Von Neumann algebra acting on the Hilbert space $L^2(\Sigma,\mu)$.*

This theorem, expressed in the category vocabulary, tells us that the category of $L^\infty$ spaces on locally compact spaces is a concrete realization of the dual category of commutative Von Neumann algebras. This of course illustrates our initial allegory: Von Neumann algebra are looked at as $L^\infty$ spaces for some "quantum locally compact space".

Among all operators, the ones corresponding to quantum observable are the self–adjoint operators[2]:

**Definition 25.** *An bounded operator $A$ on a Hilbert space is self-adjoint when $A^* = A$.*

Von Neumann algebras have a natural ordered algebra structure defined on them:

**Definition 26.** *Given $\mathfrak{A}$ a Von Neumann algebra acting on the Hilbert space $\mathcal{H}$, let $\mathfrak{A}_+$ be the Von Neumann algebra of positive operators, i.e.:*

$$\forall A \in \mathfrak{A} \quad A \in \mathfrak{A}_+ \Longleftrightarrow (\forall \xi \in \mathcal{H} \quad \langle A\xi, \xi\rangle \geq 0)$$

One shows that any positive operator is self–adjoint.

Among important objects defined on Von Neumann algebras are the traces:

**Definition 27.** *Let $\mathfrak{A}$ be a Von Neumann algebra. A linear form $\varphi$ on $\mathfrak{A}$ is a trace when:*

$$\forall (a,b) \in \mathfrak{A}^2 \quad \varphi(ab) = \varphi(ba).$$

**Definition 28.** *A trace $\varphi$ on some Von Neumann algebra $\mathfrak{A}$ is faithful when:*

$$\forall a \in \mathfrak{A}_+ \backslash \{0\} \quad \varphi(a) > 0.$$

3.2. **Finite Quantum Groupoids.** We are exposing in this section the main results of the theory of finite quantum groupoid, developed by [23]. Unfortunately, if the formalism is relatively simple, it cannot be extended to infinite quantum groupoids. Thus, we will discuss the best candidates for the general case in the next paragraph. However, more general theories are yet to be completed, which motivates us to give an account for the finite case, far better understood. It should be noted that some other definitions have been provided, and a good summary of these can be found in [13], where it is established that they are all equivalent.

[2] In fact, the natural algebras for quantum physics are C*–algebras, which are more general than Von Neumann algebras: they are those $*$–Banach algebra $*$–isomorphic to a norm-closed $*$–subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Also, self-adjointness and positivity are well defined in any $*$–Banach algebra, see [3]. We restrict ourselves here to what is immediatly needed.

3.2.1. *Introduction.* Let $\mathcal{G}$ be some non-empty *finite* groupoid, whose multiplication is denoted with the usual multiplicative formalism. As we indicated in the introduction (cf definition 1), if $\mathcal{G}$ is indeed a group, the algebra $l^\infty(\mathcal{G})$ of functions on $\mathcal{G}$ is endowed with a structure of Hopf algebra. Now, what happens to this construction when $\mathcal{G}$ is a groupoid? The antipode is unchanged, thanks to the axioms on the existence of an inverse in the definition 2. Now, the comultiplication does not make sense any more, as the product is only defined on the set of composable pairs $\mathcal{G}^{(2)}$. We are faced with two alternatives to solve this ambiguity:

- Either we try to extend the coproduct $\Delta(f)$ to all $\mathcal{G}^2$ for all $f \in l^\infty(\mathcal{G})$,
- Or we modify the definition of the comultiplication so that it is valued in another kind of product of algebras.

Obviously, the second idea seems more complicated. The first idea will fit very well the case of finite groupoids: if one wants to extend $\Delta$, the only choice which respects linearity is given by:

$$\forall f \in l^\infty(\mathcal{G}) \;\; \Delta(f) = \left[ (x,y) \longmapsto \left\{ \begin{array}{c} f(xy), \text{ when } (x,y) \in \mathcal{G}^{(2)} \\ 0 \text{ otherwise.} \end{array} \right. \right]$$

This comultiplication is still associative (cf definition1), and is still an algebra monomorphism. However, it is no more unital: in fact, it is clear that $\Delta(1)$ is the characteristic function of $\mathcal{G}^{(2)}$ so $\Delta$ is unital if, and only if $\mathcal{G}$ is a group. It is easy to check the compatibility relation with the antipode still holds too. A last remark is that, trivially, $l^\infty(\mathcal{G})$ is a Von Neumann algebra, when the adjoint operation is the complex conjugation - as it is $*$–isomorphic to $\mathbb{C}^{\#\mathcal{G}}$ acting by (left) multiplication on itself, viewed this time as a Hermitian space.

In [23], Yamanouchi introduces the following definition:

**Definition 29.** *A generalized Von Neumann Hopf algebra* $(\mathfrak{A}, \Delta, S)$ *is:*

1. *A finite dimensional Von Neumann algebra* $\mathfrak{A}$*,*
2. *A* $*$*–monomorphism* $\Delta : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ *which satisfies coassociativity:*
$$(Id \otimes \Delta) \circ \Delta = (\Delta \otimes Id) \circ \Delta,$$
*called comultiplication (the image of an element in* $\mathfrak{A}$ *is called the coproduct of the element),*
3. *A unital involutive* $*$*–antiautomorphism* $S : \mathfrak{A} \to \mathfrak{A}$ *such that:*
$$\sigma \circ \Delta \circ S = (S \otimes S) \circ \Delta,$$
*where* $\sigma$ *is the linear automorphism of* $\mathfrak{A} \otimes \mathfrak{A}$ *defined by* $\sigma(x \otimes y) = y \otimes x$ *for all elementary tensors* $x \otimes y \in \mathfrak{A} \otimes \mathfrak{A}$*.* $S$ *is called the antipode.*

We just seen that $l^\infty(\mathcal{G})$ can be endowed with a natural generalized Von Neumann Hopf algebra structure encoding the groupoid structure.

We shall now present Yamanouchi theory of generalized Kac algebra.

3.2.2. *Generalized Kac algebras.* From the perspective of operator algebras, it is interesting to carry out (abstract) harmonic analysis on "group-type" structures. This idea motivated the article of Yamanouchi [23] which we are going to summarize here. The basic ingredient for harmonic analysis is a Haar measure on the space which we wish to study. This measure shall induce a Haar integral, which is a linear form on the algebra of functions. This leads to the definition:

**Definition 30.** *A generalized Kac algebra* $(\mathfrak{A},\Delta,S,\varphi)$ *is a generalized Von Neumann hopf algebra* $(\mathfrak{A},\Delta,S)$*, together with a faithful trace* $\varphi$ *such that:*

1. $\varphi\circ S=\varphi$,
2. $\varphi$ *is left invariant, in the sense:*

$$\forall(x,y)\in\mathfrak{A}^2\quad (Id\otimes\varphi)(\Delta(x)(1\otimes y))=S(Id\otimes\varphi)((1\otimes x)\Delta(y)).$$

Let us first study the commutative case: when $\mathfrak{A}$ is commutative,it is the space of all functions on some finite set $\mathcal{G}$ (cf theorem 3). Yamanouchi establishes that indeed, one can endow $\mathcal{G}$ with a groupoid structure such that the construction of the previous paragraph gives back the generalized Kac algebra $(\mathfrak{A},\Delta,S,\varphi)$. Moreover, $\varphi$ is indeed a left-invariant integral on $\mathcal{G}$. More precisely:

**Proposition 8.** *Let* $\varphi$ *be a linear form on* $l^\infty(\mathcal{G})$*.Then* $\varphi : f\longmapsto\sum_{\gamma\in\mathcal{G}}a_\gamma f(\gamma)$ *is a faithful Haar trace if, and only if:*

$$\forall\gamma\in\mathcal{G}\quad 0<a_\gamma=a_{\gamma^{-1}}=a_{s(\gamma)}=a_{r(\gamma)}.$$

**Example 6.** Groupoid algebras. *A fundamental example of generalized Kac algebra is given by the groupoid algebra* $\mathbb{C}\mathcal{G}$ *of a finite groupoid* $\mathcal{G}$*, endowed with a Haar integral* $\varphi$ *as defined in proposition 8. The construction goes as follows:* $\mathbb{C}\mathcal{G}$ *is turned into a Hermitian space* $L^2(\varphi)$ *while endowed with the inner product* $(.\|.)$ *defined by* $(x\|y)=\varphi(y^*x)$ *for all* $(x,y)$ *in* $\mathbb{C}\mathcal{G}^2$*.* $\mathcal{G}$ *then acts on* $L^2(\varphi)$ *by:* $\gamma\in\mathcal{G}\mapsto\lambda_\gamma$ *where* $\lambda_\gamma$ *is the operator on* $L^2(\varphi)$ *defined by extending by linearity:*

$$\forall\upsilon\in\mathcal{G}\quad\lambda_\gamma(\upsilon)=\begin{cases}\gamma\upsilon,\ \text{when }(\gamma,\upsilon)\in\mathcal{G}^{(2)},\\ 0\ \text{otherwise.}\end{cases}$$

*Now, it is immediate to check that:*

$$\forall(\gamma,\theta)\in\mathcal{G}\quad\lambda_\gamma\lambda_\theta=\begin{cases}\lambda_{\gamma\theta},\ \text{when }(\gamma,\theta)\in\mathcal{G}^{(2)},\\ 0\ \text{otherwise;}\end{cases}\tag{3.1}$$

$$\forall\gamma\in\mathcal{G}\quad(\lambda_\gamma)^*=\lambda_{\gamma^{-1}}.$$

*Set* $\mathfrak{A}_{\mathbb{C}\mathcal{G}}$ *to be the linear spans of* $\{\lambda_\gamma:\gamma\in\mathcal{G}\}$*. As this last set is stable by multiplication and adjunction by equation 3.1,* $\mathfrak{A}_{\mathbb{C}\mathcal{G}}$ *is a Von Neumann algebra (note that in finite dimension, any norm closed set is weak-operator topology closed). Moreover,* $\mathbb{C}\mathcal{G}$ *is canonically* $*$*–isomorphic to* $\mathfrak{A}_{\mathbb{C}\mathcal{G}}$*, by letting* $\sum_{\gamma\in\mathcal{G}}a_\gamma\cdot\gamma\mapsto\sum_{\gamma\in\mathcal{G}}a_\gamma\lambda_\gamma$ *(which acts like the Fourier Transform). Now define the comultiplication* $\Delta_{\mathbb{C}\mathcal{G}}$ *on* $\mathfrak{A}_{\mathbb{C}\mathcal{G}}$ *by linearly extending the map* $\lambda_\gamma:\gamma\in\mathcal{G}\longmapsto\lambda_\gamma\otimes\lambda_\gamma$*. At last, the antipode is defined by:*

*Thus,* $(\mathfrak{A}_{\mathbb{C}\mathcal{G}},\Delta_{\mathbb{C}\mathcal{G}},S_{\mathbb{C}\mathcal{G}},\varphi)$ *is a cocommutative generalized Kac algebra.*

Another important result of Yamanouchi is that any cocommutative generalized Kac algebra arises from such a construction. Thus, given a finite groupoid with some Haar integral on it, there are two generalized Kac algebras derived from it: an commutative one obtained from the functions algebra and a cocommutative one obtained from the groupoid algebra. Moreover, given any (co)commutative generalized Kac algebra, one recovers a groupoid and a Haar integral, so that can always pair a commutative and a cocommutative generalized Kac algebra by associating them when they lead to the same groupoid with the same Haar integral - we shall say the algebras in such a pair are associated.

This leads us to the duality theory of Yamanouchi, which extends in the case of finite dimensional Kac algebra the duality theory of Takesaki (cf [18]). We first define the category of generalized Kac algebra:

**Definition 31.** *A homomorphism of generalized Kac algebra* $\theta : (A, \Delta_A, S_A, \varphi_A) \to (B, \Delta_B, S_B, \varphi_B)$ *is a unital* $*$*–homomorphism*$A \to B$ *such that:*

1. $(\theta \otimes \theta) \circ \Delta_A = \Delta_B \circ \theta$,
2. $\theta \circ S_A = S_B \circ \theta$.

*The class of generalized Kac algebra, together with these morphisms, is a category.*

Now we are able to state a fundamental theorem:

**Theorem 4.** *There exists an involutive contravariant functor* Dual *from the category of generalized Kac algebra into itself, such that:*

- *If* $(\mathfrak{A}, \Delta, S, \varphi)$ *is an commutative generalized Kac algebra, then* $\mathsf{Dual}[(\mathfrak{A}, \Delta, S, \varphi)]$ *is its associated cocommutative generalized Kac algebra,*
- *If* $(\mathfrak{A}, \Delta, S, \varphi)$ *is a cocommutative generalized Kac algebra, then* $\mathsf{Dual}[(\mathfrak{A}, \Delta, S, \varphi)]$ *is its associated commutative generalized Kac algebra.*

The strength of this theorem is that it gives a dual to *any* generalized Kac algebra, allowing us to indeed perform Harmonic analysis on quantum groupoids in very much the same way as for groupoids. Among other important point, note that a cocommutative generalized Kac algebra is commutative if, and only if its associated groupoid is commutative. Thus, as soon as we deal with non-commutative groupoids, we actually see non-commutative algebras appearing in any attempt of building a reasonable duality theory - the same situation occurs with groups. Thus, there is really very little difference between a quantum groupoid and a classical one from this perspective.

3.3. **A note on Infinite Quantum groupoids.** Consider now the infinite groupoid $\mathbb{R}^2$, whose multiplication is defined by $(x, y)(x, z) = (x, y + z)$ (i.e. view $\mathbb{R}^2$ as a bundle of additive groups $\mathbb{R}$ over $\mathbb{R}$). Then, notice that if we were trying to extend the example in 3.2.1, we would find that the comultiplication is non zero only on $\{((x, y), (x, z)) : (x, y, z) \in \mathbb{R}^3\}$, which has Lebesgues measure 0 in $\mathbb{R}^4$. Thus, in $L^\infty(\mathbb{R}^4)$, the comultiplication would be identically zero. The all formalism developed above collapsed, and we need to consider a tensor product adapted to infinite groupoids. The solution resides in making the comultiplication assume its values in a "restricted tensor product". This way is being explored by Vallin (cf [19]), Enock (cf [4]), and is based upon the notion of relative tensor product of Hilbert spaces carried out by Sauvageot (cf [16]).

The general idea is to look at a groupoid as a "generalized bundle of groups", or even better as a "fibered structure". The right mathematical translation of this notion is the notion of module. The use of modules as generalization of vector bundles or other kind of fibered structures are common, some can be found in [2]. In particular, a space of functions on a groupoid can be seen as a module over the space of functions of the same type over the unit space of the groupoids. This was first pointed out in [11]. The articles of Vallin and Enock previously quoted look at the Von Neumann algebra $L^\infty(\mathcal{G})$ of essentially bounded functions on the Hausdorff locally compact groupoid $\mathcal{G}$ as a module over the algebra $L^\infty(\mathcal{G}^{(0)})$. The set of axioms thus obtained defines the category of Hopf Bimodules. These objects are still studied, and require the construction of the fiber product of Von Neumann algebra, which involved too much operator algebra for this note.

An important result from [13] is that, when a Hopf Bimodule is finite dimensional (over a finite dimensional basis algebra) then this notion is equivalent to the notion of generalized Kac algebra. In particular, one retrieves the characterization of the commutative Hopf bi-modules as algebras of functions on some finite groupoid. However, a self-contained proof solely based on the formalism of Hopf bi-modules is proposed in [19] for the finite dimensional case. The proof is also given for the characterization of cocommutative Hopf-bimodules as groupoid algebras, again in finite dimension. As of today, the infinite dimensional characterization remains open, as well as a general duality theory.

## 4. A version of Quantized Poisson Groupoid

We turn at present to a more geometric approach. The procedure here is to encode the differential structure of a Poisson groupoid into an algebra. Thus, we look at the corresponding notion as an algebra over a quantum Poisson groupoid. The approach is very much a generalization of the quantization of the universal enveloping algebra of a Poisson group, and we refer to [8] for a nice account on this topic. Globally, the differential objects coding the groupoid multiplication are those (left) invariant for it, so the basic object is the Lie algebroid associated to a Poisson groupoid. Then, the Poisson structure allows us to obtain a "Hopf algebra-type" of objects - named Hopf algebroid - which is the basis of the quantization.

The description use algebras of differential operators which do not enjoy the fine properties of the operator algebras of functional analysis. Thus the description of the quantized object is highly algebraic. We finish our account with some basic results on deformation quantization of a Poisson groupoid.

### 4.1. Lie Algebroids and Hopf Algebroids.

The structure of Lie algebroid is the infinitesimal counterpart of a Lie groupoid. We refer to [17] for an extensive study of this structure. For our purpose, it is of interest to endow the Lie Algebroid of a Poisson groupoid of a supplementary structure leading to the notion of Hopf algebroid, which will serve as Lie algebroids for Quantum Poisson Groupoid.

#### 4.1.1. *Lie Algebroids.*

**Definition 32.** *A* Lie Algebroid *over a manifold $\mathcal{M}$ is a real vector bundle $E$ over $\mathcal{M}$ together with a bundle map $\rho : E \rightarrow T\mathcal{M}$ and a real Lie algebra structure $[.,.]_E$ on the set $\Gamma(E)$ of sections of $E$ such that:*

1. *The induced map $\Gamma(\rho) : \Gamma(E) \rightarrow \Gamma(\mathcal{M})$ is a Lie algebra homomorphism, (remember that $\Gamma(\mathcal{M})$ is the Lie algebra of vector fields on $\mathcal{M}$)*
2. $\forall f \in C^{\infty}(\mathcal{M})\ \ \forall (u,v) \in \Gamma(E) \quad [u, fv]_E = [u, fv]_E + (\rho(u) \cdot f)v.$

*The map $\rho$ is the anchor of the Lie algebroid.*

We refer to [17] for many examples. We shall only indicate the following important examples, which will help us quantizing groupoids:

**Example 7.** The Lie Algebroid of a Lie groupoid. *Let $\mathcal{G}$ be a groupoid. $\mathcal{G}$ acts on the left on itself as follows:*

**Proposition 9.** *Following the definition 9, let $\mu : \gamma \in \mathcal{G} \longmapsto r(\gamma)$, and then consider the map:*

$$\begin{aligned} \cdot_l : \mathcal{G} \diamond \mathcal{G} = \{(\gamma,\theta) \in \mathcal{G} \times \mathcal{G} : s(\gamma) = r(\theta)\} &\longrightarrow \mathcal{G} \\ (\gamma,\theta) &\longmapsto \gamma\theta. \end{aligned}$$

*Then $(\mu, \cdot_l)$ defines an action of the groupoid $\mathcal{G}$ on itself, named the canonical left action of $\mathcal{G}$ on itself.*

*If moreover $\mathcal{G}$ is a Lie groupoid, by definition, for all $\gamma \in \mathcal{G}$, the map $\theta \longmapsto \gamma \cdot_l \theta$ is a diffeomorphism of the manifold structure.*

Now, it is easy to check the source map is indeed invariant with respect to the canonical left action on the groupoid. Thus, we introduce:

**Example 8. Definition 33.** *Let $\mathcal{G}$ be a Lie groupoid. A vector field $X$ on $\mathcal{G}$ is* left invariant *when $dr(X) = 0$ where $dr$ is the linear tangent map for the range map $r$.*

**Proposition 10.** *A left invariant vector field on $\mathcal{G}$ is fully determined by its values along $\mathcal{G}^{(0)}$.*

**Notation 1.** *Denote by $\chi(\mathcal{G})$ be the set of left invariant vector fields on the Lie groupoid $\mathcal{G}$.*

**Proposition 11.** *$\chi(\mathcal{G})$ is a Lie subalgebra of the Lie algebra of the vector fields on $\mathcal{G}$. Moreover, any vector field in $\chi(\mathcal{G})$ is fully determines by its values along $\mathcal{G}^{(0)}$, as a corollary of proposition 10.*

*Left invariant functions are also fully determines by their values along $\mathcal{G}^{(0)}$. Conversely, any function on $\mathcal{G}^{(0)}$ can be extended (by composing it with the source map of the groupoid) to a left invariant function on $\mathcal{G}$. So we can identify the space of left invariant smooth functions with the smooth functions on $\mathcal{G}^{(0)}$. Now, $\chi(\mathcal{G})$ acts on the left invariant functions by differentiation. With the previous identification we obtain a map $\rho : \chi(\mathcal{G}) \to \Gamma(T\mathcal{G}^{(0)})$. In order to proof that $\chi(\mathcal{G})$ is a lie algebroid, we shall endow it with a vector bundle structure over $\mathcal{G}^{(0)}$. This is achieved through the following identification:*

**Proposition 12.** *If $E$ is the normal bundle of $\mathcal{G}^{(0)}$, then $\Gamma(E) \simeq \chi(\mathcal{G})$ as vector bundles. Moreover, since $\chi(\mathcal{G})$ is a Lie subalgebra of $T\mathcal{G}$, the previous isomorphism of vector bundle induces a Lie algebra structure on the sections of $E$.*

*Thus, using the previous identification which sends a left invariant vector field to a section of the normal bundle of $\mathcal{G}^{(0)}$, we have proved:*

**Proposition 13.** *The normal bundle $E$ of $\mathcal{G}^{(0)}$ is endowed with a Lie algebroid structure over $\mathcal{G}^{(0)}$, whose anchor map is $\rho$ and with its canonical lie algebra structure.*

*It is called the Lie algebroid of the Lie groupoid $\mathcal{G}$.*

A natural question, in view of the relation between Lie groups and Lie algebras, is to know when one can "integrates" a Lie algebroid to obtain a Lie groupoid of which it would be the Lie algebroid as in the previous example. Unfortunately, it is not always the case. However, a theorem of Dazord shows that every Lie algebroid obtained from the action of a Lie algebra on a manifold is integrable. We refer again to [17] for a discussion of this topic, together with the development of the theory of these objects.

Now, in preparation for our work with Poisson groupoids, we shall investigate some constructions of Lie algebroids on a Poisson manifolds, and see a duality for Lie algebroids. Let $(\mathcal{M}, \{.,.\})$ be a Poisson manifold, and $\pi$ be the associated Poisson bivector. Then, as seen previously, we have a map:

$$\breve{\pi} : T^*\mathcal{M} \longrightarrow T\mathcal{M}.$$

The following result proposition is the starting point to define a Lie algebroid over $\mathcal{M}$ reflecting the Poisson structure:

**Proposition 14.** *(cf [17]) There is a Lie bracket on $\Gamma(T^*\mathcal{M})$ such that:*

1. $\forall (f,g) \in C^\infty(\mathcal{M})^2 \quad [df, dg] = d\{f,g\}$
2. $\breve{\pi} : \Gamma(T^*\mathcal{M}) \longrightarrow \Gamma(T\mathcal{M})$ *is a Lie algebra anti-homomorphism.*

Thus:

**Proposition 15.** *Let $\mathcal{M}$ be a Poisson manifold. $T^*\mathcal{M}$ is made into a Lie algebroid with the Lie bracket on its smooth sections defined in proposition 14, and the anchor being $-\breve{\pi}$, where $\breve{\pi}$ is the associated Poisson bivector.*

*This Lie algebroid is called the Lie algebroid of the Poisson manifold $(\mathcal{M}, \{.,.\})$.*

The Lie algebroid of a Poisson manifold $(\mathcal{M}, \{.,.\})$ is not always integrable, but when this does happen, the Lie groupoid thus obtained is endowed with a compatible symplectic structure.

Now, in the case of a Poisson groupoid, the supplementary structure given by the Poisson bracket allows some interesting development in the construction of a dual Lie algebroid to the Lie algebroid of the groupoid.

In general, the dual of a Lie algebroid is not an algebroid:

**Proposition 16.** *(cf [17]) Let $E$ be a Lie algebroid over a $n$–dimensional manifold $M$, with Lie bracket on $\Gamma(E)$ denoted by $[.,.]_E$ and anchor $\rho$. Then, for some local coordinates $(x_1, \ldots, x_n)$ of $M$ and some local basis $(e_1, \ldots, e_n)$ for $\Gamma(E)$, we define the constants $(b_{i,j})_{i,j=1,\ldots n}$ and $(c_{i,j,k})_{i,j,k=1,\ldots n}$ by:*

$$\forall (i,j) \in \{1, \ldots, n\}^2 \quad [e_i, e_j]_E = \sum_{k=1}^{n} c_{i,j,k} e_k,$$

$$\forall i \in \{1, \ldots, n\} \qquad \rho(e_i) = \sum_{j=1}^{n} b_{i,j} e_j.$$

*Now, let $(\xi_1, \ldots, \xi_n)$ be the dual local basis $(e_1, \ldots, e_n)$ the dual bundle $E^*$ of $E$. Define on $C^\infty(E^*)$ the bracket $\{.,.\}_{E^*}$ by:*

$$\forall (i,j) \in \{1, \ldots, n\}^2 \quad \{x_i, x_j\}_{E^*} = 0,$$

$$\{\xi_i, \xi_j\}_{E^*} = \sum_{k=1}^{n} c_{i,j,k} \xi_k,$$

$$\{\xi_i, x_j\}_{E^*} = -b_{i,j}.$$

*Then $(E^*, \{.,.\}_{E^*})$ is a Poisson manifold. We call this vector bundle the dual of the Lie algebroid $E$.*

It is remarkable that the Poisson bracket on a Poisson groupoid $\mathcal{G}$ induces a map on its Lie algebroid $\chi(\mathcal{G})$ whose transpose (or dual map) endows the dual of $\chi(\mathcal{G})$ in the previous sense with a Lie algebroid structure. Moreover, it is straightforward the Poisson structure on the dual Lie algebroid transposes back to the algebroid structure on $\chi(\mathcal{G})$, making the duality theory for Lie algebroids of Poisson groupoids reflexive. Thus, there is an interest in summarizing the all information provided by the pair $(\chi(\mathcal{G}), \chi(\mathcal{G})^*)$ in one object:

**Definition 34.** *(cf* [21]*) Suppose that $E$ is a Lie algebroid over a manifold $\mathcal{M}$, and suppose that its dual $E^*$ also carries a structure of Lie algebroid on the same manifold. Then $(E, E^*)$ is a* Lie bialgebroid *when:*

$$\forall (X,Y) \in \Gamma(E) \quad d_*([X,Y]) = [d_*X, Y] + [X, d_*Y].$$

This notion generalizes the concept of Lie bialgebra for Poisson groups (cf [1])..

4.1.2. *Hopf Algebroids.* Hopf algebroids have been introduced in [10] as an algebraic

structure which could encompass the structure of "universal enveloping algebra of a quantum groupoid" - which we shall abbreviate by QUE-algebroids. Indeed, any universal enveloping algebra of a Lie algebroid is endowed with the following structure:

**Definition 35.** *A* (counital) Hopf algebroid $(A, B, \alpha, \beta, \Delta, \varepsilon)$ *over a (unital) ring $\mathbb{K}$ is defined by:*

1. *$A$ and $B$ are associative unital $\mathbb{K}$–algebras, and $\alpha$ and $\beta$ are homomorphisms from $B$ to $A$.*
2. *The ranges of $\alpha$ and $\beta$ commute in $A$, which implies that $A$ is naturally endowed with a $(B,B)$–bimodule*[3] *structure given by:*
$$\forall (a,b) \in B^2 \quad \forall x \in A \quad a \cdot x \cdot b = \alpha(a)\beta(b)x.$$
3. *$\Delta : A \to A \otimes_B A$ is a $(B,B)$–bimodule homomorphism satisfying* coassociativity*:*
$$(\Delta \otimes_B Id) \circ \Delta = (Id \otimes_B \Delta) \circ \Delta,^4$$
4. *$\Delta$ is also a unital $\mathbb{K}$–algebra homomorphism,*
5. *$\forall (a,b) \in A \times B \ \ \Delta(a)(\alpha(b) \otimes_B 1) = \Delta(a)(1 \otimes_B \beta(b))$,*
6. *$\varepsilon : A \to B$ is a $(B,B)$–bimodule homomorphism satisfying $\varepsilon(1) = 1$ and*
$$(\varepsilon \otimes_B Id) \circ \Delta = (Id \otimes_B \varepsilon) \circ \Delta = Id_A.$$

In the previous definition, we used for condition 6 the identification $A \otimes_B B \simeq B \otimes_B A \simeq A$. The algebra $A$ is called the *total algebra*, and the algebra $B$ is called the based algebra. $\Delta$ is called the *comultiplication*, and $\varepsilon$ the *counit* of the Hopf algebroid.

**Example 9.** *While non present in* [10] *or* [22]*, here is an easy illustration of the definition. Let $\mathcal{G}$ is a finite groupoid, then let $A_{\mathcal{G}} = \mathbb{C}^{\mathcal{G}}$ be the algebra of functions on $\mathcal{G}$, and let $B_{\mathcal{G}} = \mathbb{C}^{\mathcal{G}^{(0)}}$ be the algebra of functions on the set of units of $\mathcal{G}$. Then, let $\alpha_{\mathcal{G}}$,$\beta_{\mathcal{G}}$,$\Delta_{\mathcal{G}}$ be the respective Gelfand duals*[5] *of the source, range and multiplications map, and let $\varepsilon_{\mathcal{G}}$ maps any function on $\mathcal{G}$ to its restriction to the set of units of $\mathcal{G}$.*

*Then it is easy to check $(A_{\mathcal{G}}, B_{\mathcal{G}}, \alpha_{\mathcal{G}}, \beta_{\mathcal{G}}, \Delta_{\mathcal{G}}, \varepsilon_{\mathcal{G}})$ is a Hopf algebroid.*

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[3] A $(B,B)$–bimodule $A$ is a left and right module on $B$, compatible in the sense that: $\forall (a,b,x) \in B^2 \times A \quad (a \cdot x) \cdot b = a \cdot (x \cdot b)$.

[5] The Gelfand dual of $\mu$ is $f \longmapsto f \circ \mu$.

Now, let us turn to the case of Poisson groupoids. As we seen previously, several algebroids structures coexist. We shall now define a bialgebroid structure summarizing all this information.

**Example 10.** *(cf* [10]*) Let $A$ be an algebra. Then, if we denote by $A^{op}$ the opposite algebra, one can define the following Hopf algebroid structure on $A \otimes A^{op}$:*

1. *The source map $\alpha : A \longrightarrow A \otimes A^{op}$ is $a \longmapsto \alpha(a) = a \otimes 1$, and the target map $\beta : A \longrightarrow A \otimes A^{op}$ is $a \longmapsto \beta(a) = 1 \otimes a$,*
2. *The comultiplication $\Delta : A \otimes A^{op} \longrightarrow (A \otimes A^{op}) \otimes_A (A \otimes A^{op})$ is $a \otimes b \longmapsto (a \otimes 1) \otimes_A (1 \otimes b)$,*
3. *The counit is $\varepsilon : a \otimes b \longmapsto ab$.*

*This structure is modeled on the coarse groupoid structure on the product of a Poisson manifold by its opposite, defined in* [20].

**Example 11.** *Let $\mathcal{D}(\mathcal{M})$ be the algebra of differential operators on a smooth manifold $\mathcal{M}$. Define the following structural maps:*

1. $\forall X \in \mathcal{D}(\mathcal{M}) \quad \forall (f,g) \in C^\infty(\mathcal{M}) \quad \Delta(x)(f \otimes g) = x(fg)$,
2. *$\alpha$ and $\beta$ are the natural inclusions of $C^\infty(\mathcal{M})$ into $\mathcal{D}(\mathcal{M})$,*
3. *$\varepsilon$ projects any differential operator on its $0^{th}$-order part.*

*As establish in* [21]*, we have thus defined a bialgebroid $(\mathcal{D}(\mathcal{M}), C^\infty(\mathcal{M}), \alpha, \beta, \Delta, \varepsilon)$.*

**Example 12.** *A very important special corollary of the example 11 is the Hopf algebroid structure on the* Universal Enveloping Algebra $\mathfrak{U}A$ of a Lie algebroid $A$ *which integrates to a Poisson groupoid $\mathcal{G}$. $\mathfrak{U}A$ is then isomorphic to the subalgebra of $\mathcal{D}(\mathcal{M})$ of right invariant vector fields on $\mathcal{G}$. Then, $(\mathfrak{U}A, C^\infty(\mathcal{M}), \alpha, \beta, \Delta \restriction_{\mathfrak{U}A}, \varepsilon \restriction_{\mathfrak{U}A})$ is a bialgebroid (cf* [21]*).*

**Definition 36.** *A* coinvolutive counital Hopf algebroid $(A, B, \alpha, \beta, \Delta, S, \varepsilon)$ *is such that $(A, B, \alpha, \beta, \Delta, \varepsilon)$ is a counital Hopf algebroid and $S : A \to A$ is an anti-automorphism such that:*

1. $S \circ \beta = \alpha$,
2. *If $\mu$ is the multiplication of $A$, then $\mu \circ (S \otimes_B Id) \circ \Delta = \beta \circ \varepsilon \circ S$,*
3. *There exists a section $\gamma$ of the natural projection $A \otimes A \longrightarrow A \otimes_B A$ such that:*
   (a) *$\gamma$ is linear $A \otimes_B A \longrightarrow A \otimes A$,*
   (b) $\mu \circ (Id \otimes S) \circ \gamma \circ \Delta = \alpha \circ \varepsilon$.

**Example 13.** *Consider the example 9 and consider the map:*

$$\forall f \in A_{\mathcal{G}} \quad S_{\mathcal{G}}(f) = [\gamma \in \mathcal{G} \longmapsto f(\gamma^{-1})].$$

*Then $(A_{\mathcal{G}}, B_{\mathcal{G}}, \alpha_{\mathcal{G}}, \beta_{\mathcal{G}}, \Delta_{\mathcal{G}}, S_{\mathcal{G}}, \varepsilon_{\mathcal{G}})$ is a coinvolutive Hopf algebroid.*

**Remark 3.** *In* [21] *and previous works, the notion of Lie bialgebroid was defined as in our previous section as a pair of Lie algebroids in duality. But in* [10]*, the notion of bialgebroid is what we named counital Hopf bimodule, while there a coinvolutive counital Hopf algebroid is called a Hopf algebroid. We have used the previous terminology in view of this problem, and followed there one of the usual terminology on Hopf algebra, extended here to modules. The term coinvolutive is not very good as we do not indeed require the antipode to be involutive, but we avoided to introduce a specific notation just for these notes.*

**Example 14.** *Another broad class of examples is given by the Hopf algebra (cf. for instance* [1],[7],[8]*). Given a Hopf algebra $(A, \Delta, S, \varepsilon)$, it is trivially turned into* *a coinvolutive Hopf algebroid by letting* $(A, \mathbb{C}, \alpha, \beta, \Delta, S, \varepsilon)$ *where* $\alpha = \beta : z \longmapsto z.1$. *This is an immediate consequence of the fact groupoids are generalizations of groups: see again example 9 and consider what happens when* $\mathcal{G}$ *is indeed a group.*

4.2. **Formal Deformation of Poisson Groupoids.** We are now using the diverse notions introduced so far to indeed "quantized" a Poisson groupoid. This algebraic approach consists in quantizing the Universal Lie bialgebroid associated to a Poisson groupoid - very much as quantizing a Poisson group is quantizing its universal enveloping Lie algebra. Quantizing means that we modify the multiplication in a "as smooth as possible" way, given as much structure we have. In the coming setting, developed by [22], the requirements are purely algebraic - in contrast, for instance, of strict quantization (introduced by Pr. Rieffel) where strong continuity and structural requirements are made (cf [9] for an exposition on this topic and references).

The general concept is to start from a coinvolutive Hopf algebroid reflecting the Poisson groupoid one wanted to quantize. Then, one associates to it a bundle of coinvolutive Hopf algebroids, such that at (some) first approximation, the different structural maps are left unchanged, but in fact are modified toward some "goal" multiplication, induced by the Poisson bracket on the original object. More precisely:

**Definition 37.** *A* deformation *of a counital Hopf algebroid* $(A, B, \alpha, \beta, \Delta, \varepsilon)$ *over a field* $\mathbb{K}$ *is a bialgebroid* $(\tilde{A}, \tilde{B}, \tilde{\alpha}, \tilde{\beta}, \tilde{\Delta}, \tilde{\varepsilon})$ *over* $\mathbb{K}[[X]]$ *the ring of formal power series on* $\mathbb{K}$ *such that:*

1. $\tilde{A}$ *is isomorphic to* $A[[X]]$ *as a* $\mathbb{K}[[X]]$*–module, and admits* $1_A$ *as unit,*
2. $\tilde{B}$ *is isomorphic to* $B[[X]]$ *as a* $\mathbb{K}[[X]]$*–module, and admits* $1_B$ *as unit,*
3. $\tilde{\alpha} \equiv \alpha(\operatorname{mod} X)$, $\tilde{\beta} \equiv \beta(\operatorname{mod} X)$, $\tilde{\varepsilon} = \varepsilon(\operatorname{mod} X)$, $\tilde{\mu} \equiv \mu(\operatorname{mod} X)$, *where* $\mu$ *(respectively* $\tilde{\mu}$*) is the multiplication of* $A$ *(respectively of* $\tilde{A}$*).*
4. $\Delta \equiv \Delta(\operatorname{mod} X \otimes_B X)$,

*Moreover, all the structural maps* $\tilde{\alpha}, \tilde{\beta}, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{\mu}$ *and* $+$ *are continuous for the* $X$*–adic topology*[6]*.*

This formal deformation is a general tool used for purely algebraic quantization: as linear spaces and even as modules, a deformation is trivial in the sense it is isomorphic to the power series module on the original algebras of the bialgebroids. The point is that, as algebra, they can be quite different: the multiplication is modified.

**Remark 4.** ***In this context****, a deformation of the universal enveloping algebra of a Poisson groupoid endowed with its natural coinvolutive Hopf algebroid structure is sometimes called a quantum groupoid, but we shall prefer to view it as (a candidate for) the universal enveloping algebra of a Poisson Quantum groupoid. It is indeed unclear wether this "quantum groupoids" are related to the "quantum groupoids" of the quantum groupoid algebra, for instance. While the constructions share many similarity, their interpretation differs greatly, leading to a multiplicity a priori of the notion of Quantum Groupoids, defined by some complementary structures, or more precisely on the viewpoint adopted in each study.*

[6]The $X$–adic topology on a $\mathbb{K}[[X]]$–module $M$ is defined as being the smallest topology such that the translations are continuous and a basis of neighborhood for 0 is given by $(X^n M)_{n \in \mathbb{N}}$.

Now, one of major topic when studying deformation is the problem of the semi-classical limit. Namely, given a non-commutative object from the quantum world, is it true that, when the "order of scale" of the considered system get to be very much larger than the "quantum order of scale", the objects becomes "commutative" ? In physics, a fundamental notion is the notion of action. Without going to deep into physics, let us just say that an action is a quantity homogeneous to an energy times a time. Typically, one can express the order of magnitude of some "action" naturally associated to a physical system, like . The decision or whether the theory to apply is quantum physics or Newton/Einstein 's physics, roughly, is based upon a comparison between this "natural action" and the Planck's constant $\hbar = 1.05457266 \times 10^{-34}\,\mathrm{J\,s}$. (Other methods are to compute the Poincare wave length of the particles in the systems and compare them to some threshold value; these are of course equivalent - the second one corresponds more to the Schrodinger's wave mechanics original approach of quantum mechanics).

It is not easy to make sense of this problem. The principle according to which this should hold is known as the Bohr's principle. In our mathematical setting, the way this is approached is by means of deformations. In the previous definition, the undeterminate $h$ is playing the role of the Planck's constant. Now, instead of making the scale of the system grows, we make the value $h$ goes to 0. In view of our discussion, this simply corresponds to changing the units in which we make the measurements.

The question is then: do we obtain some kind of commutative structure when this happens? Is it "classical"? In the case of our note, we would like to know if, by such a procedure, a universal enveloping algebra of a Poisson Quantum groupoid goes down to the universal algebra of an integrable algebroid. The answer is given in [22]:

**Theorem 5.** *A universal enveloping algebra of a Poisson Quantum groupoid naturally induces a Lie bialgebroid* $(E, E^*)$ *as a classical limit.*

From there, a natural question arises: is any Lie bialgebroid a classical limit of a universal enveloping algebra of a Poisson Quantum groupoid? It is not known, and was conjectured in [22]. Some specific class of Lie bialgebroids are indeed quantizable, we shall refer to [22] for this result.

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