Berezin-Töplitz Quantization
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1 Introduction

The general idea of quantization is to find a way to pass from the classical setting to the quantum one. In the classical situation, we have a symplectic manifold (M, ω) standing for the space of "states" of some physical system. The topological data of the manifold is contained in the structure of the function algebra $C(M)$. The smooth structure on M gives rise to a distinguished subalgebra $C_0^\infty(M)$ of smooth functions vanishing at ∞, and the symplectic structure gives rise to a Poisson structure on $C_c^\infty(M)$ - the subalgebra consisting of the compactly supported smooth functions.

One of the natural generalizations of the idea of a function algebra on a manifold (as well as on more general topological spaces) is that of a C^*-*algebra*, which will now be defined.

Perhaps now is a good place to state that from now on (unless stated otherwise) **all scalars are complex**.

Definition 1 *A* ***Banach algebra*** *is a Banach space* $(\mathcal{A}, \|\|)$ *with an associative algebra structure, such that for all* $x, y \in \mathcal{A}$*, one has:* $\|xy\| \leq \|x\|\|y\|$.

Definition 2 *Let* $\mathcal{A}$ *be an algebra. An* ***involution*** *on* $\mathcal{A}$ *is an antilinear operator* $^* : \mathcal{A} \to \mathcal{A}$ *(i.e. for* $x, y \in \mathcal{A}$*,* $\lambda, \mu \in \mathbf{C}$*, one has* $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$*), such that for all* $x, y \in \mathcal{A}$*, one has* $(xy)^* = y^* x^*$*, and* $x^{**} = x$*. An algebra equipped with an involution is called a* ****-algebra****.*

Definition 3 *A* C^****-algebra*** *is a Banach algebra* $\mathcal{A}$ *with involution* **, such that for all* $x \in \mathcal{A}$*,* $\|x\|^2 = \|x^* x\|$.

If X is some locally compact Hausdorff space, then $C_0(X)$, the algebra of continuous functions on X vanishing at ∞, is a C^*-algebra, where the norm is the sup-norm ($\|f\| = max\{|f(x)| | x \in X\}$), and the involution is given by complex conjugation. This is a commutative C^*-algebra, of course. Conversely, it is known that if $\mathcal{A}$ is a commutative C^*-algebra then there is a locally compact Hausdorff space X, such that $\mathcal{A}$ is isometrically isomorphic to the algebra $C_0(X)$.

In fact, is can be shown that the correspondence $X \mapsto C(X)$ is a faithful contravariant functor from the category of compact Hausdorff spaces (with morphisms being continuous maps) to that of commutative C^*-algebras with unit. This is why C^*-algebras are sometimes referred to as "non-commutative spaces".

An example of a noncommutative C^*-algebra is $\mathcal{B}(\mathcal{H})$ - the algebra of bounded operators on a Hilbert space $\mathcal{H}$, where the involution is given by taking the adjoint, and the norm is the operator norm. Any *-subalgebra of $\mathcal{B}(\mathcal{H})$ which is norm-closed is also a C^*-algebra. In this context, one should mention the following:

Definition 4 *A **representation** of a C^*-algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ is a homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. A representation π is said to be **faithful** if $ker(\pi) = 0$.*

Theorem 1 ***(The Gelfand-Naimark representation theorem):*** *Any C^*-algebra has a faithful representation.*

Ideally, in order to pass from the classical setting to a C^*-algebra, we would like to do the following:

Given a symplectic manifold (M, ω), we would like to find some (noncommutative) C^*-algebra $\mathcal{A}$ and a linear map $f \mapsto x_f$ from some Poisson algebra over M, say, $C_c^\infty(M)$, such that:

$$x_f x_g - x_g x_f = x_{i\hbar\{f,g\}}$$

for all f, g in the Poisson algebra.

Unfortunately, it seems that this approach is infeasible. The plan we are going to discuss here replaces this requirement by slightly weaker conditions. For concreteness' sake, we will assume that we are looking at C^*-algebras that are subalgebras of $\mathcal{B}(\mathcal{H})$.

For positive numbers $\hbar$, we will attempt to construct a linear map $f \mapsto T_f^\hbar$, such that $\forall f, g$:

$$(a) \| [T_f^\hbar, T_g^\hbar] - T_{i\hbar\{f,g\}}^\hbar \| = o(\hbar)$$

$$(b) \lim_{\hbar \to 0} \|T_f^\hbar\| = \|f\|$$

The letter T here stands for "Töplitz".

Another important feature of this quantization is that the map $f \mapsto T_f^\hbar$ will be *positive*, in the sense that if $f \geq 0$ then $T_f^\hbar$ is a positive operator. Positive maps are of great importance in the theory of operator algebras, however, we will not discuss them in this paper.

References: A basic good reference on functional analysis is the textbook of Rudin [1]. Kadison and Ringrose [2] have another basic functional analysis text with emphasis on operator algebras, containing a exposition of the basics of the theory of C^*-algebras. A more advanced reference on C^*-algebras is Davidson's book [3].

2 Töplitz Operators

One fundamental phenomenon that occurs in infinite dimensional Hilbert spaces is the existence of *proper isometries*, i.e., non-surjective (and hence non-unitary) isometric linear operators. One can show that, in a certain sense, the study of such maps can be reduced to the study of the **unilateral shift**, which is defined as follows:

Definition 5 *: Let $\mathcal{H} = l^2(\mathbf{N})$. We define: $S : \mathcal{H} \to \mathcal{H}$ by*

$$S(a_n) = \begin{cases} a_{n-1} & | \; n > 0 \\ 0 & | \; n = 0 \end{cases}$$

for each $(a_n)_{n\in\mathbf{N}} \in l^2(\mathbf{N})$.

Töplitz operators come into play when one wants to study $C^*(S)$ – the C^*-algebra generated by S. While this representation of S is fairly nice, it is not very easy to see what polynomials in S, S^* will look like. It is more convenient to consider the following equivalent formulation:

Definition 6 *: The **Hardy space** $H^2 = H^2(\mathbf{T}) \subseteq L^2(\mathbf{T})$ is the space of all L^2 elements of* "analytic type"*, i.e., such that their negative Fourier coefficients vanish:*

$$H^2 = \{f \in L^2 | \hat{f}(n) = 0 \; \forall n < 0\}$$

Here $\mathbf{T}$ denotes the unit circle in the complex plane. The term "analytic type" comes from the fact that those functions are, in fact, boundary values of holomorphic functions in the unit disk.

The first simple observation about the Hardy space we should make is that it is complete, i.e. that it is a Hilbert space. The Fourier transform implements a unitary equivalence between $l^2(\mathbf{Z})$ and $L^2(\mathbf{T})$. The orthonormal basis $\{\delta_n | n \in \mathbf{Z}\}$ of l^2 corresponds to the orthonormal basis $\{z^n | n \in \mathbf{Z}\}$ (where here z just denotes the inclusion $\mathbf{T} \hookrightarrow \mathbf{C}$, and since we are only dealing with unit complex numbers, we have $z^{-1} = \bar{z}$). Under this correspondence, the one-sided space $l^2(\mathbf{N})$ corresponds to the closure of $span\{z^n | n \geq 0\}$, which is exactly H^2. The unilateral shift on $l^2(\mathbf{N})$ corresponds to multiplication by the function z.

Definition 7 *: Let $f \in \mathcal{C}(\mathbf{T})$. The* ***multiplication operator*** $M_f : L^2(\mathbf{T}) \to L^2(\mathbf{T})$ *is defined by:* $M_f g = fg$.

One can show that for any such f, M_f is indeed a bounded operator, and, we have: $\|M_f\| = \|f\|_\infty$.

Definition 8 *: Let P denote the orthogonal projection $L^2 \to H^2$, and let $f \in \mathcal{C}(\mathbf{T})$. The* ***Töplitz operator with symbol*** f *is the operator* $T_f : H^2 \to H^2$ *defined by the compression of M_f to H^2, i.e.* $T_f = PM_f\lceil_{H^2}$.

The unilateral shift S is unitarily equivalent to the Töplitz operator T_z. The C^*-algebra generated by T_z is called the **Töplitz algebra**, and is denoted by $\mathcal{T}$. The map $f \mapsto T_f$ is called the **symbol map**. This can easily be seen to be a positive linear map $\mathcal{C}(\mathbf{T}) \to \mathcal{T}$. This map, however, is not a homomorphism: in general, we do not have $T_f T_g = T_{fg}$. However, we have the following:

Theorem 2 *: $\forall f, g \in \mathcal{C}(\mathbf{T}), T_f T_g - T_{fg}$ is compact.*

Notice that this implies that $[T_f, T_g]$ is compact for all f, g. Compact operators are sometimes thought of as some "quantum" analogue of "infinitesimals". This suggests that Töplitz operators might indeed supply a reasonable framework for "quantizing" various function algebras.

Let $\mathcal{K}$ denote the C^*-algebra of all compact operators on our Hilbert space H^2. One can show that indeed $\mathcal{K} \subseteq \mathcal{T}$. $\mathcal{K}$ is an ideal in $\mathcal{T}$ and the quotient C^*-algebra $\mathcal{T}/\mathcal{K}$ is canonically isomorphic to $C(\mathbf{T})$. This can all be summarized as:

Theorem 3 *: There is a short exact sequence:*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(\mathbf{T}) \longrightarrow 0$$

The Töplitz operators discussed so far are only associated to the circle $\mathbf{T}$, which is not a symplectic manifold. But, we can now generalize the construction of Töplitz operators as follows:

Let λ denote the (normalized) Lebesgue measure on $\mathbf{T}$, thought of as as a (singular) measure on $\mathbf{C}$ (so, $\lambda(\mathbf{C} \setminus \mathbf{T}) = 0$). This gives us that $L^2(\mathbf{T}) = L^2(\mathbf{C}, \lambda)$, and, H^2 is just the $L^2(\lambda)$-norm closure of the the space of all polynomials in z (or, equivalently, of all the entire functions contained in this L^2 space).

Now, suppose we have any complex manifold M, and a (not necessarily singular) measure μ on M. We have the space $L^2(M, \mu)$, and we can define $H^2(M, \mu) \subseteq L^2(M, \mu)$ to be the closure of the set of all holomorphic functions which are contained in L^2.

For any bounded continuous function f on M we have the multiplication operator $M_f : L^2(M, \mu) \to L^2(M, \mu)$, and, if P denotes the orthogonal projection $P : L^2(M, \mu) \to H^2(M, \mu)$, we can define the Töplitz operator with symbol f to be: $T_f = PM_f \lceil_{H^2(M,\mu)}$, exactly as before.

We are now in position to describe the basic scheme of the Berezin-Töplitz quantization:

Given a Kähler manifold M, and some distinguished algebra $\mathcal{A}$ of smooth functions on M (e.g. $\mathcal{A} = C_c^\infty(M)$, but as we will see later we might want to include other functions in it), which is closed under the Poisson bracket that comes from the Kähler structure, we seek a one parameter family of measures on M, $\mu_\hbar$, for $\hbar > 0$, such that $\forall f, g \in \mathcal{A}$:

$$\|[T_f^\hbar, T_g^\hbar] - T_{i\hbar\{f,g\}}^\hbar\| = o(\hbar)$$

and

$$\lim_{\hbar \to 0} \|T_f^\hbar\| = \|f\|,$$

where $T_f^\hbar$ is the Töplitz operator with symbol f with respect to the measure $\mu_\hbar$. (cf. [9], [7], [10]).

References: A good reference for the theory of Töplitz operators is Douglas [4]. Douglas, however, only treats Töplitz operators over $\mathbf{T}$. A thorough exposition of the various H^2 spaces in several variables, and their associated Töplitz operators, can be found in Upmeier [5]. Davidson [3] discusses the structure of isometries and of the Töplitz algebra.

3 The Canonical Commutation Relations

Let $n \in \{1, 2, ..., \infty\}$. The **canonical commutation relations**, in **Heisenberg's form**, are the following set of equations:

$$[P_k, Q_l] = -i\hbar\delta_{k,l}$$

$$[P_k, P_l] = [Q_k, Q_l] = 0, \forall k, l < n$$

For the purpose of this section, we can normalize $\hbar$ to be 1.

The first problem with putting the CCRs in the context of operators on a Hilbert space (or, more abstractly speaking, in the context of C^*-algebras), is the following:

Theorem 4 *: The CCRs have no solution in any normed algebra.*

Proof: Let $\mathcal{A}$ be a normed algebra (i.e. an algebra satisfying the conditions of a Banach algebra, except for perhaps completeness).

Suppose the claim is false, so $\exists x, y \in \mathcal{A}$ such that $[x, y] = 1$. One easily verifies (by induction) that $\forall n \in \mathbf{N}, [x^n, y] = nx^{n-1}$. So,

$$n\|x^{n-1}\| = \|[x^n, y]\| \leq 2\|x^n\|\|y\| \leq 2\|x^{n-1}\|\|x\|\|y\|,$$

and therefore:

$$n \leq 2\|x\|\|y\| \ \ \forall n \in \mathbf{N},$$

which is absurd.

QED.

This might appear at first sight to deal a death blow to the whole idea of using C^*-algebras, or other Banach algebras, to study the CCRs. However, there is a way to work around this. The idea is to think of the P's and Q's in Heisenberg's CCRs not as operators, but rather as infinitesimal generators of a group. That is, we will cosider, say, P as really coming from an expression of the form "e^{itP}". It is a fact from the theory of one-parameter unitary groups that any such group (with the appropriate continuity conditions) indeed has an infinitesimal generator; however, that generator might be unbounded. This leads us to an equivalent formulation of the CCRs, known as **Weyl's form of the CCRs**.

Consider $\mathbf{R}^n$ $(n < \infty)$ with the standard inner product $(\cdot, \cdot)$. We seek two groups parametrized by the additive group $\mathbf{R}^n$, $V_{\vec{t}}, U_{\vec{s}}$ say, such that:

$$\forall \vec{t}, \vec{s} \in \mathbf{R}^n, V_{\vec{t}} U_{\vec{s}} = e^{i(\vec{s},\vec{t})} U_{\vec{s}} V_{\vec{t}}$$

One can easily verify that this new form is indeed equivalent to the Heisenberg form, by formally differentiating with respect to $\vec{t}, \vec{s}$. In order to be able to make sense of such a differentiation, we will need to also impose some continuity requirement on those groups. Before we do so, let us reformulate this in terms of one unitary group:

Instead of having the $\vec{s}, \vec{t}$ belong to the same space, we can think of them as being the first and second components of $\mathbf{R}^n \oplus \mathbf{R}^n$, with the standard symplectic form (i.e., in Darboux coordinates, the first $\mathbf{R}^n$ will be the p's and the second will be the q's). So, if ω is this form, we can think of our CCRs in Weyl's form as follows.

We seek a map associating to each vector $\xi \in (\mathbf{R}^{2n}, \omega)$ a unitary operator W_ξ over some Hilbert space $\mathcal{H}$ (or, more generally, some unitary element of a C^*-algebra), such that:

$$W_\xi W_\eta = e^{i\omega(\xi,\eta)} W_{\xi+\eta}$$

Of course, the underlying symplectic vector space need not be this standard $\mathbf{R}^{2n}$ - we can use any symplectic vector space, either finite or infinite dimensional.

We would like now to impose some continuity requirement. To do that we need to have a topology on the underlying symplectic vector space, and the symplectic form does not provide us with one. However, by picking a compatible complex structure on the underlying space, we can turn our symplectic vector space to a Hermitian inner product space, $\mathcal{Z}$, where the symplectic form is just the imaginary part of the inner product. Typically, we might want to assume that the inner product space is complete, i.e., is a Hilbert space, but this is not required for the definition of the CCRs.

The appropriate continuity requirement here is *strong continuity*:

$$\lim_{\xi \to 0} \|W_\xi f - f\| = 0, \ \forall f \in \mathcal{H}$$

The operators W_ξ are called the **Weyl operators**. The C^*-algebra they generate is called the **CCR algebra** over $\mathcal{Z}$.

References: A treatment of the CCRs can be found in Bratteli and Robinson [6]. One-parameter strongly continuous unitary groups are discussed in Rudin [1].

4 Examples

We now describe two examples of Berezin-Töplitz type quantizations.

4.1 The Quantized Unit Disks of Klimek and Lesniewski.

Here, the underlying Kähler manifold is just the unit disk in the complex plane, with the *Poincaré symplectic form*:

$$\omega = \frac{i}{2(1-|z|^2)^2} dz \wedge d\bar{z}$$

We pick a family of measures, parametrized by $\hbar$ as follows:

$$d\mu_\hbar = (1-|z|^2)^{1/\hbar} \frac{i}{2(1-|z|^2)^2} dz d\bar{z}$$

We then form the algebras of Töplitz operators with respect to those measures, following the general scheme described earlier.

Klimek and Lesniewski then prove the following:

Let f, g be functions in the Frechét space of smooth functions on the unit disk that extend, with all their derivatives, to continuous functions on the closed unit disk.

Theorem 5 *:* $\|T_f^\hbar\| \le \|f\|_\infty \le \|T_f^\hbar\| + o(1)$*, and, therefore,* $\lim_{\hbar\to 0} \|T_f^\hbar\| = \|f\|_\infty$

Theorem 6 *: There is a constant C (depending on f, g), such that:*

$$\|[T_f^\hbar, T_g^\hbar] - T_{i\hbar\{f,g\}}^\hbar\| \le C\hbar^{3/2}$$

This shows that this scheme indeed gives us a quantization in the desired sense.

As for the structure of the algebra, they show the following. $\mathbf{D}$ here denotes the closed unit disk:

Theorem 7 *: The C^*-algebra generated by $\{T_f^\hbar | f \in C(\mathbf{D})\}$ is isomorphic to the usual Töplitz algebra $\mathcal{T}$, for all $\hbar > 0$.*

Remarks:

1. The main tool used to prove the asymptotic estimates is that of *reproducing kernels*:

For each one of our measures, μ (we supress the $\hbar$ here), and for each point inside the unit disk, ζ, one can show that the evaluation functional $H^2(\mathbf{D}, \mu) \to \mathbf{C}$ fiven by $f \mapsto f(\zeta)$ is a bounded linear functional. By the Riesz representation theorem for bounded functionals on Hilbert spaces, we know that there is an H^2 function K_ζ such that:

$$f(\zeta) = \langle f, K_\zeta \rangle = \int_{\mathbf{D}} f(z)\overline{K_\zeta(z)}d\mu(z)$$

Such a K_ζ, usually thought of as a function on $\mathbf{D} \times \mathbf{D}$ $(K = K(z, \zeta))$, is called a reproducing kernel. One can work out a nice explicit formula for the reproducing kernels with respect to our $\mu_\hbar$, and in turn, this can be used to write explicit integral formulae for the Töplitz operators.

This tool is also used in the second example we give.

2. The starting point in the article of Klimek-Lesniewski is more abstract. For some constant $\alpha \in (0, 1)$, they consider the unital complex algebra P_α generated by two elements, "z" and "$\bar{z}$", with the relation:

$$[z, \bar{z}] = \alpha(1 - z\bar{z})(1 - \bar{z}z)$$

Of course, for $\alpha = 0$, this is just the usual algebra of polynomials in $z, \bar{z}$. Our $\hbar$ is related to this α by:

$$\hbar = \frac{\alpha}{\alpha + 1}$$

They then consider $*$-*representations* of this algebra, defined to be homomorphisms $\pi : P_\alpha \longrightarrow \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, such that $\pi(\bar{z}) = \pi(z)^*$. A natural thing to do in C^*-algebra theory will be to consider a universal C^*-algebra generated by this relation, C_α. Such a C^*-algebra will encode all the information about those representations. This C^*-algebra then turns out to have a faithful representation as Töplitz algebra with respect to the measure $\mu_\hbar$.

3. Another important point about this quantization is that one can show that the group $SU(1,1)/\mathbf{Z}_2$ of all Möbius transformations that preserve the unit disk acts on the quantized algebra as well, using the same formal rule (for $\gamma : \mathbf{C} \to \mathbf{C}$ defined by $\gamma(z) = (\bar{a}z + \bar{b})/(bz + a)$, we have the action, on C_α, defined by $\rho_\gamma(z) = (\bar{a}z + \bar{b})(bz + a)^{-1}$).

This strengthens the justification for calling those algebras "quantized disks".

One should also remark here about the choice of the particular symplectic form on the disk: the Poincaré form is the $SU(1,1)$-invariant form on the disk.

The authors also make use of this group action in the proofs of the asymptotic estimates mentioned above.

Reference: [7].

4.2 Coburn's Quantization of $\mathbf{C}^n$

We consider here $\mathbf{C}^n$, with the standard inner product structure. The interesting twist here is that we will quantize the function algebra $TP(\mathbf{C}^n) + C_c^\infty(\mathbf{C}^n)$, where $TP(\mathbf{C}^n)$ denotes the algebra of trigonometric polynomials over $\mathbf{C}^n$, i.e., $TP(\mathbf{C}^n) = span\{e^{iIm\langle z,\xi\rangle} \,|\xi \in \mathbf{C}^n\}$.

We denote those **characters** by $\chi_\xi(z) = e^{iIm\langle z,\xi\rangle}$.

Our family of measures now will be the Gaussian family:

$$d\mu_\hbar = \frac{e^{-|z|^2/\hbar}}{(\pi\hbar)^n} d\lambda$$

where λ denotes Lebesgue measure on $\mathbf{C}^n$.

As in the previous example, we have the following:

Theorem 8 *: Let f be a bounded uniformly continuous function on $\mathbf{C}^n$, then:*

$$\lim_{\hbar \to 0} \|T_f^\hbar\| = \|f\|_\infty$$

Theorem 9 *: $\forall f, g \in TP(\mathbf{C}^n) + C_c^\infty(\mathbf{C}^n)$ there is a constant C (depending on f, g), such that:*

$$\|[T_f^\hbar, T_g^\hbar] - T_{i\hbar\{f,g\}}^\hbar\| \leq C\hbar^2$$

This shows that this family of measures gives rise, again, to a Berezin-Töplitz quantization.

One can show the following:

Theorem 10 *: The C^*-algebra generated by $\{T_f^\hbar | f \in C_c^\infty(\mathbf{C}^n)\}$ is $\mathcal{K}(H^2(\mathbf{C}^n, \mu_\hbar))$ (the algebra of compact operators over this Hilbert space).*

As for the TP part, we define, for $\xi \in \mathbf{C}^n$,

$$W_\xi^\hbar = e^{\hbar|\xi|^2/8} T_{\chi_\xi}^\hbar$$

One can verify that $W_\xi^\hbar$ is unitary, and, that we have:

$$W_\xi^\hbar W_\eta^\hbar = e^{i\hbar Im<\xi,\eta>/4} W_{\xi+\eta}^\hbar$$

This means that the C^*-algebra generated by $\{T_{\chi_\xi}^\hbar | \xi \in \mathbf{C}^n\}$ is the CCR algebra over $\mathbf{C}^n$, with respect to the symplectic form $\omega(\xi, \eta) = (\hbar/4) Im \langle \xi, \eta \rangle$. Of course, changing $\hbar$ will give just give rise to isomorphic algebras (the isomorphism can easily be implemented by rescaling $\mathbf{C}^n$).

Put together, we see that the C^*-algebra obtained by quantizing $TP(\mathbf{C}^n) + C_c^\infty(\mathbf{C}^n)$ in this fashion is $CCR(\mathbf{C}^n) + \mathcal{K}$.

References: [8], [9].

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