Symplectic Lefschetz fibrations and the geography of symplectic 4-manifolds

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This paper is a survey of results which have brought techniques from the theory of complex surfaces to bear on symplectic 4-manifolds. Lefschetz fibrations are defined and some basic examples from complex surfaces discussed. Two results on the relationship between admitting a symplectic structure and admitting a Lefschetz fibration are explained. We also review the question of geography: which invariants (c_1^2, c_2) occur for symplectic 4-manifolds. Constructions of simply connected symplectic manifolds realizing certain of these pairs are given.

1 Introduction

It is possible to answer questions about a broad class of 4-manifolds by using the fact that they admit a special kind of structure called a Lefschetz fibration: a map from the manifold to a complex curve whose fibers are Riemann surfaces, some of them singular. The interesting data in such a fibration is the number of singular fibers and the types of these singular fibers. The types can be specified by monodromies (elements of the mapping class group of a regular fiber) associated to loops around the critical values of the fibration map in the base curve.

Lefschetz fibrations should, at least in theory, frequently be applicable to questions about symplectic 4-manifolds, in light of a theorem of Gompf that a Lefschetz fibration on a 4-manifold X whose fibers are nonzero in homology gives a symplectic structure on X and a theorem of Donaldson that, after blowing up a finite number of points, every symplectic 4-manifold admits a Lefschetz fibration.

After giving some examples from the category of complex surfaces and discussing the above results on Lefschetz fibrations, we state and sketch a proof of Gompf's theorem that fiber sums can be carried out along codimension 2 symplectic submanifolds so that the resulting manifold will still carry a symplectic structure.

Next is a brief introduction to the geography problem. The interesting part of this problem in the symplectic category is the question of existence of symplectic manifolds which are not homeomorphic to any complex surface. We will see that such manifolds do exist, and in fact all possible invariants of spin symplectic manifolds lying below the so-called Noether line are easy to realize using Gompf's symplectic fiber sum.

There are a few conventions used throughout the paper. The word *curve* always means complex curve ($\dim_{\mathbb{R}} = 2$), but *surface* means closed 2-manifold. A complex surface will always be explicitly referred to as such. Coefficients for homology and cohomology are taken in $\mathbb{Z}$ when not otherwise specified. The Euler class $e(\xi)$ of a principal $U(1) = SO(2)$ bundle ξ is the same as the first Chern class. We may evaluate $e(\xi)[S^2]$ for such a bundle over the 2-sphere and speak of the Euler number. This can be viewed in terms of building total space of the associated disc bundle to the given principal bundle from two copies of $D^2 \times D^2$. In particular, in the first factor of this product, the construction is that of S^2 as the union of the "northern" and "southern" hemispheres along the "equator." The fibers in the second factor are identified by assigning an element of $SO(2)$ to each point of the equator. By a diffeomorphism, we can choose the element of $SO(2)$ assigned to some fixed base point on the equator to be the identity. The identifications over the equator then give a map $S^1 \to SO(2)$. The isomorphism type of the principal bundle depends only on the homotopy class of this map, which is an element of $\pi_1(SO(2)) \cong \mathbb{Z}$ called the Euler number of the bundle. $PD(\cdot)$ is used in various contexts to denote the Poincaré dulaity isomorphism $H_k(X) \cong H^{n-k}(X)$ for compact manifolds. We sometimes abuse notation and identify the latter group with the deRham cohomology.

2 Building blocks in the complex category

The following definition will be needed later, but it also provides one of the most basic examples of a 4-manifold admitting a Lefschetz fibration. (This particular one has no critical points and no singular fibers.)

Definition 2.1 *A* **ruled surface** *is a compact complex surface S with a map $\pi : S \to C$ to a complex curve C such that every fiber is a complex line $\mathbb{CP}^1$.*

2.1 Lefschetz fibrations

The structure of Lefschetz fibrations comes from a decomposition of complex surfaces called a Lefschetz pencil.

If $S \subset \mathbb{CP}^n$ is a complex surface, then a generic codimension 2 linear subspace $A \subset \mathbb{CP}^n$ meets S in some finite set of points B. We may then consider two codimension 1 linear subspaces of $\mathbb{CP}^n$ which both contain the subspace A but otherwise are generic. We can specify these subspaces as the zero sets of two linear homogeneous polynomials $V(p_0)$ and $V(p_1)$.

Now for fixed $t = [t_0 : t_1] \in \mathbb{CP}^1$, consider the variety $L_t = V(t_0p_0 + t_1p_1)$. This will be a codimension 1 linear subspace of $\mathbb{CP}^n$ containing A. By a dimension count, we see that L_t intersects our original surface S in a complex curve, which may be singular. Choosing $s, t \in \mathbb{CP}^1$, note that $L_s \cap L_t = A$ so that $(L_s \cap S) \cap (L_t \cap S) = B$. That is, all of the curves obtained above intersect exactly in the finite set of points B. After blowing up each point of B, we obtain a well-defined map $X\#|B|\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ by sending each point to that $t \in \mathbb{CP}^1$ specifying the curve $L_t \cap S$ on which it lies.

Definition 2.2 *A* **Lefschetz fibration** *on a 4-manifold X is a map $\pi : X \to \Sigma$ where Σ is a closed 2-manifold having the following properties:*

1. *The critical points of π are isolated.*
2. *If $P \in X$ is a critical point of π then there are local coordinates (z_1, z_2) on X and z on Σ with $P = (0, 0)$ and such that in these coordinates π is given by the complex map $z = \pi(z_1, z_2) = z_1^2 + z_2^2$.*

In terms of this descrtiption of the critical points, a Lefschetz fibration can be thought of as a kind of complex analogue of a Morse function.

Note that if the 4-manifold X is closed ($\partial X = \emptyset$ and X is compact) then the generic fiber F of π is a closed 2-manifold, so a Riemann surface of some genus. Part of the utility of Lefschetz fibrations is that this simple property means that X can be reconstructued once we know how the fibers are identified if one travels around some loop in $\Sigma \setminus C$ where $C \subset \Sigma$ is the set of critical values of π. That is, Lefschetz fibrations admit a combinatorial description in terms of their **monodromy representation** $\rho : \pi_1(\Sigma \setminus C) \to \pi_0(\mathrm{Diff}(F))$ from the fundamental group of $\Sigma \setminus C$ into the group of homotopy classes of self-diffeomorphisms of F. The latter group is often called the **mapping class group**.

2.2 Examples

A very simple example of a Lefschetz fibration is a map from $\mathbb{CP}^2\#\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$. One begins with a point $P \in \mathbb{CP}^2$ and two generic lines containing this point, which we can take to be the vanishing sets $V(p_0)$ and $V(p_1)$ of two independent degree 1 polynomials. For any point $Q \in \mathbb{CP}^2 \setminus P$ there is a unique line $V(p_Q)$ containing both P and Q given as the vanishing set of another degree 1 polynomial. Some easy linear algebra shows that one can write $p_Q = t_0p_0+t_1p_1$ for some $[t_0 : t_1] \in \mathbb{CP}^1$. The map $Q \mapsto [t_0 : t_1]$ gives the desired fibration on $\mathbb{CP}^2 \setminus P$. Since we can view blowing up the point P as replacing the point P by the set of all lines passing through P, we obtain a well-defined fibration $\mathbb{CP}^2\#\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$. Note that this shows $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ is a ruled surface since the fiber over a point $[t_0 : t_1]$ is $V(t_0p_0 + t_1p_1)$, a $\mathbb{CP}^1$.

One can play a similar game starting with points $P_1, \dots, P_9 \in \mathbb{CP}^2$ given as the intersection $V(p_0)\cap V(p_1)$, where p_0 and p_1 are two generic homogeneous cubics in three variables. The nine intersection points are in general position. We will be interested in vanishing sets of polynomials of the form

$$\begin{aligned}&a_1X^3 + a_2Y^3 + a_3Z^3 + a_4X^2Y + a_5X^2Z+\\ &a_6Y^2Z + a_7Z^2X + a_8Z^2Y + a_9Y^2X + a_{10}XYZ.\end{aligned}$$

Picking a point $Q \in \mathbb{CP}^2 \setminus \{P_1, \dots, P_9\}$ and asking that its coordinates $[X : Y : Z]$ satisfy the above equation gives one linear relation on the a_i. In the same way $P_1, \dots, P_9$ give 9 more. Generically, we obtain in this way a homogeneous cubic which (by more linear algebra) can be written as $p_Q = t_0p_0 + t_1p_1$. Note that all points $Q' \in V(p_Q)$ give the same $[t_0 : t_1]$. Again, by viewing the blow-up of each of the P_i as replacing that point by the set of all points passing through it, we obtain a map $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ whose generic fibers are elliptic curves in $\mathbb{CP}^2$, which are topologically the two-dimensional torus T^2. A holomorphic map $\pi : S \to C$ from a complex surface to a complex curve whose generic fibers are nonsingular elliptic curves is called an **elliptic fibration**. The above is an example known as $E(1)$. There is also the notion of $\mathcal{C}^\infty$ elliptic fibration which is roughly a map from a 4-manifold to a 2-manifold which is locally diffeomorphic to a holomorphic elliptic fibration by diffeomorphisms of open subsets of S and C commuting with the given maps $S \to C$.

These two examples help one picture the more general construction. Given a holomorphic line bundle $\mathcal{L} \to X$ over a complex manifold X with two sections $\sigma_0, \sigma_1 \in \Gamma(X; \mathcal{L})$ which are transverse to the zero section, one can consider the vanishing of $t_0\sigma_0 + t_1\sigma_1$ where $[t_0 : t_1] \in \mathbb{CP}^1$ to get a map $X \setminus A \to \mathbb{CP}^1$, where $A = V(\sigma_0) \cap V(\sigma_1)$.

2.3 Fiber sum

There is a way to piece together two Lefschetz fibrations $\pi_1 : X_1 \to \Sigma_1$ and $\pi_2 : X_2 \to \Sigma_2$ whose generic fibers have the same genus to obtain a new Lefschetz fibration $\pi : X_1 \#_\phi X_2 \to \Sigma_1 \# \Sigma_2$. $\#_\phi$ denotes the fiber sum that is about to be defined and $\#$ the ordinary connected sum of the base spaces. One begins with neighborhoods ν_i of generic fibers F_i of π_i for $i = 1, 2$. These are diffeomorphic to $D^2 \times \Sigma_g$ where Σ_g is the Riemann surface with the same genus as the F_i. One then picks a diffeomorphism $\phi : S_1 \times \Sigma_g$ (orientation-reversing) of the boundaries of $X_i \setminus \nu F_i$ and identifies them via ϕ.

A special case that should be mentioned here is $E(2) = E(1) \#_\phi E(1)$ with $\phi : S^1 \times T^2 \to S^1 \times T^2$ the identity map. Topologically this is a $K3$ (or Kummer) surface, homeomorphic to a degree 4 hypersurface in $\mathbb{CP}^3$. More generally we may inductively form the elliptic surface $E(n) = E(n-1) \#_\phi E(1)$ with ϕ the identity as in the construction of $E(2)$. We will return to these examples in the discussion of geography.

3 Symplectic versions of these constructions

3.1 Lefschetz fibrations on symplectic manifolds

Symplectic manifolds (X, ω) for which $\omega \in H^2(X; \mathbb{R})$ is an integral class admit topological Lefschetz pencils via the construction for complex manifolds described in the previous section. That is, they are unions of the vanishing sets of ratios of two sections of a certain line bundle over X as the ratios vary over $\mathbb{CP}^1$. These real codimension 2 submanifolds all intersect exactly in the common zero set A of the two chosen sections. The set A has real codimension 4.

Theorem 3.1 *(Donaldson) If (V, ω) is a symplectic manifold with $[\omega]$ integral then for large k, V admits a topological Lefschetz pencil, in which the fibers are symplectic subvarieties representing the Poincaré dual of $k[\omega]$.*

To obtain the structure of a Lefschetz fibration, we will need to blow up X along this set A, which for a 4-manifold is just a finite set of points. Some remarks on the proof of the above theorem follow. First note that the requirement that ω be an integral class (which seems to show up as a hypothesis of many other results in addition to this one) is not so severe.

Proposition 3.1 *[Gom95] Any closed symplectic manifold (M, ω) admits another symplectic form ω' whose cohomology class is in the image of $H^2(M, \mathbb{Z}) \to$*

$H^2_{DR}(M)$.

Proof. Fix a metric on M and let B_ϵ be the ϵ-ball about 0 in the space of harmonic[1] 2-forms on M. Since nondegeneracy is an open condition, all of the forms in $\omega + B_\epsilon$ will be symplectic when ϵ is sufficiently small. Since $\omega + B_\epsilon$ covers an open subset of $H^2_{DR}(M)$ it contains an element of $H^2(M;\mathbb{Q})$, which we may multiply by a suitable integer to get the desired ω'. □

Holomorphic sections were used to pick out the fibers in the construction of a Lefschetz fibration on a complex manifold, so all the fibers also have the structure of complex manifolds. One can pick a compatible almost complex structure J on any symplectic manifold (X, ω), but there is no guarantee that there will be any complex line bundle with a sufficient supply of J-holomorphic sections to construct the Lefschetz pencil. Donaldson remedies this defect by considering an appropriate line bundle with sections which are "approximately holomorphic."

Since ω is an integral class, we can form a complex line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = \frac{\omega}{2\pi}$ and pick a unitary connection on $\mathcal{L}$ with curvature $-i\omega$. This lets one define a $\bar\partial$ operator on the sections of $\mathcal{L}$.

Theorem 3.2 *[Don96] Let $\mathcal{L} \to X$ be a complex line bundle over a compact symplectic manifold (X, ω) with compatible almost-complex structure, and with $c_1(\mathcal{L}) = \left[\frac{\omega}{2\pi}\right]$. Then there is a constant C such that for all large k there is a section s of $L^{\otimes k}$ with*

$$|\bar\partial s| < \frac{C}{\sqrt{k}} |\partial s|$$

on the zero set of s. Where ∂ and $\bar\partial$ are the complex linear and anti-linear parts of the derivative ∇ which is canonically defined along the vanishing of any section of $\mathcal{L}$.

We can apply the following proposition on each tangent space to the vanishing set of a section to see that for a section $s \in \Gamma(X; \mathcal{L}^{\otimes k})$ satisfying $|\bar\partial s| < |\partial s|$ we have $s^{-1}(0) \subset M$ a symplectic submanifold.

Proposition 3.2 *Suppose that $A : \mathbb{C}^n \to \mathbb{C}$ is a real linear map. Let a' and a'' be the complex linear and conjugate linear parts of A. Then if $|a''| < |a'|$ then* $\ker(A)$ *is symplectic with respect to the standard symplectic structure $\omega(v,w) =$* $\operatorname{Im}\langle v, w\rangle$*, where $\langle \cdot, \cdot \rangle$ is the standard Hermitian metric on $\mathbb{C}^n$.*

[1] This means those $\eta \in \Omega^2(M)$ such that $\triangle\eta = (dd^* + d^*d)(\eta) = 0$ where d is exterior differentiation and d^* is its adjoint with respect to the bilinear forms given on $\Omega^*(M)$ by the metric. We view $d, d^* : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$. It is a theorem that every deRham cohomology class has a unique harmonic representative.

3.2 Symplectic structure on Lefschetz fibrations

The converse to the result in the last section is considerably easier to prove. The argument below follows [GS99].

Theorem 3.3 *Suppose that X is a closed 4-manifold with a Lefschetz fibration $\pi : X \to \Sigma$. Denote the homology class of a regular fiber by $[F] \in H_2(X)$. Then X admits a symplectic structure which is symplectic on all the fibers of π if and only if $[F] \neq 0$.*

Proof. $[F] \neq 0$ is necessary, since given a symplectic form $\omega \in \Omega^2(X)$ which is also symplectic on fibers we would have $\int_F \omega > 0$ for F a regular fiber.

If $[F] \neq 0$ then there exists some $\alpha \in \Omega^2(X)$ with $\int_F \alpha > 0$. While this is a big step towards constructing a symplectic form on X which is also symplectic on the fibers, we have to make sure that $\int_{F_0} \alpha > 0$ for each closed surface F_0 contained in a singular fiber S, which is topologically a regular fiber with some loops called vanishing cycles collapsed to points.

Singular fibers are still homologous to regular fibers. If $S = \pi^{-1}(c)$ is the preimage of a critical value $c \in \Sigma$ of π, and $F = \pi^{-1}(d)$ is a regular fiber, then considering the preimage of an arc from c to d in Σ shows that $[S] = [F] \in H_2(X)$. The same argument shows that all the regular fibers are homologous.

If a singular fiber S is obtained from F by collapsing a separating vanishing cycle, then we have $S = F_0 \cup F_1$ for some closed surfaces F_0 and F_1 and $\int_F \alpha = \int_{F_0} \alpha + \int_{F_1} \alpha > 0$, so if $\int_{F_0} \alpha = r \leq 0$ then $s = \int_{F_1} \alpha > 0$. We can replace α by $\alpha' = \alpha + (-r + \frac{s}{2}) PD(F_1)$. $\int_{F_0} PD(F_1) = [F_0] \cdot [F_1] = 1$, so $\int_{F_0} \alpha' = r - r + \frac{s}{2} > 0$ and $\int_{F_1} \alpha' = s + (-r + \frac{s}{2})[F_1] \cdot [F_1] > 0$. All other fibers F of π do not intersect F_1 and so have $\int_F \alpha' = \int_F \alpha$. Since there are only finitely many isolated critical points of π we obtain, after finitely many modifications of this kind, a closed form $\zeta \in \Omega^2(M)$ such that $\int_{F_0} \zeta > 0$ for each closed surface F_0 contained in a fiber of π.

Now we need to fix this to be symplectic on fibers. Pick disjoint open balls $U_j \subset X$ about each critical point of π so that on each U_j there are local coordinates with in which we have $\pi(z_1, z_2) = z_1^2 + z_2^2$. We have the standard symplectic structure $\omega_{U_j} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ on each U_j. π is holomorphic, so if $y \in \Sigma$ then $\pi^{-1}(y) \cap U_j$ is a holomorphic curve, which means it is ω_{U_j}-symplectic.[2] For fixed y, extend the symplectic structure on $\pi^{-1}(y) \cap \omega_{U_j}$ to a symplectic structure ω_y on all of $\pi^{-1}(y)$. (Extend means pick an ω_y if $\pi^{-1}(y) \cap U_j = \emptyset$.) We can rescale each ω_y staying away from $\cup U_j$ to make $\int_F \omega_y = \int_F \zeta$ so that $[\omega_y] = [\zeta] \in H^2_{DR}(X; \mathbb{R})$ for all y. Now we need to glue these together.

[2] In general if J is an ω-compatible almost complex structure then C is J-holomorphic if and only if $\omega|_C$ is symplectic.

Pick a finite cover $\{W_j\}$ of Σ with at most one critical value in each W_j. Pick $y_j \in W_j$ for each j, choosing y_j to be the critical value for those W_j which contain a critical value. If $\pi^{-1}(W_j)$ is just a collection of regular fibers, let $r_j : \pi^{-1}(W_j) \to \pi^{-1}(y_j)$ be a retraction. If $\pi^{-1}(y_j)$ is a singular fiber, then take a retraction $r_j : \pi^{-1}(W_j) \to \pi^{-1}(y) \cup cl(U_j)$.[3] Choose also a partition of unity $\sum \rho_j = 1$ subordinate to this cover. We can define forms $\eta_j \in \Omega^2(W_j)$ for each j by

$$\eta_j(P) = \begin{cases} r^*\omega_{U_j}(P), & r(P) \in cl(U_j) \\ r^*\omega_{y_j}(P), & \text{otherwise.} \end{cases}$$

Since we made the $[\omega_y] = [\zeta]$ there exist $\theta_j \in \Omega^1(\pi^{-1}(W_j))$ so that

$$(\omega_y - \zeta)\,|_{\pi^{-1}(W_j)} = d\theta_j$$

for all j. We can then form

$$\eta = \zeta + d\left(\sum(\rho_j \circ \pi)\theta_j\right) \in \Omega^2(X).$$

This has $d\eta = d\zeta = 0$ and so is a closed form. For a given fiber $\eta_{F_y} = \sum \rho_j(y)\eta_j$ is linear combination of forms sympelctic on F_y with coefficients in $[0,1]$ and so is itself symplectic on F_y. To get a symplectic form on X, let

$$\omega_t = t\eta + \pi^*\omega_\Sigma,$$

where ω_Σ is any symplectic form you choose on the base. ω_t is closed for all t since η and ω_Σ are closed. Along fibers, ω_t agrees with $t\eta$ and so is nondegenerate. If $x \in \pi^{-1}(y)$ is not a critical point of π then for $v, w \in T_xF_y$ we have $\omega_t(v,w) = t\eta(v,w)$ since $\pi^*\omega_\Sigma$ is zero along fibers. This also shows that we have equal symplectic orthogonals $(T_xF_y)^{\omega_t} = (T_xF_y)^\eta \subset T_xX$. Since η is symplectic on F_y we have $T_xX = T_xF_y \oplus (T_xF_y)^{\omega_t}$, and $\pi^*\omega_\Sigma$ is nondegenerate on the second factor. Since nondegeneracy is an open condition, ω_t is nondegenerate on T_xX for small enough t.

The argument above worked away from the critical points of π. In particular, since $X \backslash \cup U_j$ is comapct, there is some t_0 with ω_t symplectic for $0 < t \leq t_0$. Now on the U_j we have $(\omega_t)|_{U_j} = t\omega_{U_j} + \pi^*\omega_\Sigma$. Recall that on the U_j we have complex structures on base and total space such that π is a holomorphic map. We can choose ω_Σ so that the almost complex structure on the base is compatible, and we have ω_{U_j} compatible with multiplication by i in U_j by definition of ω_{U_j}. If $v \in TU_j$ is given in the local coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$ by $v = a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial y_1} + c\frac{\partial}{\partial x_2} + d\frac{\partial}{\partial y_2}$ then

$$\begin{aligned} \omega_t(v, iv) &= t\omega_{U_j}(v, iv) + \pi^*\omega_\Sigma(v, iv) \\ &= t(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)(v, iv) + \omega_\Sigma(\pi_*v, \pi_*(iv)) \\ &= t(a^2 + b^2 + c^2 + d^2) + \omega_\Sigma(\pi_*v, i\pi_*v) \\ &> 0 \end{aligned}$$

[3] $cl(\cdot)$ means the closure.

because it is the sum of two positive terms, so all the ω_t are nondegenerate on $\cup U_j$ as well as on its complement, so we have the desired symplectic form on X with all fibers symplectic and see that it arises as a perturbation of the pullback $\pi^*\omega_\Sigma$ of a symplectic form on the base. $\square$

3.3 Blowing up

Blowing up is a familiar procedure for removing non-transverse intersections and self-intersections of curves. One views this locally at $P \in \mathbb{C}^2$ by picking a neighborhood of P diffeomorphic to $\mathbb{C}^2$ in such a way that $P = (0,0)$. Then let

$$\gamma = \{(Q,l) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid Q \in l\}.$$

Note that $\gamma \setminus \left(0 \times \mathbb{CP}^1\right)$ is diffeomorphic to $\mathbb{C}^2 \setminus \{(0,0)\}$ via some map ϕ. $E = \{0\} \times \mathbb{CP}^1$ is called the **exceptional sphere**. We can extend ϕ smoothly by defining $\phi_E = 0$. If $P \in X$ is a point in a 4-manifold, we can form $X' = (X \setminus P)\cup_\phi \gamma$ which comes with an obvious map $\sigma : X' \to X$ with E the preimage of P.

Proposition 3.3 *The topological effect of the blowup process is to replace the 4-manifold X with $X\#\overline{\mathbb{CP}^2}$.*

Proof. γ is clearly a complex line bundle over E via projection onto the second factor. Forming the blowup consists of removing a 4-ball from X and gluing in the unit disc bundle in γ over $E = S^2$. The Euler number of this disc bundle is -1. To see this, note that the function $(z_1, z_2) \mapsto z_1$ is linear on every line through the origin in $\mathbb{C}^2$ and so can be thought of as a section of $\gamma^* = \mathrm{Hom}(\gamma, \mathbb{C})$. It intersects the zero section of γ^* exactly in $[0:1]$ and is a holomorphic subset, so this intersection is positive. Now for general complex line bundles γ we have $e(\gamma^*) = -e(\gamma)$, so we see the Euler number of our particular γ is -1.

The Euler number of the normal bundle of a hyperplane in $\mathbb{CP}^2$ is $+1$, so $\overline{\mathbb{CP}^2}$ is obtained by adding a 4-ball to the boundary of a disc bundle with Euler number -1 along it's S^3 boundary, so the connect sum with $\overline{\mathbb{CP}^2}$ and the blowup process are both replacing a B^4 with a D^2 bundle over S^2 with Euler number -1. $\square$

Proposition 3.4 *If X is a symplectic 4-manifold then the blowup $X\#\overline{\mathbb{CP}^2}$ is also symplectic.*

The following notion is useful in discussing geography problems:

Definition 3.1 *A 4-manifold X is called* **minimal** *if there is no 4-manifold Y with $X = Y\#\overline{\mathbb{CP}}^2$, that is, if X is not the blowup of another manifold.*

By work of Taubes, being nonminimal is equivalent to containing a smoothly embedded S^2 with self-intersection -1. (The definition above requires this S^2 to be symplectic.) One can also speak of a kind of minimality for Lefschetz fibrations.

Definition 3.2 *A Lefschetz fibration is called* **relatively minimal** *if there is no S^2 with self-intersection -1 contained in a fiber.*

Proposition 3.5 *[Sti00] If $\pi : X \to \Sigma$ is a Lefschetz fibration with $g(\Sigma) \geq 1$ then π is a relatively minimal Lefschetz fibration if and only if X is minimal.*

Proof. Assume the result of Taubes that if X contains a sphere of -1 self-intersection there is a compatible almost complex structure on X for which there is a J-holomorphic (-1)-sphere. When X is equipped with such an almost-complex structure, π becomes a holomorphic map from the sphere to Σ and is therefore constant, so this (-1)-sphere is actually contained in one of the fibers of π. This shows relative minimality of π implies minimality of X. The converse is obvious. $\square$

3.4 Gompf's symplectic fiber sum

The main theorem of [Gom95] states that fiber sums can be carried out along symplectic submanifolds and the result will still carry a symplectic structure.

Theorem 3.4 *Let (M^n, ω_M) and (N^{n-2}, ω_N) be closed symplectic manifolds and let $j_1, j_2 : N \to M$ be symplectic embeddings with disjoint images. Suppose also that if ν_1 and ν_2 denote the normal bundles of $j_1(N)$ and $j_2(N)$ in M, respectively and e denotes the Euler class then $e(\nu_2) = -e(\nu_1)$. Then, for any choice of orientation-reversing $\Psi : \nu_1 \cong \nu_2$, the manifold $\#_\Psi M$ admits a symplectic structure ω, which is induced by ω_M after a perturbation near $j_2(N)$.*

For an even more precise statement and the details of carrying out fiber sums on pairs, see [Gom95]. A synopsis of the proof given there for the basic result above comprises the rest of this section.

We begin by setting up a model for the gluing of tubular neighborhoods of the codimension 2 submanifolds $j_1(N)$ and $j_2(N)$. The model represents the local situation after the gluing and consists of an S^2 bundle S over N so that N sits inside S as zero section and as ∞ section. There is an $SO(2)$ action on S fixing these copies of N which just rotates all the fibers around their axes. To get on more precise footing, let E be an $\mathbb{R}^2$ bundle isomorphic to ν_1. Then $\overline{E}$ is isomorphic to ν_2 since ν_1 and ν_2 are identified by the orientation-reversing map Ψ.

There is now the crucial use of the map $\iota : D^2 \setminus 0 \to D^2 \setminus 0$ given by

$$\iota(x) = x\sqrt{\frac{1}{\pi|x|^2} - 1}.$$

which antisymplectically turns the punctured 2-disc inside out. If $(\rho, \phi) = \iota(r, \theta) = \left(r\sqrt{\frac{1}{\pi r^2} - 1}, \theta\right) = \left(\sqrt{\frac{1}{\pi} - r^2}, \theta\right)$ then

$$\begin{aligned} \rho\, d\rho\, d\phi &= \rho \frac{\partial \rho}{\partial r}\, dr\, d\theta \\ &= \rho \frac{-r}{\rho}\, dr\, d\theta \\ &= -r\, dr\, d\theta \end{aligned}$$

Letting D_+ be the sub disc bundle of E above consisting of the union of all the closed normal discs of radius $\frac{1}{\sqrt{\pi}}$ and D_- the disc bundle of discs with radius $\frac{1}{\sqrt{\pi}}$ in $\overline{E}$. We form the sphere bundle S by identifying D_+ with D_- via the orientation-reversing map ι on each disc. The $SO(2)$ action mentioned earlier comes from the standard action on E and $\overline{E}$. It is clear that this commutes with ι and so gives a well-defined action on S. In the following N is sometimes referred to as a subset of D_+ or D_-. If i_0 and i_∞ denote the inclusions of N in S as the zero section and the ∞ section respectively, then $N \subset D_+$ really means $i_0(N) \subset D_+ = S \setminus i_\infty(N)$, and $N \subset D_-$ means $i_\infty(N) \subset D_- = S \setminus i_0(N)$. It is hoped that this abbreviated notation will alleviate rather than cause confusion.

The construction makes use of a symplectic form $\eta \in \Omega^2(S)$ with the following properties:

1. η is $SO(2)$-invariant.
2. $\int_F \eta = 1$ for any fiber F of S.
3. If $i_0 : N \to S$ denotes inclusion as the zero section, then $i_0^*\eta = 0$.

It is proved in [Gom95] that such an η exists and that if $\pi : S \to N$ is the projection then for some t_1 the forms

$$\omega_t = \pi^*\omega_N + t\eta$$

are $SO(2)$-invariant symplectic forms for $t \in (0, t_1]$. With this setup, consider an embedding $f : D_+ \to M$ with $f \circ i_0 = j_1$ mapping the copy of N in D_+ to our copy $j_1(N)$ in M. Such an embedding is guaranteed because we chose D_+ to have the same Euler number as the normal bundle of $j_1(N)$ in M. Since j_1 was assumed to be a symplectic embedding of N, $f|_{i_0(N)}$ is symplectic. We can apply the Moser theorem to the forms $\omega_E + t(f^*\omega_M - \omega_E)$ to get an isotopy from f to an embedding $\tilde{f}$ which is symplectic on a neighborhood of $N \subset D_+$. This isn't good enough, though. What we really want is a symplectic embedding of the whole of D_+. The following shrinking technique can be used to do this.

Proposition 3.6 *For any neighborhood W of N in (D_+, ω_{t_1}), there is a t_0 with $0 < t_0 \leq t_1$ so that for all positive $t \leq t_0$, (D_+, ω_t) embeds symplectically in W fixing N by an embedding isotopic (fixing N) to the identity on D_+.*

Proof. The argument makes use of a standard chain homotopy operator, described in Section 6.3 of [CdS01]. We now recall that construction: For $s \in [0, 1]$, let $\pi_s : D_+ \to D_+$ be given by multiplication by s on each fiber and define a vector field $X_s = \frac{d}{ds}\pi_s$. One can then define an integral operator $I : \Omega^p(D_+) \to \Omega^{p-1}(D_+)$ by

$$I(\eta) = \int_0^1 \pi_s^*(\iota_{X_s}\eta)\, ds.$$

Using the nondegeneracy of ω_t gives a time-dependent vector field Y_t by $\iota_{Y_t}\omega_t = -I(\eta)$, where $\eta \in \omega^2(D_+)$ is the restriction of the $\eta \in \omega^2(S)$ referred to above. We can then integrate Y_t to an $SO(2)$-invariant flow on compact subsets of D_+ that is defined for some open time interval about any fixed t_0. The next step is to show that this flow is defined on all of $D_+ \times [t_0, t_1]$.

For fixed $x \in D_+$ denote by $D(x)$ the disc in the fiber of D_+ containing x which is bounded by the $SO(2)$ orbit of x. Define the area $A(x) = \int_{D(x)} \eta$. Then by the definition of ω_t above the ω_t area of $D(x)$ is $\int_{D(x)} \omega_t = t \int_{D(x)} \eta = tA(x)$ since $\pi^*\omega_N$ is zero along the fibers of D_+. If F_t denotes the flow of the time-dependent vector field Y_t defined above then $F_t^*\omega_t$ is independent of t and the ω_t area of $F_t(D(x)) = D(F_t(x))$ is given by

$$tA(F_t(x)) = \int_{F_t(D(x))} \omega_t = \int_{D(x)} F_t^*\omega_t = \int_{D(x)} F_{t_0}^*\omega_{t_0} = t_0 A(x)$$

and we see that

$$A(F_t(x)) = \frac{t_0}{t} A(x).$$

In words, the disc fibers of D_+ decrease in area under the flow F_t, so points cannot escape from D_+ under the flow, and we get a well-defined $F_t : D_+ \to D_+$ for $t \in [t_0, t_1]$. (We can choose $0 < t_0 \leq t_1$ as we please and $F_{t_0} = \mathrm{Id}_{D_+}$.) So

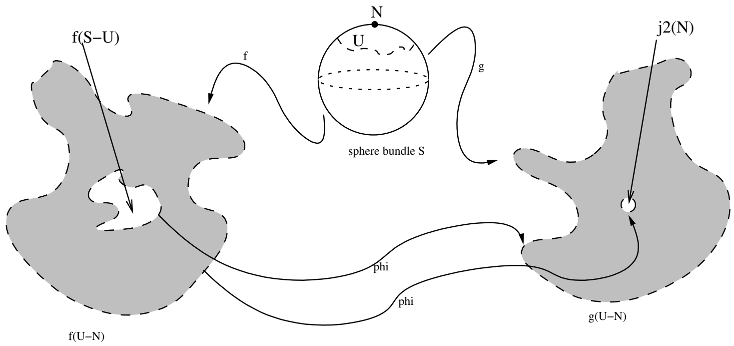

Figure 1: Fiberwise schematic for symplectic sum

we choose t_0 sufficiently small that $F_{t_1}(D_+) \subset W$. This F_{t_1} is the desired symplectic embedding of the whole of D_+ in W. $\square$

Applying this to the embedding $\tilde{f} : D_+ \to M$ which was symplectic on a neighborhood of $N \subset D_+$ we get an embedding $\hat{f} : D_+ \to M$ onto a neighborhood of $j_1(N)$ that is symplectic on all of D_+. We would like to do a similar thing for D_- but must deal with the problem that $i_\infty(N) \subset S$ is not necessarily symplectic with respect to ω_t.

Let $g : S \setminus i_0(N) \to M$ be the identification of the "northern" part of the sphere bundle with a neighborhood of $j_2(N)$ in M. It is possible to extend g^{-1} to a map λ from a neighborhood of the closure C of $g(S \setminus i_0(N))$ in M sending all points outside of C to $i_0(N)$. We can form $\zeta = \lambda^* \eta$ to get a form on this neighborhood that extends by zero to all of M. We modify ω_M to $\tilde{\omega}_M = \omega_M + t\zeta$. This is the perturbation referred to in the statement of the theorem. For small enough t, $\tilde{\omega}_t$ is symplectic on M and $j_2(N)$, and $g|_{i_\infty(N)}$ becomes a symplectic embedding. We can isotop g to be symplectic on a neighborhood U of $i_\infty(N)$. Assuming that the embedding $f : D_+ \to M$ is already symplectic on all of D_+, which we just showed can be achieved by an isotopy of f, we are in the situation depicted in Figure 1. That is, f and g symplectically identify $U \setminus N = U \setminus i_\infty(N) \subset S$ with $g(U \setminus N)$ and $f(U \setminus N)$. This latter is the same as $f(D_+) \setminus f(D_+ \setminus U)$, so we finally see that we can modify M by removing a compact neighborhood $f(D_+ \setminus U)$ of $j_1(N)$ from M together with the copy $j_2(N)$ of N and gluing the "ends" (which look like open annulus bundles over N) together by the map $\phi = g \circ f^{-1}$ which we have just seen is symplectic and which turns each annulus inside out as ι did to the punctured disc. Up to diffeomorphism, the effect is the same as the fiber sum described in the last section, but now it is furnished with a symplectic form $\tilde{\omega}_t$ for some small t. $\square$

3.5 Non-Kähler manifolds

Some basic facts about Kähler manifolds are discussed in [CdS01]. We recall the definition given there:

Definition 3.3 *A* **Kähler manifold** *is a symplectic manifold* (M, ω) *equipped with a compatible almost-complex structure* J *which is intergable.*

Recall also that complex submanifolds of Kähler manifolds are Kähler and that there is a Kähler form on $\mathbb{CP}^n$, so that all nonsingular projective varieties are Kähler.

Proposition 3.7 *The odd Betti numbers of a compact Kähler manifold are even.*

Proof. It's easy if one takes for granted that

$$H^k_{DR}(M; \mathbb{C}) \simeq \bigoplus_{l+m=k} H^{l,m}(M)$$
$$H^{l,m} \simeq \overline{H^{m,l}}$$

where the group on the left comes from tensoring the ordinary deRham cohomology with $\mathbb{C}$ and the groups on the right are the Dolbeault cohomology groups.[4] If we let $h^{l,m} = \dim_{\mathbb{C}}(H^{l.m}(M))$ then

$$\begin{aligned} b_{2k+1}(M) &= \sum_{i=0}^{2k+1} h^{i,2k+1-i} \\ &= \sum_{i=0}^{k} h^{i,2k+1-i} + h^{2k+1-i,i} \\ &= 2\sum_{i=0}^{k} h^{i,2k+1-i}. \qquad \square \end{aligned}$$

In a short note [Thu76], Thurston gives the following example of a symplectic manifold with b_1 odd. Consider the representation $\rho : \mathbb{Z} \oplus \mathbb{Z} \to SL_2\mathbb{Z}$ given by

$$\rho(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(0,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

[4] Without the Kähler hypothesis there is still the Fröhlicher spectral sequence relating these groups with the page $E^1_{p,q} = H^{p,q}(M)$. The fact used above is that this spectral sequence has $E^1 \cong E^\infty$ for a compact Kähler manifold. For more, see [GH94], which also states that there are no known complex manifolds with $E^2 \not\cong E^\infty$.

Note that $\mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(T^2)$ and the mapping class group $\pi_0(\text{Diff}(T^2)) \cong SL_2\mathbb{Z}$. Viewing this ρ as a holonomy representation gives a manifold X^4 which is a T^2 bundle over T^2. We can view this as being built from a trivial T^2 bundle over $S^1 \times I$ by identifying the fibers over the ends $S^1 \times \{0\}$ and $S^1 \times \{1\}$ via a diffeomorphism isotopic to the one specified by $\rho(0,1)$. We then have the following presentation for the fundamental group, which abelianizes to give $H_1(X)$ and so also the first Betti number $b_1(X)$.

$$\begin{aligned} \pi_1(X^4) &= \left\langle a,b,c,t \mid ab\bar{a}\bar{b}=1, ac\bar{a}\bar{c}=1, bc\bar{b}\bar{c}=1, ta\bar{t}=a, tb\bar{t}=ab \right\rangle \\ H_1(X) &= \frac{\mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \oplus \mathbb{Z}t}{b=a+b} \\ &\cong \mathbb{Z}^3 \\ b_1(X) &= 3 \end{aligned}$$

By the above proposition, the manifold so constructed cannot be Kähler. It is a symplectic manifold by the result of Section 3.2, since a T^2 bundle over T^2 is a Lefschetz fibration with no singular fibers.

4 Geography

Since one can realize any finitely presented group as the fundamental group of some closed 4-manifold, a sub-problem of the classification of closed 4-manifolds in general is deciding if any given finite presentation of a group is in fact describing the trivial group. This is known to be hard, so it is better to first ask for a classification of the simply-connected closed 4-manifolds.

In the topological category, this amounts to the classification of intersection forms, thanks to the following:

Theorem 4.1 *(Freedman) If Q is a unimodular symmetric bilinear form then there is some simply connected, closed 4-manifold X such that the intersection form $Q_X = Q$. If Q is even*[5] *then X is unique. Otherwise, there are two homeomorphism types of closed simply connected 4-manifolds having intersection form Q. At most one of these can have a smooth structure.*

Note that for a closed 4-manifold X the intersection form must be unimodular, for if $A = H_2(X)$ then $Q_X : A \to A^* = \text{Hom}\left(\frac{H_2(X)}{\text{Torsion}}, \mathbb{Z}\right) = \frac{H^2(X)}{\text{Torsion}}$ is an isomorphism by Poincaré duality.

[5] An integral form Q is called even if $Q(x,x) \in 2\mathbb{Z}$ for all x.

Theorem 4.2 *[GS99] Indefinite intersection forms are classified by their rank, signature, and parity and are equivalent to* $kH \oplus lE_8$ *where* $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* E_8 *is the Cartan matrix for the root lattice of the exceptional Lie algebra* $\mathfrak{e}_8$*. If a definite intersection form belongs to a smooth, simply connected, closed 4-manifold then it is equivalent to* $\oplus k \langle \pm 1 \rangle$*.*

So the homeomorphism type of a smooth, simply connected, closed 4-manifold is determined by a few characteristic numbers, which will be discussed further in the next section. The simple connectivity hypothesis is important. $T^2 \times T^2$ and $T^2 \times S^2$ have the same (c_1^2, c_2), but $\pi_1(T^2 \times T^2) \cong \mathbb{Z}^4 \ncong \mathbb{Z}^2 \cong \pi_1(Y)$.

A geography problem asks which pairs of characteristic numbers (c_1^2, c_2) (or equivalently (χ, σ), as will be explained soon) actually occur as the invariants of a certain class of manifolds: which are realized by simply connected complex surfaces, which are realized by symplectic manifolds, etc.

4.1 Characteristic numbers

Any almost complex manifold comes with an almost complex tangent bundle, allowing one to define the Chern classes of the manifold. Recall that they are defined inductively, with the top Chern class equal to the Euler class. The Chern class c_{n-1} is the top Chern class of the canonical complex $(n-1)$-plane bundle one can construct over the complement of the zero section in a complex n-plane bundle.

For an almost complex (hence oriented) 4-manifold X, these characteristic classes yield two important characteristic numbers

$$c_1^2 = \langle (c_1(TX) \smile c_1(TX)), [X] \rangle \qquad \text{and} \qquad c_2 = \langle c_2(TX), [X] \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $H_4(X)$ and $H^4(X)$. Similarly, the first Pontryagin class $p_1(X) \in H^4(X)$ gives a characteristic number p_1.

Note first that by definition $c_2 = \chi(X) = \chi$ is just the Euler characteristic of X. Here we will assume the Hirzebruch signature theorem, which states that $p_1 = 3\sigma$ for a smooth, closed, oriented 4-manifold. For any real vector bundle ξ, we have the definition $p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C})$. If ξ already had an almost complex structure, then there would be a natural identification $\xi \otimes \mathbb{C} \cong \xi \oplus \bar{\xi}$, where $\bar{\xi}$ denotes the bundle obtained from ξ by complex conjugation on each fiber. So we have

$$\begin{aligned} p_1(\xi) &= -c_2(\xi \oplus \bar{\xi}) \\ &= -(c_2(\xi) + c_1(\xi)c_1(\bar{\xi}) + c_2(\bar{\xi}) \\ &= -(2c_2(\xi) - c_1^2(\xi) \end{aligned}$$

$$c_1^2(\xi) \quad = \quad p_1(\xi) + 2c_2(\xi).$$

Applying this and the signature formula when ξ is the tangent bundle of a smooth, closed 4-manifold admitting an almost complex structure, we have

$$c_1^2 = 3\sigma + 2\chi.$$

Now σ, χ are defined for all 4-manifolds, so instead of using the definition in terms of characteristic classes one may just define $c_1^2 = 3\sigma + 2\chi$ and $c_2 = \chi$.

To see to what extent these characteristic numbers really characterize X, note that $\chi = 2 - 2b_1 + b_2^+ + b_2^-$ and $\sigma = b_2^+ - b_2^-$. Thus for simply connected 4-manifolds ($b_1 = 0$) the characteristic numbers give us the rank and signature of the intersection form.

Now $c_1^2 - 2c_2 = 3\sigma$. If X is spin (Roughly, if there is a trivialization of TX over the 1-skeleton of a CW structure for X which extends over the 2-skeleton.) then by Rohlin's theorem $16|\sigma$ so that $48|c_1^2 - 2c_2$, which shows that most pairs (c_1^2, c_2) are not the invariants of a spin homeomorphism type. For simply connected spin manifolds, the rank and signature completely determine the homeomorphism type. For non-spin manifolds there are two homeomorphism types but only one of these admits a smooth structure. This is using the equivalence of X having an even intersection form with the vanishing of the second Stiefel-Whitney class $w_2(TX) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$, which measures the obstruction to putting a spin structure on X.

By way of example, we summarize below the values of these invariants for some familiar 4-manifolds

	c_1^2	$c_2 = \chi$	σ	b_2^+	b_2^-	$b_1 = b_3$
$\mathbb{CP}^2$	9	3	1	1	0	0
$S^2 \times S^2$	8	4	0	1	1	0
$E(1)$	0	12	-8	3	11	0
$E(2) = K3$	0	24	-16	3	19	0
$E(n)$	0	12n	-8n	2n-1	10n-1	0
$\Sigma_g \times S^2$	$8 - 8g$	$4 - 4g$	0	1	1	2g
$T^2 \times T^2$	0	0	0	3	3	4

The usual way to pose a geography question for manifolds in a given category is to ask which (c_1^2, c_2) can be realized by minimal manifolds in that category. This is explained by the following:

Proposition 4.1 *The invariants of the blowup of a 4-manifold X are determined by the invariants of X.*

$$c_2(X \# \overline{\mathbb{CP}^2}) = c_2(X) + 1 \qquad\qquad c_1^2(X \# \overline{\mathbb{CP}^2}) = c_1^2(X) - 1.$$

Proof. The first statement follows from the fact that $c_2 = \chi(X)$ is the Euler characteristic and that for any connected sum of manifolds M and N we have $\chi(M\#N) = \chi(M) + \chi(N) - 2$.

Using the fact that the signature of the intersection form adds under connect sum and the formula for c_1^2 above, we have the easy calculation

$$\begin{aligned} c_1^2(X\#\overline{\mathbb{CP}^2}) &= 3\sigma(X\#\overline{\mathbb{CP}^2}) + 2c_2(X\#\overline{\mathbb{CP}^2}) \\ &= 3(\sigma(X) - 1) + 2(\chi(X) + 1) \\ &= c_1^2(X) - 1. \qquad \square \end{aligned}$$

In particular, the product $\Sigma_2 \times \Sigma_{k+1}$ of two Riemann surfaces has $(c_1^2, c_2) = (8k, 4k)$, so by blowing up products of Riemann surfaces we can obtain all pairs $0 \leq c_1 \leq 2c_2$ as the invariants of symplectic manifolds. However, these examples are not minimal, not spin, and not simply connected.

We just note here that there is a well-developed theory of the geography of simply connected minimal compact complex surfaces. In particular, these are projective varieties, so they are all symplectic. Thus the geography of simply connected minimal symplectic manifolds must contain that of the simply connected minimal complex surfaces. We will soon see that this is a proper containment.[6] To put the constraints on the geography of symplectic manifolds in the following sections in perspective, we mention here constraints

$$\begin{array}{ll} c_1^2 \geq \frac{c_2}{5} - 36 & \text{(the Noether line) and} \\ c_1^2 \leq 3c_2 & \text{(the Bogomolov – Miyaoka – Yau line)} \end{array}$$

on minimal complex surfaces of general type. General type means that

$$\frac{\dim \Gamma_{\mathrm{hol}}((\wedge^2 T^* X)^{\otimes k})}{k} \to \infty$$

as $k \to \infty$. (The Kodaira dimension is 2.) More details can be found in [BPVdV84].

4.2 Constraints on the geography of symplectic manifolds

One side of the geography problem is to identify regions of the (c_1^2, c_2)-plane where there is no hope of ever finding a (minimal) symplectic manifold. An important result in this direction is the so-called Noether condition, which uses only the almost-complex structure.

[6] Thurston's example of a non-Kähler symplectic manifold given above is not enough to do this because that manifold was not simply connected.

Proposition 4.2 *An almost-complex* 4*-manifold* X *has*

$$c_1^2 + c_2 \equiv 0 \pmod{12}.$$

Proof. There is an argument relying on the classification of integral unimodular symmetric bilinear forms that $c_1^2 \equiv \sigma \pmod 8$. Details can be found in [GS99]. This with $c_1^2 = 3\sigma + 2\chi$ gives $\sigma + \chi \equiv 0 \pmod 4$ so that $3\sigma + 3\chi = c_1^2 + c_2 \equiv 0 \pmod{12}$.

One can also get this for complex surfaces via the Atiyah-Singer index theorem, since the degree 2 term (the part in $H^4(X)$) of the Todd class is $\frac{1}{12}\left(c_1^2 + c_2\right)$.[7]
□

The following result rules out many of the possible pairs (c_1^2, c_2).

Theorem 4.3 *[Liu96] Let* (M, ω) *be a symplectic* 4*-manifold which is not the blowup of a ruled surface (see Definition 2.1). Then* $c_1^2(M, \omega) \geq 0$*.*

This is a consequence of the result of [Tau96] relating Seiberg-Witten invariants with Gromov invariants for symplectic 4-manifolds.

Given this lower bound, one might ask what kind of upper bound can be put on c_1^2. There is an easy result of this kind when $b_1 \leq 1$ (in particular it holds for simply connected manifolds) that does not require the presence of a symplectic structure.

Proposition 4.3 *If* X *is a* 4*-manifold with* $b_1(X) \leq 1$ *then* $c_1^2(X) \leq 5c_2(X)$*.*

Proof. Use $c_1^2 = 3\sigma + 2\chi$ and $c_2 = \chi$ as follows:

$$\begin{aligned}
c_1^2(X) &= 3\sigma(X) + 2\chi(X) \\
&= 3(b_2^+(X) - b_2^-(X)) + 2c_2(X) \\
&= 3(c_2(X) - 2 + 2b_1(X) - 2b_2^-(X)) + 2c_2(X) \\
&= 5c_2(X) + 6(b_1(X) - 1) - 2b_2^-(X) \\
&\leq 5c_2(X) \qquad \square
\end{aligned}$$

A basic question about the possible topology of symplectic 4-manifolds remains open:

[7] The author does not know whether this big machine is one that also works when the almost-complex structure is not integrable. $\chi_h = \frac{1}{12}(c_1^2 + c_2)$, rather than c_2, is sometimes used for the second characteristic number.

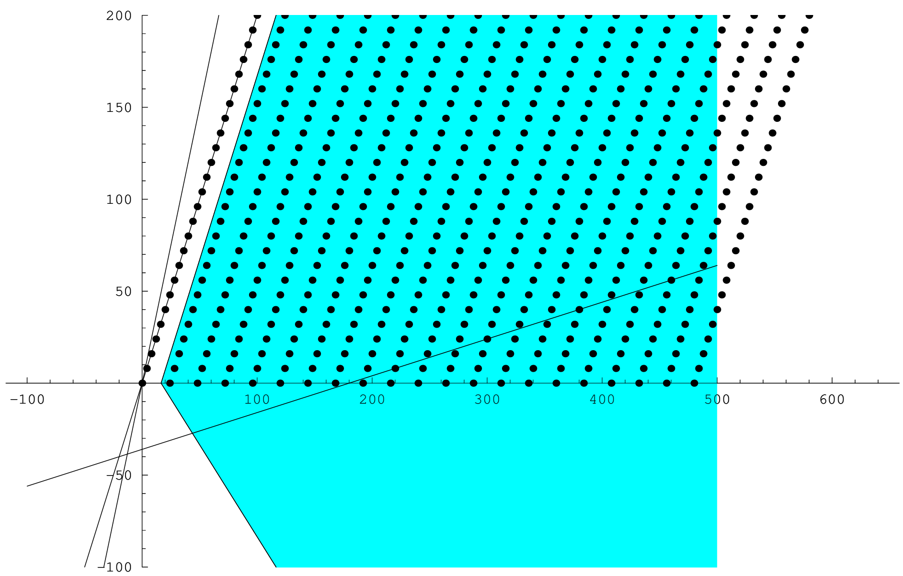

Figure 2: Spin and simply connected symplectic manifolds

Conjecture 4.1 *(Gompf) If X is a symplectic 4-manifold and X is not the blowup of a ruled surface, then the Euler characteristic $\chi(X) \geq 0$.*

The qualification is really necessary since the ruled surface $\Sigma_g \times \mathbb{CP}^1$ is a symplectic manifold with Euler characteristic $\chi\left(\Sigma_g \times \mathbb{CP}^1\right) = 4 - 4g$. Given Theorem 4.3, this conjecture would also hold if the following one does, that is, if symplectic manifolds all fall below the BMY line as do the simply connected complex surfaces.

Conjecture 4.2 *[Sti00] If X is a symplectic 4-manifold other than a sphere bundle then $c_1^2(X) \leq 3c_2(X)$.*

4.3 Constructions of symplectic 4-manifolds

A complete understanding of the geography of 4-manifolds carrying a symplectic structure requires not only constraints on the invariants of the type described above but also proofs that pairs (c_1^2, c_2) satisfying the contraints actually occur as the invariants of symplectic manifolds. The most satisfying way to do this is to construct such manifolds explicitly.

After a short detour on slopes and arbitrary fundamental groups, we will review some constructions from [Gom95] that show that the invariants at the dots in the diagram are realized by spin symplectic 4-manifolds, and that all the pairs (c_1^2, c_2) in the shaded region which satisfy the Noether condition $c_1^2 + c_2 \equiv 0 \pmod{12}$ are realized by simply connected symplectic manifolds. Not all of these are minimal, but all the pairs in the shaded region with $c_1^2 \geq 0$ can be constructed without resorting to blowing up. Convention dictates that we write pairs of invariants in the order (c_1^2, c_2) despite the fact that, also by convention, the c_1^2-axis is vertical and the c_2-axis horizontal when depicting geography graphically. The three lines in the diagram are the BMY line $c_1^2 = 3c_2$, the $\sigma = 0$ line $c_1^2 = 2c_2$, and the Noether line $c_1^2 = \frac{1}{5}c_2 - 36$. Some symplectic manifolds with positive signature are also discussed in [Gom95], but we will not discuss them here. Note, however, the abundant supply of spin symplectic manifolds lying below the Noether line. None of these are homeomorphic to minimal complex surfaces of general type.

Stipsicz notes in [Sti00] that a few of such constructions suffice to see that there are no gaps in the "slope" realizable by symplectic 4-manifolds.

Theorem 4.4 *[Sti00] If $\alpha \in [0, 3] \cap \mathbb{Q}$, then there is a minimal symplectic 4-manifold X such that $c_1^2(X) = \alpha c_2(X)$. Consequently, there is no gap in the geography of symplectic 4-manifolds in the interval $[0, 3]$.*

The proof cites three constructions of symplectic 4-manifolds for each of the three cases $\alpha \in [0, \frac{1}{5}]$, $\alpha \in [\frac{1}{5}, 2]$, and $\alpha \in [2, 3]$.

Gompf [Gom95] gives several constructions of symplectic manifolds as applications of his proof that the fiber sum operation can be carried out along codimension 2 symplectic submanifolds. One striking example of this is the following result on realizing a specified fundamental group:

Theorem 4.5 *[Gom95] Let G be any finitely presented group. Then there is a closed, symplectic 4-manifold M with $\pi_1(M) = G$.*

Proof. (Sketch) As one might expect, M is constructed via a fancy version of the construction of a 2-dimensional CW complex having arbitrary finitely presented fundamental group. Instead of beginning with a wedge of circles, one begins with the product of a closed surface and a torus. If

$$G = \langle g_1, \ldots, g_l \mid r_1, \ldots, r_k \rangle$$

then let F be the surface of genus l and $\{\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l\}$ be embedded circles representing a symplectic basis for $H_1(F)$. The relations $r_1, \ldots, r_k$ can be viewed as words in the α_i alone, specifying circles $\gamma_1, \ldots, \gamma_k$ in F.

Let $S = E(n) \setminus \nu(T^2)$ be one of the elliptic surfaces $E(n)$ minus a neighborhood of a regular fiber. Then S is simply connected[8], and a fiber sum with $E(n)$ along this regular fiber with a torus T in $F \times T^2$ will have the effect of killing $\pi_1(T)$ in the fundamental group. If μ is a meridian of the second factor of $F \times T^2$ then we can get a closed 4-manifold X with $\pi_1(X) = G$ by fiber summing a copy of $E(n)$ to each of the tori $\beta_i \times \mu$ for $i = 1, \ldots, l$ and $\gamma_i \times \mu$ for $i = 1, \ldots, k$ and also summing a copy to a torus $\{pt\} \times T^2$. This manifold will be symplectic if we can get a symplectic structure on $F \times T^2$ for which all these tori are actually symplectic submanifolds.

If ω is the product symplectic structure on $F \times T^2$ and $\eta = \pi_1^* \rho \wedge \pi_2^* \alpha$ with α a volume form on μ, ρ a 1-form on F which restricts to a volume form on all the β_i and γ_i, and $\pi_1 : F \times T^2 \to F$ and $\pi_2 : F \times T^2 \to T^2$ the projections, then we can form

$$\omega_t = \omega + t\eta,$$

and for sufficiently small t this will be symplectic on the tori $\beta_i \times \mu$ and $\gamma_i \times \mu$. Before we are really in a place to do the fiber sum we have to perturb these tori to be disjointly embedded. It is possible, but the details are omitted here.

The other issue is whether the form $\rho \in \Omega^1(F)$ we asked for above actually exists. There are situations where it does not, but Gompf avoids this problem by taking copies of T^2 with distinguished curves $a \times S^1$, $S^1 \times b$, and $S^1 \times c$ with $b \neq c$, designating a small disc $D \subset T^2$ intersecting c but not $a \cup b$, and equipping these tori with 1-forms which are volume forms on each of the three distinguished curves and are zero near D. Connect summing one of these to each edge of the graph in F formed by the α_i, β_i, and γ_i in our original setup yields a new F with new α_i, β_i, and γ_i which still gives the same fundamental group. The point is that we may assume without loss of generality that the η we wanted in the last paragraph did exist. $\square$

We now return to explaining the "map" above with the following:

Proposition 4.4 *For a fiber sum $M_1 \#_\psi M_2$ along a surface of genus g we have*

$$(c_1^2, c_2)(M_1 \#_\psi M_2) = (c_1^2, c_2)(M_1) + (c_1^2, c_2)(M_2) + (g-1)(8, 4).$$

Proof. First note that we can view $M_1 \#_\psi M_2$ as having been obtained from $(M_1 \times I) \cup (M_2 \times I)$ by attaching handles to the top level and rounding corners. This gives a cobordism from $M_1 \cup M_2$ to $M_1 \#_\psi M_2$, and since signature is a cobordism invariant we have

$$\sigma(M_1 \#_\psi M_2) = \sigma(M_1) + \sigma(M_2).$$

[8]It's because we can contract the loop we added by cutting out the codimension 2 fiber along what remains of a section ($S^2 \setminus \{pt\}$) of the elliptic fibration.

We have also $\chi(M_1\#_\Psi M_2) = \chi(M_1) + \chi(M_2) - 2\chi(N)$ where the sum is being taken along N. This gives the second component of the desired formula. The first follows easily from the formulas for σ and χ above together with $c_1^2 = 3\sigma + 2\chi$. □

Theorem 4.6 *[Gom95] Any pair of integers (m, n) satisfying the Noether and Rohlin conditions and $0 \leq m \leq 2n$ is realized as (c_1^2, c_2) of a closed spin symplectic 4-manifold.*

Proof. Recall that the Rohlin condition is $48|c_1^2 - 2c_2$. If $12|m+n$ and $48|m-2n$ then $(m, n) = (8k, 4k + 24l)$ for some $k, l \in \mathbb{Z}$. To prove the theorem, we need to realize such pairs with $k, l \geq 0$. Taking the product of two Riemann surfaces of genus $k + 1$ and 2 with the product symplectic structure gives a symplectic manifold with $(c_1^2, c_2) = (8k, 4k)$. Picking a homologically nontrivial loop in each of the factors of this product gives a Lagrangian torus. We can take l parallel copies of such a torus so that they are disjointly embedded. It is possible to perturb the symplectic form so that these Lagrangian tori become symplectic, but we omit the argument here.

Carrying out the symplectic fiber sum with l copies of $E(2)$ along its (torus) fiber adds $l(c_1^2(E(2)), c_2(E(2))) = (0, 24l)$ to our invariants, so we have seen how to realize all pairs $(8k, 4k + 24l)$ as desired. □

Theorem 4.7 *[Gom95] If (m, n) are integers with $n > 0$ and $-n + 16\frac{1}{2} \leq m \leq 2n - 33$ then there is a simply connected symplectic manifold X with $(c_1^2(X), c_2(X)) = (m, n)$.*

Proof. Like the last result, this is accomplished by fiber sums, but this time they are along surfaces of genus 2. We will need some building blocks.

Begin with a union of six lines in $\mathbb{CP}^2$ in general position. Resolve all but 8 points of intersection via deformation. Blowing up the 8 nodes will reduce the self-intersection from 36 to 4. An additional 4 blowups yeild $\mathbb{CP}^2\#12\overline{\mathbb{CP}^2}$ with a smoothly embedded surface of genus 2 having self-intersection 0. Call this manifold P. It is easily read off from the intersection form $\langle 1\rangle \oplus 12\langle -1\rangle$ that $c_2(P) = \chi(P) = 15$ and $\sigma(P) = -11$ so that $c_1^2(P) = -3$.

Consider the manifold Z described in Section 3.5. As we noted before, it is a T^2 bundle over T^2. It can be viewed as a quotient of $\mathbb{R}^4$ with the standard symplectic structure by the group generated by translations parallel to the x_1, x_2, and x_3 axes and the (symplectic) map

$$(x_1, x_2, x_3, x_4) \mapsto (x_1 + x_2, x_2, x_3, x_4 + 1).$$

It fibers over T^2 via projection onto the last two factors. Moreover, it is parallelizable, so $\chi(Z) = \sigma(Z) = 0$. Let Q be formed from two copies of Z, identifying two fibers by a 90 degree rotation $\psi(x_1, x_2) = (-x_2, x_1)$ extended to an orientation-reversing diffeomorphism of their normal bundles. We can do this so sections of the two copies of Z fit together to give a genus 2 surface with self-intersection 0 in Q.

Since the surfaces F of genus 2 in P and Q both have Euler number 0, there is an orientation-reversing diffeomorphism of their normal bundles. We can fix an identification of the normal bundle of F in P with $F \times D^2$, take k copies of $F \times p$ for k different points p close enough to $0 \in D^2$ to make $F \times p$ symplectic. We can then form the fiber sum $P\#kQ$. Now since $(c_1^2, c_2)(Q) = (0, 0)$ and we are summing along a surface of genus 2, Proposition 4.4 tells us that summing with k copies of Q increases (c_1^2, c_2) by $(8k, 4k)$. Thus we have realized the invariants $(8k - 3, 4k + 15)$ for all $k \geq 1$, and it is easy to check that all pairs satisfying the constraints in the theorem statement can be obtained from these by blowing up. (See Proposition 4.1.)

We still need to see that $P\#kQ$ is simply connected. This is achieved by showing that any loop in a copy of $Q \setminus (F \times D^2)$, where F is the surface obtained by piecing together the two sections, is trivial in $\pi_1(Q \setminus F)$. We omit the details of this here. Then note that $\pi_1(P \setminus kF)$ is generated by k meridians to the copies of F which were removed. These are homotopic to loops in the copies of Q and so to constant loops. $\square$

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