On the Kontsevich Formula for Deformation Quantization.

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1 Introduction

The aim of mechanics (both classical and quantum) is to study the evolution of a system. In classical mechanics [16] the set of possible states of a system forms a Poisson manifold M. The *observables* (physical quantities, depending on the state of the system, that we want to measure) are the smooth functions $C^\infty(M)$, a commutative algebra. In quantum mechanics [7] the set of possible states is a projective Hilbert space H. The observables are self-adjoint (although usually unbounded) operators, forming a non-commutative C^*-*algebra*.

The change from a Poisson manifold to a Hilbert space is a pretty big one. There are various methods to attempt this. (See [3] for a general definition of quantization and [1] for an introduction to geometric quantization). Deformation quantization was proposed in [2] as an alternative. Instead of building a Hilbert space from a Poisson manifold and associating an algebra of operators to it, we are only concerned with the algebra. After all, a quantum system can be totally studied in terms of its C^*-*algebra*. However, the product of classical observables is commutative, whereas the observables of a quantum system do not commute. [2] proposes to deform the commutative product in $C^\infty(M)$ to a non-commutative, associative product.

Generally, in mathematics, a deformation of an object is a family of objects depending on a parameter. (See [10]). Say X is an object in a certain category $\mathcal{C}$. A *deformation* of X is a family of objects $X_\epsilon \in Obj(\mathcal{C})$ depending on a *parameter* ϵ such that $X_{\epsilon_0} = X$ for a certain ϵ_0. In our case, we want to define a family of associative products in $C^\infty(M)$ depending smoothly on a parameter $\hbar$, such that we recover the standard pointwise commutative product when $\hbar = 0$.

We will state the problem in detail in section 2. In section 3 we will state the results of Kontsevich [15] that solve the problem and give a universal explicit formula. We also include examples and explicit calculations, leading to a simplification of Kontsevich formula. Finally, in section 4 we will see how that formula can be interpreted as a semiclassical expansion of a certain string theory using Feynman diagrams [4].

2 Deformation Quantization

2.1 Definitions

We define first Poisson manifolds, the context in which we will look for deformation products.

Definition. *A* Lie algebra *is a vector space B with a skew-symmetric bilinear operation $(f, g) \longrightarrow \{f, g\}$ satisfying the* Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

Definition. *A* Poisson algebra *is a vector space B together with two products: a commutative product*

$$(f, g) \longrightarrow fg$$

that makes B into a commutative algebra, and the Poisson bracket

$$(f, g) \longrightarrow \{f, g\}$$

that makes B into a Lie algebra, satisfying the Leibniz rule

$$\{f, g_1 g_2\} = g_1\{f, g_2\} + \{f, g_1\}g_2$$

Let M be a manifold and $A = C^\infty(M)$ be the algebra of smooth functions on M, where the product is pointwise multiplication.

Definition. *A* Poisson manifold *is a manifold M whose algebra A is a Poisson algebra with the pointwise multiplication as commutative product.*

We already have the commutative product in A. Hence, we only need to define the Poisson bracket

$$\{\ ,\} : A \otimes A \longrightarrow A$$

Since it satisfies the Leibniz rule in every component, it comes from a bivector field. That is,

$$\exists\ \alpha \in \Gamma(\Lambda^2 TX) \text{ such that } \{f, g\} = \alpha(df, dg) \quad \forall f, g \in A$$

In local coordinates $\{x^i\}$, a bivector field is represented by the functions $\alpha^{ij}(x)$ and the Poisson bracket of two functions becomes [1] $\{f, g\} = \alpha^{ij}\partial_i f \partial_j g$ where $\partial_i f = \partial f/\partial x^i$. The Jacobi identity can be written in terms of α: $[\alpha, \alpha] = 0$, [2] or, in local coordinates $\sum \alpha^{ir}\partial_r \alpha^{jk}$, where the sum is taken over the cyclic permutations of i, j, k.

It is time to begin defining a deformation product as we promised in the introduction. We start with the pointwise product on A (or the commutative product in a general algebra B) and would like to define a family of products depending on a parameter[3] $\hbar$, something like

$$\begin{aligned} \times_\hbar : A \times A &\longrightarrow A \\ (f, g) &\rightsquigarrow f \times_\hbar g \end{aligned}$$

where $\times_0$ is the commutative product. What does smooth dependence in $\hbar$ mean? We ask for the expression $\times_\hbar$ to be a power series in $\hbar$, more precisely: a formal power series (since we do not want to worry about convergence). But then $f \times_\hbar g$ is no longer in A, but in $A[[\hbar]]$

$$\times_\hbar : A \times A \longrightarrow A[[\hbar]]$$

This could be annoying, since we expect a product to eat two elements in an algebra, and give an element in the same algebra. To fix it, we can extend the product to the algebra $A[[\hbar]]$ by $\mathbb{C}[[\hbar]]$ linearity:

$$\star : A[[\hbar]] \times A[[\hbar]] \longrightarrow A[[\hbar]]$$

Notice that we changed the symbol for the product from $\times_\hbar$ to $\star$ since $\hbar$ is no longer a parameter, but a formal variable. We are now ready to define:

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[1] Einstein's convention should be assumed: repeated indices are summed.

[2] Here $[,]$ is the Schouten-Nijenhuis bracket. It is the bracket that makes $\Gamma(\Lambda TM)$ into a differential graded Lie algebra (subsection 2.2). See, for instance, [15].

[3] In Physics [4, 5], the convention is to call $i\hbar/2$ what here is called $\hbar$.

Definition. *A* deformed product *or* star product *in A is an associative, $\hbar$-adic continuous, $\mathbb{C}[\hbar]$ bilinear product*

$$\star : A[[\hbar]] \times A[[\hbar]] \longrightarrow A[[\hbar]]$$

that takes the particular value on A:

$$f \star g = \sum_{n=0}^{\infty} B_n(f,g)\hbar^n \quad \forall\, f,g \in A$$
$$f \star g|_{\hbar=0} = fg$$

where $B_n : A \times A \longrightarrow A$ are bidifferential operators.

Remarks about the definition.

1. The extension of the product to elements in $A[[\hbar]]$ using $\mathbb{C}[[\hbar]]$ linearity is given by

$$\left(\sum_{k=0}^{\infty} f_k \hbar^k\right) \star \left(\sum_{m=0}^{\infty} g_m \hbar^m\right) = \sum_{n=0}^{\infty}\left(\sum_{m+k+r=n} B_r(f_k, g_m)\right)\hbar^n$$

2. The condition that the $\star$ product recovers the commutative product for $\hbar = 0$ ($f \star g|_{\hbar=0} = fg$) is equivalent to $B_0(f,g) = fg$.

3. It is enough to ask for associativity on A (it extends then to $A[[\hbar]]$): $(f \star g) \star h = f \star (g \star h) \quad \forall\, f,g,h \in A$. This translates into the condition

$$\sum_{m+k=n} B_m(B_k(f,g),h) = \sum_{m+k=n} B_m(f, B_k(g,h)) \tag{1}$$

4. What are the operators $B_n : A \times A \longrightarrow A$? We could ask for any bilinear operator, but we want them to be bidifferential operators, i.e, bilinear maps which are differential operators with respect to each argument. Associativity will impose extra conditions.

Let's examine the first operator B_1. For $n = 1$ and $n = 2$, associativity condition (1) looks like:

$$B_1(fg,h) + B_1(f,g)h = B_1(f,gh) + fB_1(g,h) \tag{2}$$

$$\begin{aligned} &B_2(fg,h) + B_1(B_1(f,g),h) + B_2(f,g)h = \\ &\qquad = B_2(f,gh) + B_1(f,B_1(g,h)) + fB_2(g,h) \end{aligned} \tag{3}$$

From equation (2) and the same equation with cyclic permutations of f, g and h, we get that the antisymmetric part of B_1 satisfies Leibniz rule

$$B_1^-(g, hf) = hB_1^-(f, g) - B_1^-(f, h)g$$
$$\text{where } B_1^-(f, g) = 1/2(B_1(f, g) - B_1(g, f))$$

If we now add the six signed permutations of equation (3), we get exactly that B_1^- satisfies the Jacobi identity. Hence B_1^- is a Poisson bracket on M. Since we started with M, a Poisson manifold, it is natural to ask that $B_1^- = \{,\}$. Alternatively, if M is just a manifold, a deformation product turns M into a Poisson manifold.

The antisymmetric part of the coefficient of $\hbar$ in a star product is a Poisson bracket. We now want to consider the relationship between Poisson brackets and star products. Do Poisson brackets on M classify star products? Is there a star product for any Poisson bracket? How unique is it?

Definition. *A* quantization *of a Poisson manifold M is a star product on A such that $B_1^- = \{,\}$*

We will answer the questions in section 3. For the uniqueness we will need a certain action introduced in the next subsection. But first, an example:

Example: Moyal Product. Let $M = \mathbb{R}^n$ and considerer a Poisson structure with constant coefficients $\alpha = \alpha^{ij}\partial_i \otimes \partial_j$, $\alpha^{ij} = -\alpha^{ji} \in \mathbb{R}$. Then the *Moyal product* is defined as a formal exponential of α:

$$\begin{aligned} f \star g &= e^{\hbar\alpha}(f, g) = \\ &= fg \; + \; \hbar\alpha^{ij}\partial_i f \partial_j g \; + \; \frac{\hbar^2}{2}\alpha^{ij}\alpha^{kl}\partial_i\partial_k f \partial_j \partial_l g + \cdots = \\ &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left(\alpha^{i_1 j_1} \cdots \alpha^{i_n j_n}\right) \left(\partial_{i_1} \cdots \partial_{i_n} f\right) \left(\partial_{j_1} \cdots \partial_{j_n} g\right) \end{aligned} \tag{4}$$

The Moyal product is a deformation of (M, α) but this formula is only valid when α has constant coefficients.

We note that this definition of quantization may be understood to come from Dirac [9]. He suggested that, in order to quantize, we should look for an associative, non-commutative product $\star$ on A and define the commutator as $\{f, g\} = -(i/\hbar)(f \star g - g \star f)$. He meant up to $O(\hbar^2)$ (the kind of statement it is usually not made explicit in Physics). [4]

[4]Note how our $\hbar$ is $i\hbar/2$ for Dirac.

2.2 A gauge group action

We will now introduce a group acting on $A[[\hbar]]$ by *changes of coordinates.* We mean changes of coordinates in $\operatorname{Spec} A[[\hbar]] = M \times \operatorname{Spec} \mathbb{C}[[\hbar]]$ preserving the projection to $\operatorname{Spec} \mathbb{C}[[\hbar]]$. More specifically, such a change of coordinates will be defined by a map $D : A[[\hbar]] \longrightarrow A[[\hbar]]$. As with the definition of the star product, it is enough to define $D : A \longrightarrow A[[\hbar]]$, then ask for it to be $\mathbb{C}[[\hbar]]$ linear. We will write

$$D(f) = \sum_{n=0}^{\infty} D_n(f)\hbar^n$$

where $D_n : A \longrightarrow A$ are differential linear operators. The operator D is invertible if and only if D_0 is invertible. We insist on $D_0 = 1$ and its inverse $E = D^{-1}$ is defined by

$$\begin{aligned} E_0 &= 1 \\ E_n &= -\sum_{m=0}^{n-1} E_m D_{n-m} \text{ for } n > 0 \end{aligned}$$

Definition. *D as defined above is called a* gauge transformation *in A. The set os such D is naturally a group.*

Given a star product $\star$ (defined by operators B_n) and a gauge transformation D (with inverse E) we can think of D as a change of formal coordinate and call $\star'$ the product in the new coordinate: $\star' = D(\star)$.

$$\begin{tikzcd} A[[\hbar]]\times A[[\hbar]] \arrow[r, "\star"] \arrow[d, "D\times D"] & A[[\hbar]] \arrow[d, "D"] \\ A[[\hbar]]\times A[[\hbar]] \arrow[r, "\star'"] & A[[\hbar]] \end{tikzcd}$$

Defined in this way, $\star'$ is clearly associative. It is indeed a new star product:

$$\begin{aligned} f \star' g &= \sum_{n=0}^{\infty} C_n(f,g)\hbar^n \\ C_n(f,g) &= \sum_{m+k+l+j=n} D_m B_k(E_l f, E_j g) \end{aligned} \tag{5}$$

Hence we have an action of gauge transformations on star products. We are interested in star products only up to gauge equivalence. This equivalence relation is closely related to the *Hochschild complex*, which we now describe.

Definition. *For $n \geq 0$ define*

$$C^n(A) = \{ f : A^n \longrightarrow A \mid f \text{ multilinear, differential operator } \}$$

The Hochschild complex is the complex with the chains and the differential

$$\begin{aligned}
d_n : & C^n(A) \longrightarrow C^{n+1}(A) \\
(d_n(\phi))(f_0, \dots, f_n) = & f_0\phi(f_1, \dots, f_n) - \\
& - \phi(f_0 f_1, \dots, f_n) \; - \; \phi(f_0, f_1 f_2, \dots, f_n) + \\
& + (-1)^n \phi(f_0, \dots, f_{n-1} f_n) \; + \; (-1)^{n+1} \phi(f_0, \dots, f_{n-1}) f_n
\end{aligned}$$

Note that we could define this for any associative algebra A.

Lemma 2.1. *Let M be any manifold and $A = C^\infty(M)$. Then $d^2 = 0$.*

Theorem 2.2. *The Hochschild complex has cohomology $H^n_{Hoch}(A) = \Lambda^n TM$, the algebra of smooth multivector fields on M.*

Looking back to equation (2) we see that associativity of the star product at order 1 is equivalent to B_1 being closed in the Hochschild complex. Apply a gauge transformation on $\star$. With the notation previously introduced, equation (5) at first order $n = 1$ is:

$$C_1(f, g) = B_1(f, g) - f D_1(g) + D_1(fg) - D_1(g) \tag{6}$$

Thus $C_1 - B_1 \in H^1_{Hoch}(A)$. In short, the first operator of a star product, up to gauge equivalence, is a first cohomology class of the Hochschild complex. Note in equation (6) that the gauge transformation only affects the symmetric part of B_1 (since a Hochschild 1-coboundary is always symmetric). We can always find a gauge transformation that kills the symmetric part. Hence, up to gauge equivalence, we may assume that B_1 is skew-symmetric. We can rewrite this in many fancy ways:

Lemma 2.3.

1. *The skew-symmetric part of B_1 is gauge invariant.*
2. *In any gauge equivalence class of star product there is one whose operator B_1 is skew-symmetric.*

3. *A Poisson manifold can be quantized (i.e. find a star product with* $B_1^- = \{,\}$*) if, and only if, it can be quantized with* $B_1 = \{,\}$*.*

4. *If we quantize a Poisson manifold then, with a gauge transformation, we can get any symmetric part* B_1^+*.*

Thus, we will only look for star products where B_1 is skew-symmetric.

3 Kontsevich Formula

3.1 The formality conjecture and some results of Kontsevich

Bayen et al [2] stated the problem of deformation quantization; Kontsevich conjectured a solution [13], proved it and gave an explicit formula [15]. Although we will not reproduce his proof here, we will state the conjecture that leads to the solution. His proof (that we will not deal with) is in terms of the cohomology of the Hochschild complex. To state things properly, we need some definitions (see [13, 15]).

Definition. *A* differential graded Lie algebra *or* DGLA *over a field* k *of characteristic zero is a complex*

$$g = \bigoplus_{k\in\mathbb{Z}} g^k \qquad d : g^k \longrightarrow g^{k+1} \qquad d^2 = 0$$

which is also a graded Lie algebra with respect to the same gradation

$$\begin{aligned}
&[\ ,\ ] : g^k \times g^m \longrightarrow g^{k+m}\\
&[x_1,[x_2,x_3]] + (-1)^{s_3(s_1+s_2)}[x_3,[x_1,x_2]] + (-1)^{s_1(s_2+s_3)}[x_2,[x_3,x_1] = 0\\
&[x_2,x_1] = (-1)^{1+s_1 s_2}[x_1,x_2] \quad \forall\, x_i \in g^{s_i}
\end{aligned}$$

plus a compatibility axiom:

$$d[x_1,x_2] = [dx_1,x_2] + (-1)^{s_1}[x_1,dx_2]$$

The Hochschild complex is the first example. We defined in subsection 2.2 the chain and the differential. The bracket is

$$[\phi_1,\phi_2] = \phi_1 \circ \phi_2 - (-1)^{k_1 k_2}\phi_2 \circ \phi_1$$

for $\phi_i \in C^{k_i}(A)$, where the product $\circ$ is defined by:

$$(\phi_1 \circ \phi_2)(x_0, \ldots, x_{k_1+k_2}) =$$
$$= \sum_{j=0}^{k_1} (-1)^{ik_2} \phi_1(x_0, \ldots, x_{i-1}, \phi_2(x_j, \ldots, x_{j+k_2}), x_{j+k_2+1}, \ldots, x_{k_1+k_2})$$

Another example is the algebra of smooth multivector fields on a manifold $\Gamma(\Lambda TM)$. This complex happens to be isomorphic to the Hochschild cohomology complex. We can finally define a concept of *quasi-isomorphism* among DGLAs. The actual definition is a bit long (see [15]) but, in few words, two DGLAs are quasi-isomorphic if there is a morphism between them that induces isomorphism on cohomology. Then Kontsevich stated [13]:

Theorem 3.1 (Kontsevich Formality Conjecture). . *The Hochschild complex and its cohomology are quasi-isomorphic as DGLAs.*

It was proven in [15], so is no longer a conjecture. Its importance relies on the following fact:

Theorem 3.2. *If a smooth manifold satisfies the formality conjecture, then any Poisson structure on it can be quantized.*

See [19, 20] for a short account of a proof.

Finally, the classification of star products up to gauge equivalence comes from:

Theorem 3.3. *Let M be a smooth manifold and $A = C^\infty(M)$. Then there is a natural one-to-one correspondence between star products on M modulo gauge equivalence and equivalence classes of deformations of the null Poisson structure on M.*

A deformation of the null Poisson structure on M is a formal bivector field

$$\alpha(\hbar) = \sum_{k=1}^{\infty} \alpha_k \hbar^k \in \Gamma(\Lambda^2 TM)[[\hbar]]$$

such that $[\alpha, \alpha] = 0 \in \Gamma(\Lambda^3 TM)[[\hbar]]$. We consider equivalence classes of those with respect to the action of formal paths in the diffeomorphism group $Diff(M)$ starting at the identity diffeomorphism. In particular, any Poisson structure $\alpha_{(0)}$ gives a path $\alpha(\hbar) = \alpha_{(0)} \cdot \hbar$ and hence, according to the previous theorem, a canonically well defined equivalence class of star products. As a consequence, α does come from a star product. Our problem is solved.

For a friendly statement of these results, as well as a list of historical partial advances in the problem of deformation quantization, see [11].

3.2 A explicit universal formula for the deformation of a Poisson structure

We recall the problem: we have a bivector field α in a manifold M, defining a Poisson bracket in A. In local coordinates this is $\{f,g\} = \alpha^{ij}\partial_i f \partial_j g$. We want to define a product $\star : A \times A \longrightarrow A[[\hbar]]$ such that $f \star g = fg + \hbar\{f,g\} + O(\hbar^2)$. The results in subsection 3.1 solve the existence and uniqueness problem. We, now, will discuss an explicit universal formula to construct the star product from [15]. This formula is valid when M is an open subset of $\mathbb{R}^d$ (see remark at the end of this subsection, however) and comes in local coordinates.

The operator B_n is defined as a sum of contributions from a certain class of graphs.

Definition. *An* admissible graph Γ of order n *is a finite oriented graph with the following properties and notation:*

- Γ *has* $n+2$ *vertices,* V_Γ*, labeled* $1, \ldots, n, L, R$*.* L *and* R *are just symbols for Left and Right.*
- Γ *has* $2n$ *edges,* E_Γ*, labeled* $i_1, j_1, i_2, j_2, \ldots, i_n, j_n$*.*
- *The edges* i_k *and* j_k *start at the vertex* k *and do not end at the vertex* k*.*
- *The edges* i_k *and* j_k *end at different vertices from each other.*

Let $\mathcal{G}_n$ *be the set of all admissible graphs of order* n*, and* $\mathcal{G}$ *the set of all admissible graphs with any finite order. If* $\Gamma \in \mathcal{G}_n$*, we write* $|\Gamma| = n$*.*

Equivalently, an admissible graph or order n is a pair of maps $i, j : \{1, \cdots, n\} \longrightarrow \{1, \cdots, n\} \cup \{L, R\}$ where every map has no fixed points and both maps are distinct at every point. There are $((n+1)n)^n$ such graphs. For instance, the graph of order 2 with vertices $i_1 = (1,2)$, $j_1 = (1,L)$, $i_2 = (2,L)$

and $j_2 = (2, R)$ is:

$$
\begin{tikzcd}
\bullet 1 \arrow[r, "i_1"] \arrow[d, "j_1"'] & \bullet 2 \arrow[dl, "i_2"] \arrow[d, "j_2"] \\
\bullet L & \bullet R
\end{tikzcd}
\tag{7}
$$

To each such graph, we are going to assign a weight (number) ω_Γ and a bidifferential operator $B_\Gamma : A \times A \longrightarrow A$. We will define the star product as:

$$f \star g = \sum_{\Gamma \in \mathcal{G}} \hbar^{|\Gamma|} \omega_\Gamma B_\Gamma(f, g) \tag{8}$$

The corresponding operators B_n are then:

$$B_n(f, g) = \sum_{\Gamma \in \mathcal{G}_n} \omega_\Gamma B_\Gamma(f, g) \tag{9}$$

To define the action of the operator $B_\Gamma(f, g)$ associated to a graph Γ use the following algorithm: write $\alpha^{i_k j_k}$ for every vertex k; write f for the vertex L, and write g for the vertex R. Then, for every edge l, add ∂_l in front of the symbol corresponding to the vertex where the edge l ends. For instance, for the graph Γ of our previous example (7), the operator is:

$$
\begin{aligned}
B_\Gamma(f, g) &= \alpha^{i_1 j_1} \partial_{i_1} \alpha^{i_2 j_2} \; \partial_{i_2} \partial_{j_1} f \; \partial_{j_2} g \\
B_\Gamma &= \alpha^{i_1 j_1} \partial_{i_1} \alpha^{i_2 j_2} \; \partial_{i_2} \partial_{j_1} \otimes \partial_{j_2}
\end{aligned}
\tag{10}
$$

A general formula can be formally written, but we will omit it since it is not enlightening. (See [15])

The weight ω_Γ is more complicated. Let $\mathcal{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ be the upper half-plane endowed with the Lobachevsky metric. For $n \geq 1$, let $C_n(\mathcal{H}) = \{(p_1, \ldots, p_n) \in \mathcal{H}^n \mid p_i \neq p_j \text{ for } i \neq j\}$. Given $p, q \in \mathcal{H}$, $p \neq q$ let $l(p, q)$ be the unique geodesic line from p to q, let $l(p, \infty)$ be the vertical line through p, and define $\phi(p, q)$ to be the angle at p from $l(p, \infty)$ to $l(p, q)$ measured counterclockwise. This then defines a map $\phi : C_2(\mathcal{H}) \longrightarrow S^1$. Incidentally, this map can be extended to a certain compatification of $C_2(\mathcal{H})$ (see [15]). However, since we are not proving the associativity of this formula here, suffice it to say that we extend it by continuity when needed (eg. for $p \in \mathbb{R}$). Now, given a graph $\Gamma \in \mathcal{G}_n$, we can define a 2n-form in $C_n(\mathcal{H})$, and the weight will be a multiple of its integral. More specifically, $\beta_\Gamma \in \Lambda^{2n}(C_n(\mathcal{H}))$ is defined by

$$\beta_\Gamma(u_1, \ldots, u_n) = \bigwedge_{l \in E_\Gamma} d\phi(u_{\pi_1(l)}, u_{\pi_2(l)}) \tag{11}$$

and then

$$\omega_\Gamma = \frac{1}{n!(2\pi)^{2n}} I_\Gamma \qquad I_\Gamma = \int_{C_n(\mathcal{H})} \beta_\Gamma \tag{12}$$

Here, for an edge $l \in E_\Gamma$, $\pi_1(l)$ is the origin of the edge and $\pi_2(l)$ is the end of the edge. Take $u_L = 0$ and $u_R = 1$. We consider the standard orientation in $\mathcal{H}$ and the corresponding one in $C_n(\mathcal{H})$. Wedge product in the definition of β_Γ is taken in the order $i_1, j_1, \ldots, i_n, j_n$. This completes the definition of the star product. We can now state:

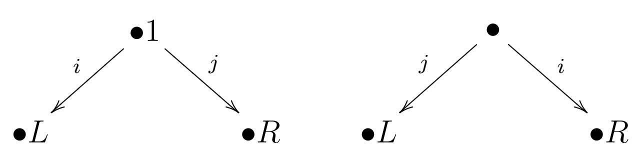
Theorem 3.4. *Let α be a Poisson bivector field in a domain of $\mathbb{R}^d$. Then the formula (8), together with (10) and (12), defines an associative product. If we change coordinates, we obtain a gauge equivalent star product.*

For a proof of associativity see [15]. To perform the integral (12) (and to prove that it is absolutely convergent) it is convenient to use a change of variables. Instead of the variables $\{Re(u_k), Im(u_k)\}_{k=1}^n$, use the variables $\{\phi_l\}_{l\in E_\Gamma}$, where $\phi_l = \phi(u_{\pi_1(l)}, u_{\pi_2(l)})$. Then equation (12) becomes:

$$\omega_\Gamma = \frac{1}{n!(2\pi)^{2n}} \int_{R_n} \bigwedge_{l\in E_\Gamma} d\phi_l$$

where R_n is a subset of $[0, 2\pi]^n$ defined by inequalities (eg $\phi_{i_1} < \phi_{j_1}$). The integral then becomes $(2\pi)^{2n}$ times a (hopefully "simple") number q, and $\omega_\Gamma = q/n!$. For those graphs for which we have an explicit calculation, q is always rational.

Let's make some explicit calculations, at least untill order 1. (See [12] for a complete calculation in a certain family of graphs). For $n = 0$ there is only one admissible graph, whose contribution is fg. For $n = 1$ there are two graphs, say Γ_1 and Γ_1'.



Γ_1 $\qquad$ Γ_1'

They differ in switching the two edges. One of them gives the bidifferential operator $B_{\Gamma_1}(f, g) = \alpha^{ij}\partial_i f \partial_j g$ and the weight

$$\omega_{\Gamma_1} = \frac{1}{(2\pi)^2} \int_{\mathcal{H}} d\phi(u, 0) \wedge d\phi(u, 1)$$

We can use a explicit formula for ϕ:

$$\phi(p,q) = arg((q-p)(p-\overline{q})) = \frac{1}{2i}\log\frac{(q-p)(\overline{q}-p)}{(q-\overline{p})(\overline{q}-\overline{p})} \tag{13}$$

The formula extends to 0 and 1 by continuity as mentioned. (Note also that $\phi(u,0) = \arg(u)/2$). Now we could use brute force and integrate with the variables $x = Re(u)$, $y = Im(u)$, but it is more ingenious to use $\phi_0 = \phi(u,0)$, $\phi_1 = \phi(u,1)$ and integrate in $R = \{0 \le \phi(u,0) \le \phi(u,1) \le 1\}$:

$$\omega_{\Gamma_1} = \frac{1}{(2\pi)^2}\int_R d\phi_0 d\phi_1 = \frac{1}{(2\pi)^2}\frac{(2\pi)^2}{2} = \frac{1}{2}$$

The diagram Γ_1' is obtained from Γ_1 by switching the two edges. This just changes the orientation of the two form and hence $\omega_{\Gamma_1'} = -\omega_{\Gamma_1}$ The contribution to the star product at first order in $\hbar$ is

$$\frac{\hbar}{2}\alpha^{ij}(\partial_i f \partial_j g - \partial_j f \partial_i g) = \hbar\ \alpha^{ij}\ \partial_i f \partial_j g \ = \ \hbar\{f,g\}$$

because $\alpha^{ij} = -\alpha^{ji}$.

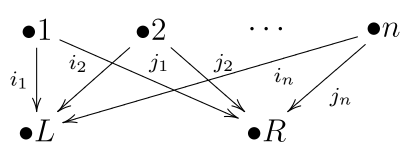
Remark: We have given a construction in local coordinates. The formula is only invariant under affine transformation. If we change coordinates, as stated in the theorem, we get a gauge equivalent star product. But as the formula 8 transform in a very complicated way under diffeomorphisms, it is nontrivial to get a global formula for a generic Poisson manifold. See [6, 5] for a description.

3.3 A simplification of Kontsevich formula

Note The ideas behind this subsection, including definitions 3.5 and 3.6, as well as theorem 3.8 appeared in Kathotia's thesis and were later published in [12]. Unaware of his work when writing this paper, our notation is quite different.

If we are given a Poisson structure and want to use formula (8) directly, we may need to calculate the contribution of infinitely many graphs. But there are simplifications we can make. Let's start with the simplest non trivial example: the Moyal product.

Example: back to the Moyal product .
We expect to recover equation (4) when we apply formula (8) to the case of constant functions α^{ij}. In such a case, a graph with an edge ending in a vertex other than L or R will have zero contribution, since it will include a term of the form $\partial_i \alpha^{jk}$ that vanishes. At order n, we only need to consider graphs where every vertex $k = 1, \cdots, n$ has two edges ending in L and R. There are 2^n such graphs, differing in the order of the pair of edges starting at each vertex. But they will all have the same contribution, as we saw in the explicit calculus at order 1. If we exchange the two edges starting at, say, k, the form β_Γ changes its sign, and the whole contribution does not change since α^{ij} are skew-symmetric. Let us call Γ_n the graph of order n where every vertex k has the first edge to L and the second to R:



For the case $n = 1$ we recover what we already called Γ_1 in the previous subsection. Then the bidifferential operator B_n in formula 9 will be in this case:

$$
\begin{aligned}
& B_n(f,g) = 2^n\, \omega_{\Gamma_n} B_{\Gamma_n}(f,g) = \\
& = \frac{2^n}{n!(2\pi)^{2n}} \left(\int_{C_n(\mathcal{H})} \beta_{\Gamma_n} \right) \left(\alpha^{i_1 j_1} \cdots \alpha^{i_n j_n} \right) \left(\partial_{i_1} \cdots \partial_{i_n} f \right) \left(\partial_{j_1} \cdots \partial_{j_n} g \right)
\end{aligned}
$$

But the 2n-form in this case decomposes as:

$$
\beta_{\Gamma_n}(u_1, \cdots, u_n) = \bigwedge_{l=1}^{n} \big(d\phi(u_l, u_0) \wedge d\phi(u_l, u_1) \big) = \bigwedge_{l=1}^{n} \beta_{\Gamma_1}(u_l) \,,
$$

and we already knew the value of the integral for the case $n = 1$, so:

$$
\begin{aligned}
\frac{2^n}{n!(2\pi)^{2n}} \int_{C_n(\mathcal{H})} \beta_{\Gamma_n} &= \frac{2^n}{n!(2\pi)^{2n}} \left(\int_{\mathcal{H}} \beta_{\Gamma_1} \right)^n = \\
&= \frac{2^n}{n!(2\pi)^{2n}} \left(\frac{(2\pi)^2}{2} \right)^n = \frac{1}{n!} \,,
\end{aligned}
$$

recovering equation (4):

$$
\begin{aligned}
f \star g = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \big(\alpha^{i_1 j_1} \cdots \alpha^{i_n j_n}\big) \big(\partial_{i_1} \cdots \partial_{i_n} f\big) \big(\partial_{j_1} \cdots \partial_{j_n} g\big) = \\
= \exp\{\hbar \alpha_{ij} \partial_i \otimes \partial_j\}(f,g) = \exp\{\hbar\, 2\omega_{\Gamma_1} B_{\Gamma_1}\}
\end{aligned}
$$

In this example we have been able to construct the whole star product with just one graph. The graphs Γ_n are somehow built from the graph Γ_1, and so are their contributions. Although in general we will need more than one graph, we can generalize this idea of decomposing admissible graphs as "sum" of simpler ones.

Definition 3.5. *Given two admissible graphs $\Gamma \in \mathcal{G}_m$ and $\Gamma' \in \mathcal{G}_k$ we define its* sum *as the admissible graph $\Gamma + \Gamma' \in \mathcal{G}_{m+k}$ resulting of putting together the two graphs, relabeling the vertices of Γ' from $1, 2, \ldots, k$ to $m+1, m+2, \ldots, m+k$ and identifying the two vertices labeled L and the two vertices labeled R. For $n \in \mathbb{N}$, $n \cdot \Gamma$ means $\underbrace{\Gamma + \cdots + \Gamma}_{n}$.*

For instance, if Γ and Γ' are

$$
\begin{tikzcd}
\bullet 1 \arrow[r, "i_1"] \arrow[d, "j_1"'] & \bullet 2 \arrow[dl, "i_2"'] \arrow[d, "j_2"] \\
\bullet L & \bullet R
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet 1 \arrow[r, bend left, "j_1"] \arrow[dr, "i_1"] & \bullet 2 \arrow[l, bend left, "j_2"] \arrow[d, "i_2"] \\
\bullet L & \bullet R
\end{tikzcd}
$$

then $\Gamma + \Gamma'$ is

$$
\begin{tikzcd}
\bullet 1 \arrow[r, "i_1"] \arrow[dr, "j_1"] & \bullet 2 \arrow[d, "i_2"] \arrow[dr, "j_2"] & \bullet 3 \arrow[r, bend left, "j_3"] \arrow[d, "i_3"] & \bullet 4 \arrow[l, bend left, "j_4"] \arrow[dl, "i_4"] \\
& \bullet L & \bullet R &
\end{tikzcd}
$$

With the notation of the Moyal example, $\Gamma_n = n \cdot \Gamma_1$. This operation is clearly associative and has an identity, namely the only admissible graph of order 0, making $\mathcal{G}$ into a semigroup.

Definition 3.6. *A* simple graph *is an admissible graph which is connected once we have erased the vertices L, R and every edge ending in them. Equivalently, an admissible graph is simple if and only if every pair of vertices* *labeled by numbers are connected though a path that avoids L and R. We call $\widetilde{\mathcal{G}}$ the set of all simple graphs, and we call $\widetilde{\mathcal{G}}_n$ the set of all simple graphs of order n.*

For instance, in the previous example Γ and Γ' are simple, but $\Gamma + \Gamma'$ is not. One would be tempted to say that the simple graphs generate $\mathcal{G}$ as a semigroup. This is not true. Consider the graph $\Gamma + \Gamma'$ in the previous example and exchange the vertices 2 and 3. The resulting graph is not simple, but it can not be written as a sum of simple graphs, according to our definition of sum. Hence simple graphs generate $\mathcal{G}$ as a semigroup, only if we consider graphs up to exchanging the numbered vertices. We will call this quotient $\mathcal{G}/\sim$. More formally, there is a natural action of the symetric group S_n on $\mathcal{G}_n$ and

$$\mathcal{G}/\sim = \bigcup_{n=0}^{\infty} \mathcal{G}_n/S_n$$

Likewise, we define simple graphs up to exchanging the numbered vertices $\widetilde{\mathcal{G}}/\sim$. We have $\widetilde{\mathcal{G}}/\sim \subset \mathcal{G}/\sim$ For a graph $\Gamma \in \mathcal{G}$, we call its class $\underline{\Gamma} \in \mathcal{G}/\sim$. The sum is well defined in this quotient, and $\mathcal{G}/\sim$ is a commutative semigroup.

Lemma 3.7. *1. An admissible graph Γ is simple if and only if every graph in $\underline{\Gamma}$ cannot be expressed as a proper sum of other admissible graphs (i.e., we cannot exchange the numbered vertices and then express it as a proper sum of admissible graphs).*

2. In every class of admissible graph there is one that can be expressed uniquely (up to reordering) as a sum of simple graphs (i.e., every admissible graph, after exchanging the numbered vertices, can be expressed uniquely as a sum of simple graphs).

3. $\widetilde{\mathcal{G}}/\sim$ generates $\mathcal{G}/\sim$ as a semigroup.

4. $\mathbb{C}[\mathcal{G}/\sim]$, the commutative semigroup algebra (formal linear combinations of elements in $\mathcal{G}$), is the free, commutative algebra generated by $\widetilde{\mathcal{G}}/\sim$ and is graded by the order.

Our goal is to reduce the sum over $\mathcal{G}$ to a sum over $\widetilde{\mathcal{G}}$. It is immediate from the definition (11) of the form β_Γ, that

$$\beta_{\Gamma+\Gamma'}(u_1, \ldots, u_{m+k}) = \beta_\Gamma(u_1, \ldots, u_m) \wedge \beta_{\Gamma'}(u_{m+1}, \ldots, u_{m+k})$$

and hence $I_{\Gamma+\Gamma'} = I_\Gamma \cdot I_{\Gamma'}$. I is a semigroup homomorphism, and so is B (the bidifferential operator associated to a graph). Hence the term $\omega_{\Gamma+\Gamma'} B_{\Gamma+\Gamma'}$ equals, up to a numerical coefficient, to $\omega_\Gamma \omega_{\Gamma'} B_\Gamma B_{\Gamma'}$. This enables us to rewrite Kontsevich formula (8) rearranging terms.

Theorem 3.8. *Under the conditions for theorem 3.4, the star product can be rewritten as*

$$\begin{aligned} f \star g =& \sum_{\Gamma \in \mathcal{G}} \hbar^{|\Gamma|} \omega_\Gamma B_\Gamma(f,g) = \\ =& \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\Gamma_1,\dots,\Gamma_N \in \widetilde{\mathcal{G}}} \hbar^{|\Gamma_1|+\dots+|\Gamma_N|}\, \omega_{\Gamma_1} \cdots \omega_{\Gamma_N} B_{\Gamma_1} \cdots B_{\Gamma_N}(f,g) = \\ =& \exp\left(\sum_{\Gamma \in \widetilde{\mathcal{G}}} \hbar^{|\Gamma|} \omega_\Gamma B_\Gamma \right) (f,g) \end{aligned} \tag{14}$$

Proof. We already explained that the "monomials" appearing in the two series in equation (14) are the same. Hence, we just need to prove that the coefficient of every monomial is the same in both sides of the equality. A term appearing in the left is of the form $\hbar^{|\Gamma|}\omega_\Gamma B_\Gamma$ for certain $\Gamma \in \mathcal{G}$. Let's decompose it as a sum of simple graphs $\Gamma = r_1\Gamma_1 + \ldots + r_s\Gamma_s$ where $r_i \in \mathbb{N}$ and $\Gamma_i \in \widetilde{\mathcal{G}}$. Put $|\Gamma_i| = n_i$ and $m = r_1 n_1 + \ldots r_s n_s$. Then we know:

$$\hbar^{|\Gamma|}\omega_\Gamma B_\Gamma = \frac{1}{m!(2\pi)^{2m}}\, \hbar^m\, I_{\Gamma_1}^{r_1} \cdots I_{\Gamma_s}^{r_s}\, B_{\Gamma_1}^{r_1} \cdots B_{\Gamma_s}^{r_s} \tag{15}$$

where $I_{\Gamma_1}^{r_1}$ (resp. $B_{\Gamma_1}^{r_1}$) means the product of I_{Γ_1} (resp. B_{Γ_1}) with itself r_1 times. That term comes from $\Gamma = r_1\Gamma_1 + \ldots + r_s\Gamma_s$, but the same term will occur when we exchange the numbered vertices. Hence, we will get this contribution times the number of elements in the class $\underline{\Gamma}$:

$$\frac{m!}{(n_1!)^{r_1} \cdots (n_s!)^{r_s}}\, \frac{1}{r_1! \cdots r_s!}\, \hbar^{|\Gamma|}\omega_\Gamma B_\Gamma \tag{16}$$

Now to the other member of equation (14). We get this contribution for $N = r_1 + \ldots + r_s$. Again the contribution appears a number of times (for reordering the simple graphs) and we know the contribution of each term, leading to:

$$\frac{N!}{r_1! \cdots r_s!}\, \frac{1}{N!} \left(\frac{I_{\Gamma_1}\hbar^{n_1}}{n_1!(2\pi)^{2n_1}} \right)^{r_1} \cdots \left(\frac{I_{\Gamma_s}\hbar^{n_s}}{n_s!(2\pi)^{2n_s}} \right)^{r_s} B_{\Gamma_1}^{r_1} \cdots B_{\Gamma_s}^{r_s} \tag{17}$$

And we just check that (17) equals (16) when we substitute (15). $\square$

In the case of the Moyal product there was only one graph in $\widetilde{\mathcal{G}}$ whose contribution did not vanish. In general there are still infinitely many, but still fewer, at least at every order, than in $\mathcal{G}$.

4 Kontsevich Formula as a Feynman path integral expansion

Anyone who has read quantum field theory and sees the Kontsevich formula for deformation quantization thinks immediately of Feynman diagrams. [5] In this section we will try to explain why. First, we will "define" some elements from field theory, what a problem in classical field theory is, how to get to quantum field theory, and how we try to solve it. In this process, Feynman diagrams will appear. Next, we will introduce a particular sigma model in which the expansion in Feynman diagrams of the expected value of a certain observable is exactly equation (8). *Disclaimer:* This section is going to be, at best, very sketchy.

4.1 Some ideas from field theory

For a detailed, self-contained introduction to quantum field theory (in particular, Feynman diagrams), see [18].

First step: classical field theory.

In classical field theory we start with a manifold D, representing space-time, and a certain bundle E over D. The *fields* are the sections of this bundle $\Gamma(E)$. The *observables* (the physical quantities we are interested in) form an algebra O constructed from $\Gamma(E)$. Observables are usually formal polynomials in the fields and their derivatives. Locally, by chosing an atlas and trivializing the bundle, we may represent fields by functions on D.

Next we define an *action*, a functional consisting of an integral over D of

[5] Even Kontsevich. Although his exposition in [15] is mathematically rigorous and makes no reference to quantum field theory, he started with Feynman diagrams (see [14]).

a top form built from the fields.

$$S : \Gamma(E) \longrightarrow \mathbb{R}$$
$$\phi \rightsquigarrow S[\phi] = \int_D \mathcal{L}[\phi]$$

$\mathcal{L}[\phi]$, in local coordinates, will look like a polynomial on the fields and their derivatives. The state of the system is a field that minimizes the action. We call it the *classical solution*. This is a variational problem that we solve using Euler-Lagrange equations. The value of an observable in the system is its value in the classical solution.

We can generalize the problem to more complicated fields, which can always be seen as sections of a more complicated bundle. First, we usually have more than one field. This is equivalent to starting with various bundles. Consider that E is the tensor product of them. Second, we impose *boundary conditions* on the fields. This is equivalent to restricting the sections in $\Gamma(E)$ we accept, or going to a subbundle of E. Thirdly, we may have a *sigma model*. A sigma model means that we not only have sections of a bundle as fields, but also include maps of D in a certain manifold M to construct the algebra of observables. But notice that maps $D \longrightarrow M$ are the sections of the trivial bundle $D \times M$ over D. Finally, we have to mention that D can be a manifold with corners instead of a smooth manifold.[6]

Towards quantization: we need a state.

Let's move on to quantization [7]. We want a deformation of the algebra of observables O, which we will call $O_\hbar$. But we want to deform not only the algebra. In the classical level we had some more information: the classical solution. Recall that the classical solution is a particular field that represents the state of the system. Such a field can be thought as a linear map

$$\langle\,\rangle \, : O \longrightarrow \mathbb{C}$$

defined by evaluation. In the language of C^*-algebras, this is called a *state*. The name comes from quantum mechanics, where the space of states of a

[6] As a matter of fact, since field theory is not totally defined or formalized, there are lots of generalizations.

[7] This is even less clear than in mechanics. As a matter of fact, a common practice, once the bundle is trivialized, is to promote the fields from functions, to operators and start playing around with them without defining the Hilbert space they are acting on.

system is a projective Hilbert space. Given a state (a vector v) and an observable (an operator A), the value of the observable in that state is $\langle v|Av\rangle$, where $\langle\ |\ \rangle$ is the Hilbert product. Thus, a state is a map from the algebra of observables to the complex numbers. Then we generalize the concept of a state to certain kinds of maps. (See [8] for a very clear exposition). In our problem, we want to deform the algebra and state $(O, \langle\ \rangle)$ to $(O_\hbar, \langle\ \rangle_\hbar)$. The state $\langle\ \rangle_\hbar$ is also called the *ground state* or the *vacuum*. Although there are other states in the C^*-algebra $O_\hbar$ that have physical meaning, we first study only the ground state. We then try to express the value of an observable in any state in terms of values of other observables in the ground state. This means that we only need to calculate $\langle A\rangle_\hbar$ for $A \in O_\hbar$. From now on, we will omit the index $\hbar$.

If we are able to calculate the value of the ground state, this is, if we are able to calculate the value of any observable in the ground state $\langle A\rangle$, we have solved our problem. We want to emphasize that $\langle A\rangle$ has physical meaning. In the laboratory we can study scattering processes or reactions and measure their cross sections (a way of defining their probability of happening). These cross sections can be expressed in terms of the values $\langle A\rangle$.

Remember that an observable $A \in O$ is a function of the fields $\Gamma(E)$, namely a polynomial in them and their derivatives. Let us write $A = F_A(\phi)$. We declare that

$$\langle A\rangle \;=\; \frac{\int\, [D\phi]\; e^{-(i/\hbar)S[\phi]}\; F_A(\phi)}{\int\, [D\phi]\; e^{-(i/\hbar)S[\phi]}} \tag{18}$$

The integral $\int[D\phi]$ is an integral over the space of fields. This is the *Feynman path integral* and is simply not yet defined (and we do not expect to find a simple definition for it: the space of fields is infinite dimensional). Nevertheless, we will try to make sense of this integral. We will manipulate it formally untill we reach an expression that makes sense and we will keep that expression as a definition.

What does this integral mean?

The space of fields is infinite dimensional because it is a space of functions on D. A (complicated) way to make sense of the definition is to discretize D and take a limit as the discretization tends to the continuum (see [17]). We will proceed in a different way. Let us reconsider the action. We said that it was of the form

$$S[\phi] = \int_D \mathcal{L}[\phi] \ ,$$

where $\mathcal{L}[\phi]$ is a top form built (in local coordinates) as a polynomial in the fields and their derivatives. We separate it into two parts,

$$S = S_f + \lambda S_I \qquad S_f[\phi] = \int_D \mathcal{L}_f[\phi] \qquad S_I[\phi] = \int_D \mathcal{L}_I[\phi] \ ,$$

where $\mathcal{L}_f$ is the part of the action of degree ≤ 2 on the fields, and $\lambda\mathcal{L}_I$ is the rest. We assume that there is some global constant multiplying that rest, namely λ, that we call the *coupling constant*. We do this because the *free* action (S_f) is easier to understand and we expect to treat the *interaction* (S_I) as a perturbation of it (though to do so, we must take the coupling constant to be small). We perform a Taylor expansion in equation (18):

$$\langle A \rangle \ = \ \sum_{N=0}^{\infty} \frac{(-i\lambda/\hbar)^n}{n!} \ \frac{\int \ [D\phi] \ e^{-(i/\hbar)S_f[\phi]} \ (S_I[\phi])^n \ F_A(\phi)}{\int \ [D\phi] \ e^{-(i/\hbar)S[\phi]}} \tag{19}$$

Recalling that $S_I[\phi]$ and $F_A[\phi]$ are polynomials in the fields and their derivatives, we use a trick from statistical mechanics. We define the *partition function*,

$$Z[\alpha] = \int \ D[\phi] \ e^{-(i/\hbar)S_f + \int_D \alpha(x)\phi(x)} \ , \tag{20}$$

and claim that we can formally calculate the terms of (19) as logarithmic derivatives of (20). For instance

$$\frac{\partial}{\partial\alpha(x)} \log Z[\alpha]|_{\alpha=0} = \ \frac{\int \ [D\phi] \ e^{-(i/\hbar)S_f[\phi]} \ \phi(x)}{\int \ [D\phi] \ e^{-(i/\hbar)S_f[\phi]}}$$

What is equation (20)? It is an integral of an exponential over the (infinite dimensional) space of fields. In the exponent we have another integral of a polynomial of degree two in the fields and their derivatives. We can think of the integral in the exponent as an infinite sum, with the derivatives being coefficients. We know how to integrate the finite dimensional equivalent of equation (20) with constant coefficients:

$$\int_{\mathcal{R}^d} d^n x \ e^{x^T A x + B^T x} = \sqrt{\frac{\pi^n}{\det A}} \ e^{B^T A^{-1} B} \ ,$$

so long as the matrix A is invertible. Let's just pretend it is still valid in (20). What is $x^T Ax$ now? It is the degree 2 part of S_f. A is an infinite matrix whose entries are differential operators. Finding its inverse is a problem in partial differential equations: we must find its *Green's function*, which here is called a *propagator*.

Then a miracle occurs...

...and we have gotten a recipe to make sense of $\langle A \rangle$. In equation (18) each term will be a sum of terms involving derivatives and finite dimensional integrals built from the propagators. The amazing fact is that we get a very "simple" recipe in the end. In fact, we can write $\langle A \rangle$ as a sum of contributions labeled by certain diagrams or graphs. The contribution of each diagram should be a finite dimensional integral with propagators. The algorithm to define which diagrams we should consider in $\langle A \rangle$ and what the contribution of each of them is what we call an *expansion in Feynman diagrams*. The *order* of a diagram (power of the coupling constant in its contribution) is related to the number of vertices it has.

It is to be noted that the contribution of each diagram depends only on the action S, whereas *which* diagrams we need to consider depends on A. So, if we want to calculate $\langle A \rangle$ for different observables A in the same theory (that is, the same S) we need to derive the rules for the contribution of each diagram only once. Then, for each A we just add the contributions of a different family of diagrams.

Of course we encounter more problems. Perhaps the main one is that the finite dimensional integrals to calculate (once we have gotten rid of the infinite dimensional one) diverge. This leads to the problem of *renormalization*, which we will not discuss here.

4.2 A sigma model

It should be plausible by now that equation (8) may come from an expansion in Feynman diagrams. We introduce now a particular quantum field theory in which that expansion will appear. See [4] for the real description of the theory.

In the classical situation, the manifold D is the unit disc $D = \{u \in \mathbb{C} \mid |u| \leq 1\}$ (a manifold with boundary). We introduce two fields: X and η. X is a map from D to a Poisson manifold M, $X : D \longrightarrow M$. (The Poisson manifold M should be thought of as data. We have a different theory for every M). η is a 1-form on D taking value in the pullback by X of the cotangent bundle of M

$$\begin{tikzcd} X^*T^*M \arrow[r] \arrow[d] & T^*M \arrow[d] \\ D \arrow[r, "X"] & M \end{tikzcd}$$

In other words, $\eta \in \Gamma(X^*T^*M \otimes T^*D)$. However, we do not consider all sections. We add the boundary conditions that η has to vanish on tangent vectors to the boundary ∂D. This has defined E.

Next we need an action. There are two natural ways of building a two form on D from our fields. First, we can use the differential of X:

$$D \xrightarrow{X} M$$

$$TD \xrightarrow{dX} TM$$

Since $dX \in \Gamma(X^*TM \otimes T^*D)$ we can build $\eta \wedge dX \in \Gamma(\Lambda^2 T^*D)$. Secondly, we have a bivector field α in the Poisson manifold M. $(\alpha(X))(\eta, \eta)$ defines a 2-form in D, too. The action is:

$$S[X, \eta] = \int_D \eta \wedge dX + \frac{1}{2}(\alpha(X))(\eta, \eta) \tag{21}$$

Take local coordinates $\{u^\mu\}_\mu$ on D and $\{x^i\}_i$ on M. Let d be the dimension of M. Then X is given by d functions $X^i(u)$, and η by d differential 1-forms $\eta_{i\mu}(u)du^\mu$. The action now reads:

$$S[X, \eta] = \int du^\mu du^\nu \left(\eta_{i\mu}(u) \frac{\partial X^i}{\partial u^\nu}(u) \; + \; \frac{1}{2} \alpha^{ij}(X(u)) \eta_{i\mu}(u) \eta_{j\nu}(u) \right)$$

The main result in [4] is that we can recover the Kontsevich formula for the star product (equation (8)) from a path integral over the space of fields. Namely, given two functions $f, g \in C^\infty(M)$, their star product at a point $x \in M$ is

$$f \star g\ (x) = \int_{X(\infty)=x} f(X(1))\ g(X(0))\ e^{(i/\hbar)S[X,\eta]}\ dX d\eta \tag{22}$$

Here 0, 1, ∞ are three cyclically ordered points on the unit circle (which we secretely view as the completed real line by stereographic projection). Cyclically ordered means that if we start from 0 and move on the circle counterclockwise we first meet 1 and then ∞. The path integral is over all $X : D \longrightarrow M$ and $\eta \in \Gamma(X^*T^*M \otimes T^*D)$ subject to boundary conditions. We had already asked that $\eta(u)(\xi) = 0$ if $u \in \partial D$ and ξ is tangent to ∂D (i.e. η vanishes at vectors tangent to the boundary of D) and we now add the boundary condition $X(\infty) = x$.

If you are now looking for the free and the interaction part in the action (equation 21) wait a moment. The quantization in this case does not work well if we do as we explained in the last subsection. Before attempting anything we have to go through various steps in which we add new auxiliary fields and change the action. First we add a *ghost* field and *antifields* [8] and change the action according to some procedure. The word *ghost* refers only to a grading. Fields have now a double grading: their order as a form on D and their *ghost number*. Every field also has a complementary field: its *antifield*. Finally we have to add *superfields* (odd variables) and change the action again.

After all that, we get a free action with two differential operators to invert. One of them has Green's function $(1/2\pi)\psi(z, w)$ and the other has Green's function $G(w, z) = \frac{1}{2\pi}\left(*d_z\psi(z, w) \oplus d_z(z, w)\right)$ where $d_z = dz\frac{\partial}{\partial z} + d\overline{z}\frac{\partial}{\partial \overline{z}}$ and $*$ is the Hodge dual in D from k-forms to $(2-k)$-forms with respect to some metric. The variables live in the upper half plane, which is how we think of D now. The function ϕ is

$$\phi(z, w) = \frac{1}{2i}\log\frac{(z-w)(z-\overline{w})}{(\overline{z}-w)(\overline{z}-\overline{w})}$$

which is exactly $-\phi$ in equation (13), i.e. the function that we used to calculate the weight of a diagram in Kontsevich formula. We finally can calculate the expected value of the observable

$$A = f(X(1))g(X(0))\delta_x(X(\infty))$$

with the interaction part of the action, and we will get an expansion in Feynman diagrams that reproduces exactly equation 8, with $i\hbar/2$ instead of $\hbar$ (by a physics convention). The role of the coupling constant λ is played here by $\hbar$, and the order of a diagram is the exponent of $\hbar$ in its contribution.

[8] The reason is to take into account a certain symmetry.

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