Math 110 Homework 8
Partial Solutions

If you have any questions about these solutions, or about any problem not solved, please ask via email or in office hours, etc.

3.2.17 Suppose that $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $C = \begin{bmatrix} x & y & z \end{bmatrix}$. Then $A = \begin{bmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{bmatrix}$ surely has rank at most 1, since its row space is spanned by the one vector (x, y, z). On the other hand, suppose that A has rank 1. Then the row space of A is spanned by one non-zero vector, say (x, y, z). In this case, every row is a multiple of (x, y, z). So $A = \begin{bmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{bmatrix}$ for some $a, b, c \in F$. But then $A = BC$ as above.

3.2.19 Given such A and B, we have that L_A is a linear transformation from F^n to F^m, and that L_B is a linear transformation from F^p to F^n. Since $\text{rank}(L_A) = \text{rank}(A) = m$ and $\text{rank}(L_B) = \text{rank}(B) = n$, we have that L_A and L_B are onto. Thus $L_{AB} = L_A L_B$ is onto and has rank m. Thus $\text{rank}(AB) = m$.

3.2.21 Since A is $m \times n$ of rank m, L_A maps from F^n *onto* F^m. We construct a transformation $T\colon F^m \to F^n$ such that $L_A \circ T$ is the identity. Then $T = L_B$ for some matrix B, and this is the matrix we seek as then $I_m = [L_A \circ T] = [L_A L_B] = [L_{AB}] = AB$. We construct T by prescribing its behavior on a basis for F^m. Let $\{e_1, \ldots, e_m\}$ be the standard basis for F^m. For each e_i, there exists $x_i \in F^n$ such that $L_A(x_i) = e_i$ since L_A is onto. Define T by setting $T(e_i) = x_i$ for each i. Then $L_A(T(e_i)) = e_i$ for every i, and so $L_A \circ T$ is the identity on F^m. This proves the claim and the result.

3.3.10 This statement is true. If the $m \times n$ coefficient matrix A has rank m, then $L_A\colon F^n \to F^m$ is onto. Thus (by previous hw problem) *any* equation $Ax = b$ has a solution.