Math 110 Homework 13
Partial Solutions

If you have any questions about these solutions, or about any problem not solved, please ask via email or in office hours, etc.

6.1.10 Suppose that $\langle x, y\rangle = 0$. Then $||x+y||^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = ||x||^2 + ||y||^2$.

6.1.18 Suppose that $\langle\cdot,\cdot\rangle'$ defines an inner product. If $x \in N(T)$, then $\langle x, x\rangle' = \langle T(x), T(x)\rangle = 0$. Thus $x = 0$, and so T is one-to-one. On the other hand, suppose that T is one-to-one. For any $x, y, z \in V$, we have $\langle T(x+y), T(z)\rangle = \langle T(x)+T(y), T(z)\rangle = \langle T(x), T(z)\rangle + \langle T(y), T(z)\rangle$, and for every $x, y \in V$ and every $c \in F$, $\langle T(cx), T(y)\rangle = \langle cT(x), T(y)\rangle = c\langle T(x), T(y)\rangle$. This establishes the linearity of $\langle\cdot,\cdot\rangle'$. Further, for any $x, y \in V$, $\overline{\langle T(x), T(y)\rangle} = \langle T(y), T(x)\rangle$. Finally, suppose that $x \neq 0$. Then $T(x) \neq 0$ since T is one-to-one. Thus $\langle x, x\rangle' = \langle T(x), T(x)\rangle > 0$. Thus $\langle\cdot,\cdot\rangle'$ is an inner product.

6.2.7 One direction is easy. If $z \in W^\perp$ then $\langle z, v\rangle = 0$ for every $v \in \beta$. On the other hand, suppose that $\langle z, v\rangle = 0$ for every $v \in \beta$. If $w \in W$ is any vector, then there exist $a_1, \ldots, a_n \in F$ and $v_1, \ldots, v_n \in \beta$ such that $w = a_1v_1 + \cdots + a_nv_n$. Then $\langle z, w\rangle = a_1\langle z, v_1\rangle + \cdots + a_n\langle z, v_n\rangle = 0$. Thus $z \in W^\perp$.

6.2.13
(a) Suppose that $S_0 \subseteq S$. Let $x \in S^\perp$. Then $\langle x, y\rangle = 0$ for every $y \in S$. Then surely $\langle x, y\rangle = 0$ for every $y \in S_0$. Thus $x \in S_0^\perp$.

(b) Suppose that $x \in S$. Then surely $\langle x, y\rangle = 0$ for every $y \in S^\perp$ by definition of $S^\perp$. But then $x \in (S^\perp)^\perp$. Since $(S^\perp)^\perp$ is a subspace of V, we also have $\text{span}(S) \subseteq (S^\perp)^\perp$.

(c) We already have that $W \subseteq (W^\perp)^\perp$. Suppose now that $x \in (W^\perp)^\perp$. If $x \notin W$, then there exist unique $w \in W$ and $0 \neq z \in W^\perp$ such that $x = w + z$. But then $\langle x, z\rangle = \langle w+z, z\rangle = \langle w, z\rangle + \langle z, z\rangle = \langle z, z\rangle > 0$ since $w \in W$, $z \in W^\perp$ and $z \neq 0$. But this contradicts the fact that $x \in (W^\perp)^\perp$. Thus we must have $x \in W$ and $W = (W^\perp)^\perp$.

(d) We already know that any $x \in V$ can be written uniquely as $x = w + z$ for $w \in W$ and $z \in W^\perp$. This is enough to show that $V = W \oplus W^\perp$.

6.2.14 For the first statement, suppose that $x \in (W_1 + W_2)^\perp$. Since $W_1, W_2 \subseteq W_1 + W_2$, we have that $x \in W_1^\perp$ and $x \in W_2^\perp$. Thus $(W_1 + W_2)^\perp \subseteq W_1 \perp \cap W_2^\perp$. On the other hand, suppose that $x \in W_1^\perp \cap W_2^\perp$. Then, for any $v \in W_1 + W_2$, we may write $v = y + z$ for some $y \in W_1$ and some $z \in W_2$. Then $\langle x, v\rangle = \langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle = 0$. Thus $x \in (W_1 + W_2)^\perp$ and so $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$.

For the second statement, notice that, by the above, $(W_1^\perp + W_2^\perp)^\perp = W_1^{\perp\perp} \cap W_2^{\perp\perp} = W_1 \cap W_2$ since W_1 and W_2 are finite-dimensional. By applying $\perp$ to both sides, we then have $W_1^\perp + W_2^\perp = W_1 \cap W_2$.

6.2.17 Suppose that $\langle T(x), y\rangle = 0$ for every $x, y \in V$. Then $T(x) = 0$ for every $x \in V$ and so $T = T_0$. If $\langle T(x), y\rangle = 0$ for every x, y in a basis, then we again conclude (by a previous hw problem) that $T(x) = 0$ for every x in a basis. But this again implies that $T = T_0$.

6.3.6 Notice that $(T + T^*)^* = T^* + T^{**} = T^* + T = T + T^*$ and that $(TT^*)^* = T^{**}T^* = TT^*$.

6.3.8 Note that $T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I$. Thus T^* is invertible with inverse $(T^{-1})^*$.

6.3.12 Notice that $x \in R(T^*)^\perp$ if and only if $\angle x, T * (y)\rangle = 0$ for every $y \in V$ if and only if $\langle T(x), y\rangle = 0$ for every $y \in V$ if and only if $T(x) = 0$ if and only if $x \in N(T)$.