SOME THEOREMS ON TOPOLOGICAL MANIFOLDS

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We list here some theorems about topological manifolds, most of which were announced in [7], [8], with proofs to appear in [9], [10], [11]. In addition, see [12], [13], [15], [6].

First is the theorem on existence and uniqueness (Hauptvermutung) of PL structures on manifolds [7], [13]:

<u>Theorem 1</u>: Let Q^q be a q-dimensional topological manifold and let C be a closed subset of Q. Suppose that a neighborhood of C has a PL structure Σ_0. Let $q \geq 6$ or $q = 5$ if $\partial Q \subset C$. If $H^4(Q,C;Z_2) = 0$, then Q has a PL structure Σ which agrees with Σ_0 near C.

Given the PL structure Σ, then the PL structures (up to isotopy) on Q which agree with Σ near C are classified by $H^3(Q,C;Z_2)$.

<u>Definition</u>: Let Σ and Θ be two PL structures on Q. Then Σ and Θ are said to be equivalent (up to homotopy, or up to isotopy) if there exists a PL homeomorphism $f:Q_\Sigma \to Q_\Theta$ (which is homotopic, or isotopic, to the identity). In the relative case, $\Sigma = \Theta$ near C and the homotopy, or isotopy, fixes a neighborhood of C.

Sullivan proved the following theorem on uniqueness up to homotopy [20], [1], [17].

<u>Theorem 2</u>: Let Σ and Θ be two PL structures on Q which agree near C. Suppose $H^4(Q,C;Z)$ has no 2-torsion. Suppose one of the following conditions holds:

(i) $\partial Q \subset C$, $q \geq 5$, Q compact, and $\pi_1(Q) = \pi_1(\partial Q)$,

(ii) $\partial Q \not\subset C$, $q \geq 6$, Q compact, and $\pi_1(Q) = \pi_1(\partial Q)$,

(iii) Q non-compact, $q \geq 6$ and $\pi_1(Q) = 0$.

Then there exists a proper homotopy $h_t : Q_\Sigma \to Q_\Theta$, $t \in [0,1]$ from the identity to a PL homeomorphism h_1 with h_t = identity near C for all $t \in [0,1]$.

The two uniqueness theorems are related as follows: let $\beta : H^3(Q,C;Z_2) \to H^4(Q,C;Z)$ be the Bockstein homomorphism coming from the exact sequence of coefficients $0 \to Z \to Z \to Z_2 \to 0$ (which is the sequence $0 \to \pi_4(G/PL) \to \pi_4(G/TOP) \to \pi_3(TOP/PL) \to 0$). Assume the conditions of Theorem 2. Then if Q has PL structures Σ and Θ corresponding to $[\Sigma], [\Theta] \in H^3(Q,C;Z_2)$, then Σ and Θ are equivalent (rel C) up to homotopy iff $\beta([\Sigma]) = \beta([\Theta])$.

This fails however for $Q = S^3 \times R^2$, which has two PL structures, because Theorem 2 fails in this case. The uniqueness half of Theorem 1 fails for closed 3-manifolds, but it is a good conjecture that Theorem 1 holds in dimensions > 3.

Let τ_Q be the tangent q-bundle of Q which is classified by a map $f : Q \to B_{TOP(q)}$. There is a Serre fibration $B_{CAT(q)} \to B_{TOP(q)}$ with fiber $TOP(q)/CAT(q)$ where CAT is either the PL or DIFF (differential) category.

<u>Theorem 3 ([7],[13],[6])</u>: Let $r \geq k$ and $r \geq 5$. Then the stabilization map

$$s : \pi_k(TOP(r), PL(r)) \to \pi_k(TOP(r+1), PL(r+1))$$

is an isomorphism and

$$\pi_k(TOP(r), PL(r)) = \pi_k(TOP, PL) = \begin{cases} 0 & k \neq 3 \\ Z_2 & k = 3 \end{cases}.$$

Thus by obstruction theory, f lifts to a map $f' : Q \to B_{PL(q)}$ (meaning τ_Q reduces to a PL bundle) if an obstruction in $H^4(Q,C;\pi_3(TOP(q),PL(q)) = 0$; if f' exists, then the homotopy classes of liftings of f (equals the concordance classes of PL reductions of τ_Q) correspond bijectively to $H^3(Q,C;Z_2)$. Theorem 1 now follows from the classification theorem below.

The complex $\mathrm{Cat}(Q \text{ rel } C;\Sigma_0)$ of CAT structures on Q which are equal to Σ_0 near C is defined to be the (semi-simplicial, Kan) complex of which a typical d-simplex is a CAT structure Γ on $\Delta^d \times Q$ such that projection onto Δ^d is a CAT submersion and $\Gamma = \Delta^d \times \Sigma_0$ near $\Delta^d \times C$. If $C = \emptyset$, we write $\mathrm{Cat}(Q)$.

Similarly, let $\mathrm{Lift}(f \text{ rel } C, f_0)$ be the (semi-simplicial, Kan) complex of lifts of $f: Q \to B_{TOP(q)}$ to maps $f': Q \to B_{CAT(q)}$, where f' agrees near C with the lifting f_0 induced by Σ_0.

<u>Theorem 4 (Classification Theorem [10])</u>: The natural map $\theta: \mathrm{Cat}(Q \text{ rel } C, \Sigma_0) \to \mathrm{Lift}(f \text{ rel } C, f_0)$ is a homotopy equivalence for $q \neq 4$, and $q \neq 5$ if $\partial Q \not\subset C$.

Lashof [12], Morlet [15], and C. Rourke have also proved versions of the classification theorem. We discuss ingredients of the proof below.

Let Γ be a CAT structure on $I\times Q$ and let $0\times\Sigma$ be its restriction to $0\times Q$. Suppose $\Gamma = I\times\Sigma$ near $I\times C$.

<u>Theorem 5 (Concordance implies isotopy) [9]</u>: Let $q \geq 6$ or $q = 5$ if $\partial Q \subset C$. There exists an isotopy $h_t: I\times Q \to I\times Q$, $t \in [0,1]$, fixing $0\times Q$ and a neighborhood of $I\times C$, such that $h_0 =$ identity and $h_1: I\times Q_\Sigma \to (I\times Q)_\Gamma$ is a CAT isomorphism. Furthermore h_t can be chosen arbitrarily close to the identity.

<u>Theorem 6 (Sliced concordance implies isotopy) [9]</u>: Suppose $q \neq 4$, and $q \neq 5$ if $\partial Q \not\subset C$. Also suppose the projection $(I\times Q)_\Gamma \to I$ is a CAT submersion. Then h_t exists as in Theorem 5. Furthermore $h_t(s\times Q) = s\times Q$ for all $s \in I$. This assertion holds if, more generally, a pair (Δ^d, Λ) replaces the pair $(I, 0)$, where Δ^d is the standard d-simplex and Λ is a contractible subcomplex such that $\Gamma|\Lambda\times Q = \Sigma\times Q$.

Theorem 6 readily implies the sliced concordance extension theorem :

Theorem 7: Let Q' be an open subset of Q containing C. Suppose $q \neq 4$, and $q \neq 5$ if $\partial Q \not\subset C$. Then the restriction map $Cat(Q \text{ rel } C,\Sigma) \to Cat(Q' \text{rel } C,\Sigma)$ is a Kan fibration.

Theorems 5 and 6 are established by decomposing Q_Σ into small handles and then applying inductively a version where Q is an open k-handle $R^k \times R^n$, $q = k+n$, and $C = (R^k - \text{int}B^k) \times R^n$. This version differs in that h_1 needs to be a CAT imbedding only near $I \times (R^k \times B^n)$ and the smallness condition is replaced by the condition that h_t fix all points outside a compactum (which is independent of t).

The handle version of Theorem 5 is most efficiently established by use of a variant of the Main Diagram of [7] (see also [9], [6]) which uses only the s-cobordism theorem.

In case CAT = PL (but not DIFF), the Alexander isotopy device (invalid for DIFF) and the s-cobordism theorem can be used to strengthen Theorem 5 to read: given the data of Theorem 5, the semi-simplicial space of concordances of the given structure Σ on Q, rel $I \times C$, is contractible (a d-simplex is a PL structure Γ on $\Delta^d \times I \times Q$ such that projection on Δ^d is a submersion, $\Gamma | \Delta^d \times 0 \times Q = \Delta^d \times 0 \times \Sigma$ and Γ equals $\Delta^d \times I \times \Sigma$ near $\Delta^d \times I \times C$).

For the handle version of Theorem 6, first note that the (many parameter) TOP isotopy extension theorem will deduce it from the statement that $(\Delta^d \times B^k \times R^n)_\Gamma$ is CAT isomorphic to $\Delta^d \times (B^k \times R^n)_\Sigma$ by a map respecting projection to Δ^d. This statement follows from a similar isomorphism of $(\Delta^d \times B^k \times (R^n - 0))_\Gamma$ with $\Delta^d \times (B^k \times (R^n - 0))_\Sigma$ which is established via a useful technical lemma.

Lemma. This lemma will hold for CAT equals TOP as well as PL or DIFF. Consider a CAT manifold E with two ends equipped with

a) a CAT submersion $p: E \to \Delta^d$ (the leaves $F_u \equiv p^{-1}(u)$, $u \in \Delta^d$, are CAT manifolds, possibly with boundary),

b) a proper CAT map $\pi : E \to R$ such that for each pair of integers a,b with $a < b$ the preimage $F_u(a,b) \equiv (\pi | F_u)^{-1}(a,b) = F_u \cap \pi^{-1}(a,b)$ of the open interval (a,b) of real numbers is a CAT product of a compact manifold with R, or at least has the following engulfing property:

* $F_u(a,b)$ has two ends ε_-, ε_+ and if U_-, U_+ are given open neighborhoods of $\varepsilon_-, \varepsilon_+$, then there exists a CAT self-isomorphism h of $F_u(a,b)$ fixing points outside some compactum such that $h(U_-) \cup U_+ = F_u(a,b)$.

Then $p : E \to \Delta^d$ is a CAT product bundle.

To prove this one can apply the d-parameter CAT isotopy extension theorem and an elementary engulfing argument to show that (*) holds "globally" with E in place of $F_u(a,b)$. Then glue together the ends of E as in [18] to deduce the result from the known fact that every <u>proper</u> CAT submersion is a CAT bundle map.

The Classification Theorem now follows from the immersion theory machine [2]. Theorem 7 is the key tool; with it, the machine works easily once we observe that the classification theorem holds for zero-handles, and this amounts to identifying $\mathrm{Cat}(R^q)$ with the complex of CAT structures on the trivial R^q bundle over a point.

A somewhat different method of proving Theorem 1 (see [7], [6]) involves a stable version of the classification theorem. We have seen that the stable tangent bundle reduces if an obstruction in $H^4(Q,C;\pi_3(TOP,PL) = Z_2)$ vanishes, and concordance classes of stable reductions are classified by $H^3(Q,C;Z_2)$. By Milnor's arguments in [14], a stable reduction gives a PL structure on $Q \times R^s$ for some s and concordance classes of stable reductions correspond to concordance classes of PL structures on $Q \times R^s$. Application of the concordance-implies-isotopy Theorem (Theorem 5) and the Product Structure Theorem below finish this proof of Theorem 1.

Theorem 8 (Product Structure Theorem). Let $q \geq 6$ or $q = 5$ if $\partial Q \subset C$, and let Σ_0 be a CAT structure near C. Let Θ be a CAT structure on $Q\times R^s$ which agrees with $\Sigma_0 \times R^s$ near $C\times R^s$. Then Q has a CAT structure Σ, extending Σ_0 near C, and $\Sigma\times R^s$ is concordant to Θ modulo $C\times R^s$.

Moreover, there is an ε-isotopy $h_t : Q_\Sigma \times R \to (Q\times R^s)_\Theta$ with h_0 = identity, h_1 PL, and h_t = identity near $C\times R^s$, where $\varepsilon : Q \times R^s \to R$ is a continuous function.

Note that the theorem fails for closed 3-manifolds; e.g. $S^3\times R^2$ has two PL structures but S^3 has only one.

The Product Structure Theorem is equivalent to the Concordance-implies-isotopy Theorem plus the Annulus Theorem [5]; the equivalence is not too hard to prove [9].

The classical PL-DIFF Product Structure Theorem (Cairns-Hirsch Theorem) [3] follows easily from the TOP-CAT versions of the Product Structure Theorem and Concordance-implies-isotopy Theorem. By using the Sliced-concordance-implies-isotopy Theorem (Theorem 6) in addition, we may also recover the PL-DIFF version of Concordance-implies-isotopy [4], [16]. (Since $\Gamma_i = 0$ for $i \leq 6$, there is nothing to prove in the low dimensions not covered by Theorems 5-8.)

The Product Structure Theorem is particularly significant because of Theorems 9, 10, and 11 below which follow easily, (see [11], [6]).

Theorem 9. Let M^m be a TOP manifold. Then M has a well defined simple homotopy type (infinite if M is non-compact [19]) which agrees with the usual definition if M is PL or is a handlebody. This implies that the Whitehead torsion of a homeomorphism is zero [7].

Theorem 10 (Transversality). Let $\xi^n = E(\xi^n) \overset{\pi}{\to} X$ be an n-plane bundle over a topological space X and let $f : M^m \to E(\xi^n)$ be a continuous function. Then if $m \neq 4$ and $m-n \neq 4$, f is homotopic to a map f_1 which is transverse to the zero-section of ξ (this means f_1^{-1}(0-section) is an (m-n)-manifold P with a normal bundle in M equal to $(\pi f_1|P)^*(\xi)$). If f is transverse near a closed set $C \subset M$, then the homotopy equals f near C.

Theorem 11. If $m \geq 6$, then M^m is a TOP handlebody. (If $m = 6$ and $\partial M \neq \emptyset$, then we obtain M by adding handles to ∂M). Equivalently, M admits a Morse function $f: M \to R$ (that is, f is locally of the form $x_1^2 + \ldots + x_\lambda^2 - x_{\lambda+1}^2 - \ldots - x_m^2$).

On the other hand, in dimension 4 or 5 (or both), there exists a manifold which is not a handlebody and thus has no Morse function.

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