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SMOOTHING LOCALLY FLAT IMBEDDINGS OF DIFFERENTIABLE MANIFOLDS†

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INTRODUCTION

IS A LOCALLY flat imbedding of a differentiable (combinatorial) manifold M^n into a differentiable (combinatorial) manifold Q^q ambient isotopic to a differentiable (combinatorial) imbedding?

A negative answer is obtained in the differentiable case by using exotic differentiable structures on spheres. The simplest example is to let Q^q be the ordinary q-sphere S^q, M be an exotic q-sphere Σ^q, and the imbedding be the identity map; but there does not exist a differentiable imbedding of Σ^q onto S^q.

However in the combinatorial case, it is possible that all locally flat imbeddings are ambient isotopic to combinatorial imbeddings, although the best known result is the following of Gluck [4]: If M^n is a finite polyhedron with $2n + 2 \leqq q$, then a locally flat imbedding is ambient isotopic to a combinatorial imbedding.

It is the purpose of this paper to give an affirmative answer in the differentiable case with dimensional restrictions as follows:

MAIN THEOREM. *Let M^n and Q^q be differentiable manifolds with M^n compact, $2q > 3(n+1)$, and $q \geqq 7$. Let $f: M \to Q$ be a locally flat imbedding such that either $f(M) \subset \operatorname{int} Q$ or f is proper ($f(\partial M) \subset \partial Q$ and $f(\operatorname{int} M) \subset \operatorname{int} Q$ and $q \geqq 8$. Let f be differentiable on a neighborhood of a differentiable submanifold N^n of M^n. Let $\varepsilon > 0$. Then there exists an ambient ε-isotopy $F_t : Q \to Q$, $t \in [0, 1]$, satisfying*

(1) *$F_0 =$ identity,*

(2) *$F_1 f$ is a differentiable imbedding,*

(3) *$F_t =$ identity on a neighborhood of $f(N)$ and on the complement of an ε-neighborhood of $f(M)$ for all $t \in [0, 1]$,*

(4) *$|F_t(x) - x| < \varepsilon$ for all $x \in Q$ and $t \in [0, 1]$.*

It follows that the differentiable imbeddings are dense in the set of locally flat imbeddings.

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Haefliger [5] has shown that two differentiable imbeddings which are isotopic are also ambient diffeotopic with the above restrictions. Hence the problem of classifying locally flat imbeddings up to ambient istotopy is equivalent to classifying differentiable imbeddings up to diffeotopy.

(It should be mentioned here that in [8] the author misquoted Haefliger [5]. The author claimed in [8] that α was an isomorphism; but α is not monic. An example of Hsiang and Szczarba [6] provides two differentiable imbeddings of a 22-dimensional manifold in R^{37} which are not diffeotopic.)

§1 contains definitions and fixes notation. In §2 and §3 we will be concerned with the problem of extending a δ-isotopy between two differentiable imbeddings of M into Q to an ambient ε-isotopy between the imbeddings. This will be possible (Theorem 2.1) if $2q > 3(n+1)$ and δ is chosen small enough.

Theorem 2.1 provides the crucial tool in §4 for proving a special case of the Main Theorem when M is an n-ball. In §5 the Main Theorem follows easily from the special case using the handle-body decomposition of M.

§1.

In Euclidean space R^n, rB^n will denote the n-ball of radius r and rS^{n-1} the $(n-1)$-sphere of radius r. $\bar{U}$ will denote the closure of U. If $X \subset Y$, then $N_\varepsilon(X) = \{y \in Y | d(y, X) < \varepsilon\}$ where d is a metric on Y.

M^n will always be a compact differentiable n-manifold and Q^q a differentiable q-manifold. Differentiable will mean C^∞-differentiable.

Let $f: M \to Q$ be a continuous map and M_0 be a subset of M. If we say that $f|M_0$ is an imbedding, we mean that f is one-to-one between points of M_0 and $f(M_0)$ and $(f|M_0)^{-1}$ is continuous at each point of $f(M_0)$. On the other hand, if we say that f is an imbedding on M_0, we mean that $f|M_0$ is an imbedding and $f(M_0) \cap f(M - M_0) = \varnothing$.

An ε-isotopy $f_t: M \to Q$, $t \in [0, 1]$, is an isotopy satisfying $|f_t(x) - f_0(x)| < \varepsilon$ for all $x \in M$, $t \in [0, 1]$. An ambient isotopy $f_t: Q \to Q$, $t \in [0, 1]$, is always assumed to satisfy $f_0 =$ identity. A diffeotopy is a differentiable isotopy.

§2.

Let $f_0, f_1 : M \to Q$ be two differentiable imbeddings connected by a δ_1-isotopy $f_t : M \to Q$, $t \in [0, 1]$. If $2q > 3(n+1)$, it is possible to approximate f_t by a δ_2-diffeotopy $F_t : M \to Q$, $t \in [0, 1]$ for which $F_0 = f_0$ and $F_1 = f_1$ (see [5, p. 48]). Then can we find an ambient ε-diffeotopy $G_t : Q \to Q$, $t \in [0, 1]$, such that $G_1 F_0 = F_1$, where ε is a function of δ_1 and δ_2?

The answer is yes if it is also known that $|(\partial F_t/\partial x_i) - (\partial F_0/\partial x_i| < \delta_2$ for all $t \in [0, 1]$ where the x_i, $i = 1, \ldots, n$, are coordinates in some coordinate neighborhood on M (see [10]). (In other words, F_t must be close to F_0 in the C^1-topology on the space of differentiable imbeddings of M in Q.)

Without the extra condition on the partial derivatives, the answer is unknown. However we obtain the following related result in which δ is a function of ε.

THEOREM 2.1. *Let $f_0: M \to Q$ be a differentiable imbedding. Let $2q > 3(n+1)$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that if $f_t: M \to Q$, $t \in [0, 1]$, is a δ-homotopy for which f_1 is a differentiable imbedding, then f_t can be replaced by an ambient ε-diffeotopy $F_t: Q \to Q$, $t \in [0, 1]$, satisfying*

(1) $F_1 f_0 = f_1$

(2) *F_t = identity on $Q - N_\varepsilon(f_0(M))$ for all $t \in [0, 1]$.*

If $\partial M \neq \emptyset$, *we assume that $f_t = f_0$ in a neighborhood of ∂M for all $t \in [0, 1]$, and then F_t also satisfies*

(3) *F_t = identity on a neighborhood $f_0(\partial M)$ in Q for all $t \in [0, 1]$.*

The proof is given in §3.

We will need the following slight generalizations of a theorem of Haefliger [5, Theorem 6.1]. The first part of Theorem 2.2. below is an immediate corollary of Haefliger's result.

THEOREM 2.2. *Let $\varepsilon > 0$ and $2q \geqq 3(n+1)$. Let $f: M \to Q$ be a continuous map. Then there exists $\delta > 0$ such that if there is an imbedding $F: M \to Q$ with $|f(x) - F(x)| < \delta$ for all $x \in M$, then f can be approximated via an ε-homotopy by a differentiable imbedding $g: M \to Q$ with $|f(x) - g(x)| < \varepsilon$.*

If f is already a differentiable imbedding on a δ-neighborhood of a closed set C, then g may be chosen equal to f on C.

THEOREM 2.3. *Let $\varepsilon > 0$ and $2q > 3(n+1)$. Let f_0 be a differentiable imbedding of M into Q. Then there exists $\delta > 0$ such that if a homotopy $f_t: M \to Q$ satisfies (1) f_1 is a differentiable imbedding, (2) $|f_t(x) - f_0(x)| < \delta$ for all $x \in M$, $t \in [0, 1]$, then f_t can be approximated by a diffeotopy $F_t: M \to Q$, $t \in [0, 1]$, such that $F_0 = f_0$, $F_1 = f_1$ and $|F_t(x) - f_t(x)| < \varepsilon$ for all $x \in M$, $t \in [0, 1]$.*

If $f_t = f_0$ on a δ-neighborhood of a closed set C for all $t \in [0, 1]$, then F_t may be taken equal to f_0 on C.

Proof of Theorems 2.2 *and* 2.3. Familiarity with [5] is assumed. The proof is essentially the same as that on pp. 79–81 of [5]. We need only replace the Lemma on p. 79 with the Lemma below.

Let $\rho > 0$ (respectively $\rho' > 0$) be real numbers such that any two points of M^n (respectively Q^q) which are closer than ρ (respectively ρ') are joined by a unique geodesic whose length is the distance between the points, We observe that

$$|F(x) - F(y)| \leqq |F(x) - f(x)| + |f(x) - f(y)| + |f(y) - F(y)| < |f(x) - f(y)| + 2\delta.$$

LEMMA. *For some δ, there exists a generic map $h: M \to Q$ such that*

(a) *$|h(x) - f(x)| < \varepsilon/4$ for all $x \in M$,*

(b) *if $h(x) = h(y)$, then $|x - y| < \rho$, and, if x' and y' lie on the unique geodesic between x and y, then $|h(x') - h(y')| < \varepsilon/4$.*

Proof. By continuity, we can pick α, $0 < \alpha < \rho$, such that if $|x - y| < \alpha$, then $|f(x) - f(y)| < \varepsilon/8$ for all $x, y \in M$. Since F is an imbedding and M is compact, we can pick $\beta > 0$ and choose δ small enough so that if $|F(x) - F(y)| < \beta + 2\delta$, then $|x - y| < \alpha$. Hence if $|f(x) - f(y)| < \beta$, then $|F(x) - F(y)| < \beta + 2\delta$ and therefore $|x - y| < \alpha$.

Let h be a generic map of M into Q which agrees with f on a neighborhood of C such that $|f(x) - h(x)| < \min(\varepsilon/16, \beta/2)$ for all $x \in M$. Clearly (a) holds. If $h(x) = h(y)$, then $|f(x) - f(y)| < |f(x) - h(x)| + |h(x) - h(y)| + |h(y) - f(y) < \beta/2 + 0 + \beta/2 = \beta$, so $|x - y| < \alpha < \rho$. If x' and y' lie on the geodesic joining x and y, then $|x' - y'| < \alpha$. Thus $|h(x') - h(y')| \leqq |h(x') - f(x') + |f(x') - f(y')| + |f(y') - h(y')| < \varepsilon/16 + \varepsilon/8 + \varepsilon/16 = \varepsilon/4$. So (b) holds and the Lemma is proved.

The remainder of the proof of Theorem 2.2 proceeds as in [5, pages 80, 81]. (The ε-homotopy between f and g arises naturally from Haefliger's method of proof.)

For Theorem 2.3, we can prove a similar Lemma for homotopies, and then again proceed as in [5, pages 80, 81].

We will need the following form of the diffeotopy extension theorem, due originally to Thom [12]. $n \leqq q$ is the only restriction on the dimensions of M and Q which is necessary.

THEOREM 2.4. *Let $f_t: M \to Q$, $t \in [0, 1]$, be a diffeotopy such that for some $M_0 \subset M$, $f_t = f_0$ on $M - M_0$ for all $t \in [0, 1]$. Suppose that $\cup_{t \in [0,1]} f_t(M_0)$ is in the interior of $Q_0 \subset Q$. Then there exists an ambient diffeotopy $F_t: Q \to Q$ satisfying*

(1) *$F_0 = $ identity*

(2) *$F_t f_0 = f_t$,*

(3) *$F_t = $ identity on $Q - Q_0$ for all $t \in [0, 1]$.*

This statement of the theorem can be found in [9], along with an elementary proof.

§3.

Proof of Theorem 2.1 In view of Theorem 2.4, the way to get an ambient ε-diffeotopy is to extend a diffeotopy which moves points only inside a disjoint union of ε-balls. Hence, we change f_t to a homotopy g_t, $t \in [0, n+1]$, where each g_t, $t \in [k, k+1]$, moves points only inside a disjoint union of small sets. It is necessary for g_k to be a differentiable imbedding for $k = 0, 1, \ldots, n+1$, so that we may approximate each g_t, $t \in [k, k+1]$, by a diffeotopy G_t, using Theorem 2.3. (See Fig. 1.)

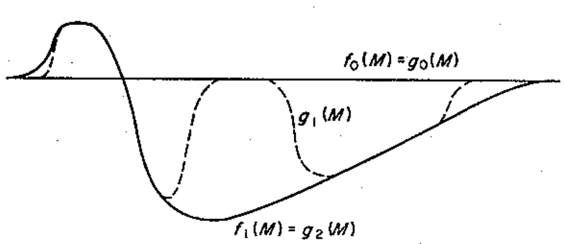

FIG. 1.

To sketch the first stage more precisely, suppose that $\{U^{0,j}\}, j = 1, 2, \ldots, m_0$, is a disjoint collection of small open subsets of M. Let $\gamma : M \to [0, 1]$ be a continuous function which in one on $U^{0,j}$ for each j and is zero except at points near $U^{0,j}$. Then $\hat{g}_t(x) = f_{\gamma(x)\cdot t}(x)$ is a homotopy for which $\hat{g}_1 = f_1$ on $U^{0,j}$ and $\hat{g}_t = f_0$ except at points near $U^{0,j}$. By Theorem 2.2, $\hat{g}_1$ is homotopic to a differentiable imbedding $g_1 : M \to Q$ with $g_1 = f_1$ on $U^{0,j}$ and $g_1 = f_0$ except near $U^{0,j}$. There is an obvious homotopy g_t, $t \in [0, 1]$, between $g_0 = f_0$ and g_1. Then using Theorem 2.3, g_t is replaced by an ambient diffeotopy G_t, $t \in [0, 1]$. This procedure is repeated for other sets $U^{i,j}$ until all of $f_0(M)$ is moved to $f_1(M)$. The details are complicated and tedious, so the reader might well skip them during a first reading of the paper.

Pick a C^1-triangulation T of M and let μ be the mesh of T. Let T^1 be the first derived subdivision. Order the vertices of T, $\{v^{0,j}\}, j = 1, 2 \ldots, m_0$, Let $U^{0,j}$ be the interior of the star of $v^{0,j}$ in T^1 for each j. Order the vertices of T^1 which are barycenters of 1-simplices of T, $\{v^{1,j}\}, j = 1, \ldots, m_1$, and let $U^{1,j}$ be the interior of the star of $v^{1,j}$ in T^1. Continue, ending with $U^{n,j}$ being the interior of the star in T^1 of the barycenter of an n-simplex of T.

The $\{U^{i,j}\}$, $i = 0, \ldots, n, j = 1, \ldots, m_i$, form an open cover of M with the property that no point of M belongs to more than $n + 1$ of the $U^{i,j}$. Let

$$U^{i,j} = U^{i,j}_{-4}, \ldots, U^{i,j}_{-1}, U^{i,j}_{0}, \ldots, U^{i,j}_{2n+2}$$

be open n-balls such that $\overline{U}^{i,j}_{k+1} \subset U^{i,j}_k$ for each i, j, k and $\{U^{i,j}_{2n+2}\}$ covers M. Let $U^i_k = \cup_{j=1}^{m_i}(U^{i,j}_k)$.

Let

$$\rho = \tfrac{1}{3}\min\{d(f_0(M - U^{i,j}_k), f_0(\overline{U}^{i,j}_{k+1})), d(f_0(\overline{U}^{i,j_1}), f_0(\overline{U}^{i,j_2}))\},$$

where the minimum is taken over all possible i, j, and k; and all j_1 and j_2 with $j_1 \neq j_2$ and where $d(X, Y)$ denotes the distance between X and Y.

For each i, we need many U^i_k, since we often have a mapping f of U^i_k into Q which is an imbedding on U^i_{k+1}; in approximating f by an imbedding, we can keep f fixed only on U^i_{k+2}. Thus, many approximations need many refinements.

Pick α, $0 < \alpha < \rho$; α will be determined more specifically later. Let

$$\alpha_0 = \delta < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1}$$

be real numbers such that, in Theorem 2.2, if $\varepsilon = \alpha_{k+1}$, then $\delta = \sum_{i=0}^{k} \alpha_i$ satisfies the theorem, and such that $\sum_{i=0}^{n+1} (n + 2 - i) \cdot \alpha_i < \alpha$.

Let $\gamma^0 : M \to [0, 1]$ be a continuous function such that $\gamma^0 = 0$ on $M - U^0_{-1}$ and $\gamma^0 = 1$ on $\overline{U}^0_0$. Let $\hat{h}^0_t(x) = f_{t\cdot\gamma^0(x)}(x)$, for all $x \in M$, $t \in [0, 1]$. Then

(1) $\hat{h}^0_0 = f_0$,
(2) $\hat{h}^0_1 = f_0$ on $M - U^0_{-1}$,
(3) $\hat{h}^0_1 = f_1$ on $\overline{U}^0_0$,
(4) $|\hat{h}^0_t(x) - \hat{h}^0_0(x)| < \delta = \alpha_0$,
(5) $\hat{h}^0_1$ is a differentiable imbedding on $(M - U^0_{-2}) \cup \overline{U}^0_{+1}$.

Proof of (5). $\hat{h}_1^0$ equals an imbedding, f_0 or f_1, on $(M - U_{-1}^0) \cup \overline{U}_0^0$. Let $x \in M - [(M - U_{-1}^0) \cup \overline{U}_0^0]$. Then the distance from $f_0(x)$ to $f_0((M - U_{-2}^0) \cup \overline{U}_1^0)$ is greater than 3ρ. Since $\hat{h}_0^0 = f_0$, we have $|\hat{h}_1^0(y) - f_0(y)| < \alpha_0 < \alpha < \rho$ for all $y \in M$. Therefore the distance from $\hat{h}_1^0(x)$ to $\hat{h}_1^0((M - \overline{U}_{-2}^0) \cup \overline{U}_1^0)$ is greater than $3\rho - 2\rho = \rho > 0$. Thus the images under $\hat{h}_1^0$ of $(M - U_{-2}^0) \cup \overline{U}_1^0$ and its complement in M are disjoint, and (5) holds. (See Fig. 2. The integers $-3, \ldots, 2$ denote the images of the left end points of $\overline{U}_{-3}^0, \ldots, \overline{U}_2^0$.)

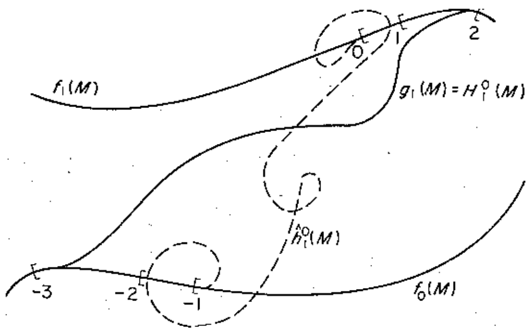

FIG. 2.

We apply Theorem 2.2 to $\hat{h}_1^0$ to get $H_t^0: M \to Q$, $t \in [0, 1]$, satisfying

(1) $H_0^0 = \hat{h}_1^0$,

(2) H_1^0 is a differentiable imbedding,

(3) $H_t^0 = H_0^0 = \hat{h}_1^0$ on $M - \overline{U}_{-3}^0$ and on $\overline{U}_1^0$ for all $t \in [0, 1]$,

(4) $|H_t^0(x) - \hat{h}_1^0(x)| < \alpha_1$ for all $x \in M$, $t \in [0, 1]$.

Next we define $g_t: M \to Q$, $t \in [0, 1]$, by

$$g_t = \begin{cases} \hat{h}_{2t}^0 \text{ for } t \in [0, \frac{1}{2}], \\ H_{2t-1}^0 \text{ for } t \in [\frac{1}{2}, 1]. \end{cases}$$

g_t satisfies

(1) $g_0 = f_0$,

(2) $g_1 = H_1^0$ is a differentiable imbedding,

(3) $g_t = g_0$ on $M - U_{-3}^0$ and on $\overline{U}_2^0$,

(4) $g_1 = f_1$ on $\overline{U}_2^0$,

(5) $|g_t(x) - g_0(x)| < \alpha_0 + \alpha_1$, for all $x \in M$, $t \in [0, 1]$.

Thus g_t moves $f_0(\overline{U}_2^0)$ to $f_1(\overline{U}_2^0)$. This is the first of $n + 1$ steps; next we would move $g_1(\overline{U}_4^1)$ to $f_1(\overline{U}_4^1)$, and so on.

We must define the obvious homotopy from g_1 to f_1 which keeps $g_1(\overline{U}_0^0) = f_1(\overline{U}_0^0)$ fixed. First let $\check{h}_t^0(x) = f_\tau(x)$ where $\tau = \gamma^0(s) + t(1 - \gamma^0(x))$. Then

(1) $\check{h}_0^0 = \hat{h}_1^0$,

(2) $\check{h}_1^0 = f_1$,

(3) $\check{h}_t^0 = \hat{h}_1^0 = f_1$ on $\overline{U}_2^0$ for all $t \in [0, 1]$.

Now define

$$h_t^1 = \begin{cases} H_{1-2t}^0 \; t \in [0, \frac{1}{2}] \\ \check{h}_{2t-1} \; t \in [\frac{1}{2}, 1]. \end{cases}$$

Then we have

(1) $h_0^1 = g_1$ (because $h_0^1 = H_1^0 = g_1$)
(2) $h_1^1 = f_1$,
(3) $h_t^1 = h_0^1 = f_1$ on $\bar{U}_2^0$ for all $t \in [0, 1]$,
(4) $|h_t^1(x) - h_0^1(x)| < \alpha_0 + \alpha_1$ for all $x \in M$, $t \in [0, 1]$.

(4) holds because $\bar{h}_t^0$ moves points no more than f_t does which is less than $\alpha_0 = \delta$, and H_t^0 moves points no more than α_1.

The proof will proceed by induction. Suppose we have g_t: $M \to Q$, $t \in [0, k]$, satisfying

(1) $g_0 = f_0$,
(2) g_i is a differentiable imbedding for $i = 0, 1, \ldots, k$,
(3) $g_t = g_i$ on $M - U_{-3}^i$ for all $t \in [i, i+1]$, $i = 0, 1, \ldots, k-1$,
(4) $g_k = f_1$ on $\bigcup_{i=0}^{k-1} \bar{U}_{2k}^i$,
(5) $|g_t(x) - g_0(x)| < \sum_{i=0}^k (k - i + 1) \cdot \alpha_i$.

We observe that g_t, $t \in [0, 1]$, which was defined earlier, satisfies these properties and starts the induction.

Also suppose we have defined h_t^k: $M \to Q$, $t \in [0, 1]$, satisfying

(1) $h_0^k = g_k$,
(2) $h_1^k = f_1$,
(3) $h_t^k = h_0^k = f_1$ on $\bigcup_{i=0}^{k-1} \bar{U}_{2k}^i$ for all $t \in [0, 1]$,
(4) $|h_t^k(x) - h_0^k(x)| < \sum_{i=0}^k \alpha_i$,

We see that $h_t^0 = f_t$, or h_t^1 as recently defined, satisfies these conditions and therefore starts the induction.

Now we want to define g_t, $t \in [k, k+1]$, and h_t^{k+1}, $t \in [0, 1]$. Let γ^k: $M \to [0, 1]$ be a continuous function such that $\gamma^k = 1$ on $\bar{U}_0^k$ and $\gamma^k = 0$ on $M - U_{-1}^k$. Let $\hat{h}_t^k(x) = h_{t \cdot \gamma^k(x)}^k(x)$ for all $x \in M$, $t \in [0, 1]$. This homotopy satisfies

(1) $\hat{h}_0^k = g_k$,
(2) $\hat{h}_1^k = g_k$ on $M - U_{-1}^k$,
(3) $\hat{h}_1^k = f_1$ on $(\bigcup_{i=0}^{k-1} \bar{U}_{2k}^i) \cup \bar{U}_0^k \supset \bigcup_{i=0}^k \bar{U}_{2k}^i$,
(4) $|\hat{h}_t^k(x) - \hat{h}_0^k(x)| < \sum_{i=0}^k \alpha_i$,
(5) $\hat{h}_1^k$ is a differentiable imbedding on $(M - U_{-2}^k) \cup (\bigcup_{i=0}^k \bar{U}_{2k+1}^i)$.

Proof of (5). $\hat{h}_1^k$ equals an imbedding, g_k or f_1, on $(M - U_{-1}^k) \cup \bigcup_{i=0}^k \bar{U}_{2k}^i$. Let

$$x \in M - [(M - U_{-1}^k) \cup \bigcup_{i=0}^k \bar{U}_{2k}^i].$$

Then the distance from $f_0(x)$ to $f_0((M - \bar{U}_{-2}^k) \cup \bigcup_{i=0}^k \bar{U}_{2k+1}^i$ is greater than 3ρ. Since $\hat{h}_0^k = g_k$ and $g_0 = f_0$, we have

$$|\hat{h}_1^k(y) - f_0(y)| \leqq |\hat{h}_1^k(y) - \hat{h}_0^k(y)| + |g_k(y) - g_0(y)| < \sum_{i=0}^k \alpha_i + \sum_{i=0}^k (k - i + 1) \cdot \alpha_i$$
$$= \sum_{i=0}^k (k - i + 2) \cdot \alpha_i < \alpha < \rho,$$

for all $y \in M$. Therefore the distance from $\hat{h}_1^k(x)$ to $\hat{h}_1^k([M - U_{-2}^k) \cup (\bigcup_{i=0}^k \bar{U}_{2k+1}^i)$ is greater than $3\rho - 2\rho = \rho > 0$. Thus the images under $\hat{h}_1^k$ of $(M - U_{-2}^k) \cup (\bigcup_{i=0}^k \bar{U}_{2k+1}^i)$ and its complement in M are disjoint, and (5) holds.

Now we apply Theorem 2.2 to $\hat{h}_1^k$, with $\varepsilon = \alpha_{k+1}$, and $\delta = \sum_{i=0}^k \alpha_i$ and get $H_t^k: M \to Q$, $t \in [0, 1]$ satisfying

(1) $H_0^k = \hat{h}_1^k$,
(2) H_1^k is a differentiable imbedding,
(3) $H_t^k = H_0^k = \hat{h}_1^k$ on $M - U_{-3}^k$ and on $\bigcup_{i=0}^k \bar{U}_{2k+2}^i$,
(4) $|H_t^k(x) - H_0^k(x)| < \alpha_{k+1}$.

Then we define g_t, $t \in [k, k+1]$, by

$$g_t = \begin{cases} \hat{h}_{2t-2k}^k & t \in [k, k+\frac{1}{2}] \\ H_{2t-2k-1}^k & t \in [k+\frac{1}{2}, k+1], \end{cases}$$

satisfying

(1) $g_0 = f_0$,
(2) g_1 is a differentiable imbedding for $i = 0, \ldots, k+1$, (because $g_{k+1} = H_1^k$),
(3) $g_t = g_i$ on $M - U_{-3}^i$ for $t \in [i, i+1]$, $i = 0, \ldots, k$, (because $g_t = g_k$, $t \in [k, k+1]$, on $(M - U_{-1}^k) \cap (M - U_{-3}^k) = M - U_{-3}^k$),
(4) $g_{k+1} = f_1$ on $\bigcup_{i=0}^k \bar{U}_{2k+2}^i$ (because $g_{k+1} = H_1^k = \hat{h}_1^k = f_1$ on $\bigcup_{i=0}^k \bar{U}_{2k+2}^i$),
(5) $|g_t(x) - g_0(x)| < \sum_{i=0}^{k+1}(k - i + 2) \cdot \alpha_i$ for all $x \in M$, $t \in [0, k+1]$ (because, $|g_t(x) - g_0(x)| \leqq |g_k(x) - g_0(x)| + |\hat{h}_t^k(x) - \hat{h}_0^k(x)| + |H_{t'}^k(x) - \hat{h}_1^k(x)| < \sum_{i=0}^k (k - i + 1) \cdot \alpha_i + \sum_{i=0}^k \alpha_i + \alpha_{k+1} = \sum_{i=0}^{k+1}(k - i + 2) \cdot \alpha_i$, where $t' = t$ or 1.)

Now we must define $h_t^{k+1}: M \to Q$, $t \in [0, 1]$. But first, let $\check{h}_t^k(x) = h_\tau^k(x)$ where $\tau = \gamma^k(x) + t(1 - \gamma^k(x))$. Then

(1) $\check{h}_0^k = \hat{h}_1^k$,
(2) $\check{h}_1^k = f_1$,
(3) $\check{h}_t^k = \hat{h}_1^k = f_1$ on $\bigcup_{i=0}^k \bar{U}_{2k}^i$ for all $t \in [0, 1]$.

Then define

$$h_t^{k+1} = \begin{cases} H_{1-2t}^k & t \in [0, \frac{1}{2}] \\ \check{h}_{2t-1}^k & t \in [\frac{1}{2}, 1], \end{cases}$$

which satisfies

(1) $h_0^{k+1} = g_{k+1}$, (because $h_0^{k+1} = H_1^k = g_{k+1}$),
(2) $h_1^{k+1} = f_1$, (because $h_1^{k+1} = \check{h}_1^k = f_1$),
(3) $h_t^{k+1} = h_0^{k+1} = f_1$ on $\bigcup_{i=0}^k \bar{U}_{2k+2}^i$ for all $t \in [0, 1]$ (because on $\bigcup_{i=0}^k \bar{U}_{2k+2}^i$, we have $h_t^{k+1} = \hat{h}_1^k = f_1 = g_{k+1} = h_0^{k+1}$ for all $t \in [0, \frac{1}{2}]$, and $h_t^{k+1} = \check{h}_{2t-1}^k = f_1$ for all $t \in [\frac{1}{2}, 1]$),
(4) $|h_t^{k+1}(x) - h_0^{k+1}(x)| < \sum_{i=0}^{k+1} \alpha_i$, (because H_t^k moves points at most α_{k+1}, and $\check{h}_t^k$ moves points no more than h_t^k does, which is $\sum_{i=0}^{k+1} \alpha_i$).

Then, by induction we obtain $g_t: M \to Q$, $t \in [0, n+1]$ satisfying

(1) $g_0 = f_0$,
(2) g_i is a differentiable imbedding for $i = 0, 1, \ldots, n+1$,
(3) $g_t = g_i$ on $M - U_{-3}^i$ for all $t \in [i, i+1]$, $i = 0, 1, \ldots, n$,
(4) $g_{n+1} = f_1$ on $\bigcup_{i=0}^n \bar{U}_{2n+2}^i = M$,
(5) $|g_t(x) - g_0(x)| < \sum_{i=0}^{n+1}(n - i + 2) \cdot \alpha_i < \alpha$, for all $x \in M$, $t \in [0, n+1]$.

Pick $\beta > 0$, to be determined later. We want to apply Theorem 2.3 to g_t, $t \in [k, k+1]$, with $\beta = \varepsilon$. Note that $g_t = g_k$ on $M - U^k_{-3}$ which is a neighborhood of $M - U^k_{-4}$. Since $|g_t(x) - g_k(x)| < |g_t(x) - g_0(x)| + |g_k(x) - g_0(x)| < \alpha + \alpha = 2\alpha$, and 2α may have been chosen small enough to satisfy Theorem 2.3, we obtain a diffeotopy $G_t: M \to Q$, $t \in [k, k+1]$, satisfying

(1) $G_k = g_k$ and $G_{k+1} = g_{k+1}$,
(2) $|G_t(x) - g_t(x)| < \beta$ for all $x \in M$, $t \in [k, k+1]$,
(3) $G_t = G_k = g_k$ on $M - U^k_{-4}$.

By doing this for $k = 0, 1, \ldots, n$, we get a diffeotopy $G_t: M \to Q$, $t \in [0, n+1]$, which also satisfies

(4) $G_0 = f_0$, and $G_1 = f_1$,
(5) $|G_t(x) - f_0(x)| \leqq |G_t(x) - g_t(x)| + |g_t(x) - g_0(x)| < \beta + \alpha$ for all $t \in [0, n+1]$, $x \in M$.

(5) implies

(5′) $|G_t(x) - G_{t'}(x)| < 2(\beta + \alpha)$ for all $t, t' \in [0, n+1]$, $x \in M$.

Consider G_t, $t \in [k, k+1]$. By choosing $2(\beta + \alpha) < \rho$ we see that $G_t(\overline{U}^{k,j}_{-4}) \subset N_\rho(G_k(\overline{U}^{k,j}_{-4})) = N^{k,j}$ for all $t \in [k, k+1]$, $j = 1, 2, \ldots, m_k$. Since the $G_k(\overline{U}^{k,j}_{-4})$, $j = 1, \ldots, m_k$, are more than 3ρ apart from each other. it follows that the $N^{k,j}$, $j = 1, \ldots, m_k$, are disjoint. The diameter of $N^{k,j}$ is less than $\mu + 2\rho$.

By applying Theorem 2.4 to G_t, $t \in [k, k+1]$, k $= 0, 1, \ldots, n$, with $M_0 = \overline{U}^k_{-4}$ and $Q_0 = \bigcup_{j=1}^{m_k} N^{k,j}$, we obtain an ambient diffeotopy $F^k_t: Q \to Q$, $t \in [0, 1]$, satisfying

(1) $F^k_0 =$ identity,
(2) $F^k_t G_k = G_{k+t}$,
(3) $F^k_t =$ identity on $Q - \bigcup_{n=1}^{m_k} N^{k,j}$ for all $t \in [0, 1]$.

In particular, $F^k_t(N^{k,j}) \subset N^{k,j}$ for $j = 1, \ldots, m_k$, $t \in [0, 1]$. By choosing $\mu + 2\rho < \varepsilon/(n+1)$, we see that F^k_t also satisfies

(4) $|F^k_t(x) - x| < \varepsilon/(n+1)$, for all $x < Q$, $t \in [0, 1]$.

Finally, let $F_t: Q \to Q$, $t \in [0, 1]$ be defined by

$$F_t = \begin{cases} F^0_{(n+1)\cdot t} & t \in [0, 1/(n+1)] \\ F^1_{(n+1)\cdot t} \cdot F^0_1 & t \in [1/(n+1), 2/(n+1)] \\ \vdots & \\ F^n_{(n+1)t} F^{n-1} \cdots F^1_1 \cdot F^0_1 & t \in [n/(n+1), 1]. \end{cases}$$

We verify the conclusions of Theorem 2.1.

(1) $F_1 f_0 = F_1 f_0 = F^n_1 \cdot F^{n-1}_1 \cdots F^1_1 \cdot F^0_1 \cdot G_0 = F^n_1 \cdots F^1_1 \cdot G_1 = \cdots = F^n_1 \cdot G_n = {}_{n+1} = f_1$.
(2) $F_t =$ identity outside $N_\varepsilon(f_0(M))$ because each F^k_t is fixed outside $N_\varepsilon(f_0(M))$. This holds since $G_k(\overline{U}^{k,j}_{-4}) \subset N_{\beta+\alpha}(f_0(M))$ which implies that $N^{k,j} = N_\rho(G_k(\overline{U}^{k,j}_{-4})) \subset N_{\beta+\alpha+\rho}(f_0(M)) \subset N_\varepsilon(f_0(M))$.

(3) If $\partial M \neq \emptyset$, and $f_t = f_0$ in a neighborhood of ∂M, then it is clear from the proof that each map constructed can be made equal to f_0 in a perhaps smaller neighborhood of ∂M.

This completes the proof.

Remark. Since the ambient diffeotopy F_t was constructed in a local fashion, it is clear that Theorem 2.1 holds even if M is not compact. Extra care must be taken with the approximations; ε, δ, ρ, α_i and β all must be chosen as functions from M to the positive real numbers. The $U_k^{i,j}$ form a locally finite open covering of M on which ε, δ, ρ, α_i and β have minimums. The proof for non-compact M is then very similar to the compact case.

§4.

THEOREM 4.1. *Let f be a locally flat imbedding of $2B^n$ into the interior of Q^q, with $2q > 3(n+1)$. Let U be a neighborhood of $f(B^n)$ in Q. Then f can be extended to a stable imbedding of R^q (where $2B^n$ is the usual subset of R^q) into U. (In all cases, stable structures are those provided by the differentiable structures.)*

Proof. Case 1: Suppose that $Q^q = R^q = U$. Then (see [2, Theorem 1′] or [11]) there is an ambient isotopy $H_t: R^q \to R^q$, $t \in [0, 1]$, such that $H_0 =$ identity, $H_1 f(x) = x$, for all $x \in 2B^n$, and H_t is fixed outside a bounded set for all $t \in [0, 1]$. Then $(H_1)^{-1}$ is a stable imbedding of R^q in R^q extending f.

Case 2: Let Q^q be any compact, differentiable manifold. It is possible to extend f to an imbedding of R^q in U (this is true for any q and n). $f(R^q)$ is diffeomorphic to R^q by a diffeomorphism h. Then $hf: B^n \to R^q$, according to Case 1, can be extended to a stable imbedding $(hf)': R^q \to R^q$. Then $h^{-1}\cdot(hf)'$ is a stable imbedding of R^q into U, extending f.

THEOREM 4.2. *Let $f: R^q \to R^q$ be a stable homeomorphism with $q \geqq 7$. Let $\varepsilon(x): R^q \to (0, \infty)$ be a continuous function. Then there exists an isotopy $f_t: R^q \to R^q$, $t \in [0, 1]$, satisfying*

(1) $f_0 = f$

(2) f_1 is a diffeomorphism

(3) $|f_t(x) - f(x)| < \varepsilon(x)$ for all $x \in R^q$, $t \in [0, 1]$.

Proof. Let $\varepsilon_r = \min\{\varepsilon(x) | x \in rB^q\}$. Since f is stable, $f|\text{int}\, 2B^q$ may be approximated (see [3]) by a differentiable imbedding $\bar{g}_2: \text{int}\, 2B^q \to R^q$ such that $\bar{g}_2$ extends to a homeomorphism $g_2: R^q \to R^q$ where $g_2 = f$ on $R^q - \text{int}\, 2B^q$ and $|g_2(x) - f(x)| < \delta_2$ for a δ_2 to be chosen. f is isotopic to g_2 via the usual isotopy g_t, $t \in [1, 2]$, where

$$g_t(x) = \begin{cases} g_2\left(\left(2-t\right)\cdot g_2^{-1} f\left(\dfrac{x}{2-t}\right)\right) & \text{for} \quad t < 2 \\ g_2(x) & \text{for} \quad t = 2. \end{cases}$$

Certainly δ_2 can have been chosen small enough so that $|g_t(x) - f(x)| < \varepsilon_2/3$ for all $x \in R^q$ and $t \in [1, 2]$.

Next, we approximate g_2 by a homeomorphism $g_3: R^q \to R^q$ for which $g_3|\text{int}\, 3B^q$ is differentiable, $g_3 = g_2 = f$ outside int $3B^q$, $g_3 = g_2$ on int B^q, and $|g_3(x) - g_2(x)| < \delta_3$. As before, there is an isotopy g_t, $t \in [2, 3]$, between g_2 and g_3, and δ_3 may have been chosen so that $|g_t(x) - g_2(x)| < \varepsilon_3/3$ for all $t \in [2, 3]$, $x \in R^q$.

Inductively, we can construct a sequence of homeomorphisms, $\{g_i\}$, and isotopies $\{g_t, t \in [i-1, i]\}$ where $g_i|\text{int}\, iB^q$ is differentiable, $g_i = f$ outside $\text{int}\, iB^q$, $g_i = g_{i-1}$ on $\text{int}(i-2)B^q$, and $|g_t(x) - g_{i-1}(x)| < \varepsilon_i/3$ for all $t \in [i-1, i]$, $x \in R^q$.

Clearly $f_1 = \lim g_i$ is a diffeomorphism Also, since we can assume that $\lim_{|x| \to \infty} \varepsilon(x) = 0$, then $\lim_{i \to \infty} \varepsilon_i = 0$ and hence $g_\infty = \lim_{t \to \infty} g_t = f_1$. Let $f_t = g_s$, $s = (1+t)/(1-t)$, $t \in [0, 1]$. Then (1) and (2) hold and (3) is easily verified (for $x \in rB^q - \text{int}(r-1)B^q$, we have

$$|f(x) - g_t(x)| \leqq |f(x) - g_{r-1}(x)| + |g_{r-1}(x) - g_r(x)| + |g_r(x) - g_{r+1}(x)| + |g_{r+1}(x) - g_t(x)|$$

$$\leqq 0 + \frac{\varepsilon_r}{3} + \frac{\varepsilon_{r+1}}{3} + \frac{\varepsilon_{r+2}}{3} \leqq \varepsilon_r \leqq \varepsilon(x),$$

since $g_{r+1}(x) = g_k(x) = f_1(x)$ for integers $k \geqq r+1$).

Theorem 4.2 is a corollary of the following two theorems (letting $L^{q-1} = \partial B^q$). The theorems are not needed here, so the proofs are omitted, but appear in [7].

THEOREM 4.3. *Let $f: R^q \to R^q$ be a stable imbedding. Then there exists an isotopy $f_t: R^q \to R^q$, $t \in [0, 1]$, satisfying* (1)

(1) $f_0 = f$,

(2) *f_1 is differentiable on B^q, and*

(3) *$f_t = f$ on $R^q - 2B^q$ for all $t \in [0, 1]$.*

THEOREM 4.4. *Let L^{q-1} be a closed, compact, differentiable manifold and $f: L \times [0, 1) \to Q$ be an imbedding with $f|L \times 0$ differentiable. Let $\varepsilon > 0$. Then there exists an isotopy $F_t: L \times [0, 1) \to Q$ satisfying*

(1) $F_0 = f$,

(2) *F_1 is differentiable on $L \times [0, 1)$,*

(3) *$|F_t(x, s) - f(x, s)| < \varepsilon$ and*

(4) *$F_t = f$ on $L \times 0$.*

If in addition f is defined on $L \times 1$, we can require that $F_t = f$ on $L \times 1$.

Let $f': 2B^n \to Q^q$ be a locally flat imbedding with $2q > 3(n+1)$. By Theorem 4.1, f' extends to a stable imbedding f'' of R^q into Q. Let $f = f''|\text{int}\, 2B^q$. If $f'(2B^n)$ lies in an n-submanifold N of Q, we can assume that $f(\text{int}\, 2B^q) \cap N = f(\text{int}\, 2B^n)$. For $f: \text{int}\, 2B^q \to Q$, we have the following Theorem:

THEOREM 4.5. *Let $\varepsilon > 0$, $q \geqq 7$ and $2q > 3(n+1)$. Let C^{n-1} be a compact, differentiable submanifold* of $\partial B^n = S^{n-1}$. *Suppose that f is differentiable on a neighborhood U of C^{n-1} in B^n. Then there exists an ambient ε-isotopy $F_t: Q \to Q$, $t \in [0, 1]$, satisfying*

(1) *$F_1 f$ is differentiable on* int B^n *and a neighborhood of C^{n-1} in B^n,*

(2) *F_t = identity $N_\varepsilon(f(B^n)) \cap f(\text{int}\, B^q)$.*

Proof. First we apply Theorem 4.2 to $f|\text{int}\, B^q: \text{int}\, B^q \to f(\text{int}\, B^q) \subset Q$ using the continuous map $\varepsilon: \text{int}\, B^q \to (0, \infty)$ which satisfies $\lim_{|x| \to 1} \varepsilon(x) = 0$. This provides an isotopy f_t which extends to an isotopy $G_t: \text{int}\, 2B^q \to Q$ such that $G_0 = f$, $G_1|\text{int}\, B^q$ is differentiable, and $G_t = f$ on int $2B^q$ − int B^q for all $t \in [0, 1]$. However G_1 is no longer differentiable on C^{n-1}. The object is to move $G_1(U^1)$ back to $f(U^1)$, keeping G differentiable on int B^n, where U^1 is a smaller neighborhood of C^{n-1} than U. The method is quite similar to the proof of Theorem 2.1 so only the outline is given here (see [7] for details).

Let Z^{n-1} be a neighborhood of C^{n-1} in ∂B^n for which $\bar{Z}$ is a manifold and $\bar{Z} \subset U \cap \partial B^n$. For some real number a, $0 < a < 1$, f is differentiable on a neighborhood in B^n of the submanifold $Z \times [a, 1]$ imbedded in B^n by $(x, t) = tx$ for $x \in \bar{Z} \subset \partial B^n$ and $t \in [a, 1]$. Subdivide the interval $[a, 1]$ by

$$a < (a+1)/2 < (a+2)/3 < \cdots < (a+k)/(k+1) < \cdots < 1.$$

Denote the interval

$$\left[\frac{a+k}{k+1}, \frac{a+k+1}{k+2}\right]$$

by I_k.

We now apply Theorems 2.1, 2.2, 2.3 and the techniques of Theorem 2.1 to G_t on disjoint neighborhoods of the manifolds $\bar{Z} \times I_{2k}$, $k = 0, 1, \ldots, \infty$, where $\delta = \delta_{2k} = \min\{\varepsilon(x) | x \in \bar{Z} \times I_{2k}\}$ is chosen small enough for Theorem 2.1. That is, we find an imbedding $G_2 : \text{int } 2B^q \to Q$ which equals G_1 except on disjoint neighborhoods of $\bar{Z} \times I_{2k}$, equals f on $\bar{Z} \times I_{2k}$, $k = 0, 1, \ldots$, and is differentiable on int B^q. Then we find an isotopy G_t, $t \in [1, 2]$ between G_1 and G_2, differentiable on int B^q. (See Fig. 3.)

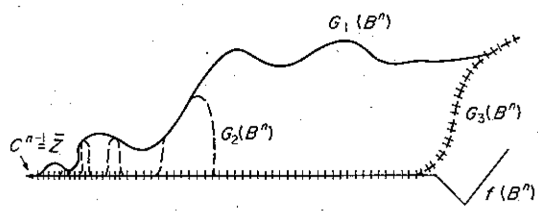

FIG. 3.

Then by the same methods, we obtain G_3, an imbedding equal to f on a neighborhood U^1 of $\bar{Z} \times [a, 1]$ in B^n, equal to G_2 elsewhere and hence differentiable on int B^n union U^1. An isotopy G_t, $t \in [2, 3]$, differentiable on int B^q, is constructed as before. Then $F_t : Q \to Q$, $t \in [0, 1]$, defined by $F_t = G_{3t} f^{-1}$, is the desired ambient isotopy.

§5.

Proof of Main Theorem. M can be represented as a handlebody $(B^n; f_1, k_1, \ldots, f_r, k_r)$, where

$M_0 = B^n$,

$M_1 = M_0 \bigcup_{f_1} B^{k_1} \times B^{n-k_1}$, where $f_1 : S^{k_1-1} \times B^{n-k_1} \to M_0$ is a differentiable imbedding,

⋮

$M = M_r = M_{r-1} \bigcup_{f_1} B^{k_r} \times B^{n-k_r}$, where $f_r : S^{k_r-1} \times B^{n-k_r} \to M_{r-1}$ is a differentiable imbedding.

It is possible to extend f_i and M_i (to $\hat{M}_i$) so that

$\hat{M}_0 = 2B^n$,

$\hat{M}_1 = \hat{M}_0 \bigcup_{f_1} 2B^{k_1} \times 2B^{n-k_1}$, where $f_1 : (2B^{k_1} - \text{int } B^{k_1}) \times 2B^{n-k_1} \to M_0$ is a differentiable imbedding such that $f_1(S^{k_1-1} \times B^{n-k_1}) \subset \partial M_0$,

$\vdots$

$M = M_r = M_{r-1} \bigcup_{f_r} 2B^{k_r} \times 2B^{n-k_r}$, where $f_r : (2B^{k_r} - \text{int } B^{k_r}) \times B^{n-k_r} \to M_{r-1}$ is a differentiable imbedding such that $f_r(S^{k_r-1} \times 2B^{n-k_r}) \subset \partial M_{r-1}$.

We can assume that f is a differentiable imbedding of all of $2B^{k_r} \times 2B^{n-k_r}$ in M. (See Fig. 4.)

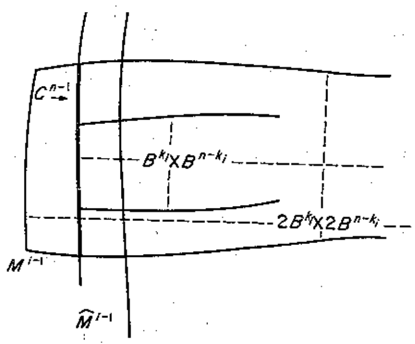

FIG. 4.

Case 1: Assume that $\partial M = N = \varnothing$.

We apply Theorem 4.5 to $f|\hat{M}_0$ with $C^{n-1} = \varnothing$, ε replaced by $\varepsilon/(r+1)$ and B^n replaced by $2B^n = \hat{M}_0$. We obtain an ambient isotopy $G_t: Q \to Q$, $t \in [0, 1]$, such that

(1) $G_0 = \text{identity}$,

(2) $G_1 f$ is differentiable on int $2B^q \supset M_0$,

(3) $|G_t(x) - x| < \varepsilon/(r+1)$ for all $x \in Q$, $t \in [0, 1]$,

(4) $G_t = \text{identity}$ on $Q - N_{\varepsilon/(r+1)}(f(\hat{M}_0))$ for all $t \in [0, 1]$.

Suppose for $0 < i < r$ we have an ambient isotopy $G_t: Q \to Q$, $t \in [0, i]$, satisfying

(1) $G_0 = \text{identity}$,

(2) $G_1 f$ is differentiable on a neighborhood of M_{i-1},

(3) $|G_t(x) - x| < i \cdot \varepsilon/(r+1)$ for all $x \in Q$, $t \in [0, i]$,

(4) $G_t = \text{identity}$ on $Q - N_{i \cdot \varepsilon/(r+1)}(f(\hat{M}_{i-1}))$ for all $t \in [0, i]$.

Again we apply Theorem 4.5, this time to $G_i f f_i : 2B^{k_i} \times 2B^{n-k_i} \to Q$. Again we replace ε by $\varepsilon/(r+1)$ and B^n by $B^{k_i} \times 2B^{n-k_i}$ in Theorem 4.5. We let C^{n-1} be $S^{k_i-1} \times B^{n-k_i}$, and observe that $G_i f f_i$ is differentiable on a neighborhood of this set (see Fig. 4). Then we obtain an ambient isotopy $\hat{G}_t: Q \to Q$, $t \in [0, 1]$, satisfying

(1) $\hat{G}_0 = \text{identity}$,

(2) $\hat{G}_1 G_i f f_i$ is differentiable on int $(B^{k_i} \times 2B^{n-k_i})$ and on a neighborhood of $S^{k_i-1} \times 2B^{n-k_i}$,

(3) $|\hat{G}_t(x) - x| < \varepsilon/(r+1)$ for all $x \varepsilon Q$, $t \varepsilon [0, 1]$,
(4) $\hat{G}_t =$ identity on $Q - N_{\varepsilon/(r+1)}(G_i f f_i(B^{k_i} \times 2B^{n-k_i}))$ and on $G_i f(M - f_i(B^{k_i} \times 2B^{n-k_i}))$.

Define $G_t: Q \to Q$, $t \in [i, i+1]$, by $G_t = \hat{G}_{i-t} \cdot G_i$. Then
(1) $G_0 =$ identity,
(2) $G_{i+1} f$ is differentiable on a neighborhood of M_i,
(3) $|G_t(x) - x| < \nu$ for all $x \in Q$, $t \in [0, 1]$ where $\nu = (i+1) \cdot \varepsilon/(r+1)$.
(4) $G_t =$ identity on $Q - N\nu(f(\hat{M}_i))$ for all $t \in [0, i+1]$.

By induction we obtain $G_t: Q \to Q$, $t \in [0, r+1]$. Then let $F_t: Q \to Q$, $t \in [0, 1]$, be defined by $F_t = G_{(r+1)t}$, $t \in [0, 1]$. This F_t clearly satisfies the theorem.

Case 2: Suppose that $\partial M \neq \emptyset$ and $f(\partial M) \subset \operatorname{int} Q$. ∂M is collared in M, i.e. there is a differentiable imbedding $g: \partial M \times I \to M$ with $g(x, 0) = x$, for all $x \in \partial M$. Let $M^* = M - g(\partial M \times [0, \frac{1}{2}))$. The argument in Case 1 provides an ambient isotopy F_t which smooths the image under f of a neighborhood U of M^* in M. Then we follow F_t by an ambient ε-isotopy H_t, $t \in [0, 1]$, which shrinks $F_1 f(M)$ inside $F_1 f(U)$. Then $H_1 F_1 f(M)$ is a differentiable imbedding. (H_t is obtained by shrinking $F_1 f(M)$ locally using the pairs $(V, V \cap F_1 f(M))$, which are homeomorphic to $(R^q, R^{2n} - R^n_+)$, obtained from the local flatness of $F_1 f(M)$.)

Case 3: Suppose that a neighborhood U in M of a submanifold K of ∂M is imbedded differentiably by f. The arguments of Cases 1 and 2 provide an ambient isotopy smoothing $f(M)$ and fixing a smaller subneighborhood U' of U containing K.

Case 4: Suppose that $N \neq \emptyset$. Apply Case 3 to $M - \operatorname{int} N$, taking care that the ambient isotopy fixes all of $f(N)$.

Case 5: Finally, suppose f is proper. Then we smooth $f(\partial M)$ in ∂Q via an ambient isotopy of Q, noting that $2(q-1) > 3((n-1)+1)$, and recalling that $q - 1 \geqq 7$. Then using the collar of ∂Q in Q, we can smooth the image of a neighborhood U of ∂M in M. Then Case 3 finishes the proof.

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