SMOOTHING LOCALLY FLAT IMBEDDINGS[1]

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The fundamental imbedding problem for manifolds is to classify the imbeddings of an n-manifold into a q-manifold under ambient isotopy. We announce here that the differentiable and topological cases of this problem for differentiable manifolds are the same if $2q>3(n+1)$ and $q\geqq 8$.

This follows from Theorem 2 below which states that a locally flat imbedding of a compact differentiable manifold M^n into a differentiable manifold Q^q is ambient isotopic to a differentiable imbedding if $2q>3(n+1)$ and $q\geqq 8$. Since this ambient isotopy may be chosen arbitrarily close to the identity map, the set of differentiable imbeddings is dense in the set of locally flat imbeddings of M^n in Q^q.

It will then follow that two locally flat imbeddings of M^n into Q^q are ambient isotopic if they are homotopic; hence the classification problem reduces to a problem in homotopy theory.

THEOREM 1. *Let $f\colon B^n\to \operatorname{int} Q^q$ be a locally flat imbedding of the unit n-ball into Q^q. Such an f always extends to $f\colon R^q\to \operatorname{int} Q^q$. Let C^{n-1} be a compact differentiable submanifold of $\partial B^n=S^{n-1}$, and suppose that f is differentiable on a neighborhood of C^{n-1} in B^n. Let $q\geqq 7$, $2q>3(n+1)$ and $\epsilon>0$. Then there exists an ambient ϵ-isotopy $F_t\colon Q^q\to Q^q$, $t\in[0, 1]$, satisfying*

(1) *$F_0=$ identity,*

(2) *F_1f is differentiable on* $\operatorname{int} B^n$ *and on a neighborhood of C^{n-1} in B^n,*

(3) *$F_t=$ identity on $Q-N_\epsilon(f(B^n))$ and on $f(R^n-\operatorname{int} B^n)$ for all $t\in[0, 1]$,*

(4) *$|F_t(x)-x|<\epsilon$ for all $x\in Q^q$ and $t\in[0, 1]$. ($N_\epsilon(X)$) is the set of points within ϵ of X.)*

THEOREM 2. *Let $f\colon M^n\to Q^q$ be a locally flat imbedding such that either $f(M^n)\subset \operatorname{int} Q^q$ and $q\geqq 7$ or $f^{-1}(\partial Q^q)=\partial M^n$ and $q\geqq 8$. Let $2q>3(n+1)$ and $\epsilon>0$. Then there exists an ambient ϵ-isotopy $F_t\colon Q^q\to Q^q$, $t\in[0, 1]$, satisfying*

(1) *$F_0=$ identity,*

(2) *F_1f is a differentiable imbedding,*

[1] This is an announcement of a portion of the author's dissertation at the University of Chicago written under Professor Eldon Dyer.

(3) $F_t = $ *identity on* $Q - N_\epsilon(f(M^n))$ *for all* $t \in [0, 1]$,
(4) $|F_t(x) - x| < \epsilon$ *for all* $x \in Q^q$ *and* $t \in [0, 1]$.

The proof follows from Theorem 1 by considering the handlebody decomposition of M^n, and smoothing the imbedding of one handle at a time.

Only imbeddings of M^n into Q^q satisfying $f(M^n) \subset \operatorname{int} Q^q$ or $f^{-1}(\partial Q^q) = \partial M^n$ will be considered. Let T be the set of equivalence classes of locally flat imbeddings of M^n into Q^q under equivalence by ambient isotopy. Similarly, let $D(C)$ be the set of equivalence classes of differentiable (combinatorial) imbeddings of M^n into Q^q under equivalence by ambient diffeotopy (ambient combinatorial isotopy). Let H be the homotopy classes of locally flat imbeddings of M^n into Q^q. H is a subset of $[M^n, Q^q]$, the homotopy classes of maps of M^n into Q^q. Then we have the following commutative diagram where the maps are the natural projections.

$$\begin{tikzcd}
D \arrow[d, "\pi"'] \arrow[dr, "\alpha"] & & \\
T \arrow[r, "\beta"] & H \arrow[r, "\subset", "i"'] & {[M, Q]} \\
C \arrow[u, "\rho"] \arrow[ur, "\gamma"'] & &
\end{tikzcd}$$

β is clearly onto for all n and q. Gluck has shown [1] that ρ and γ, and hence β and βi are isomorphisms for $q \geqq 2n+2$. Haefliger has shown [2] that π is a monomorphism and that α is an isomorphism if $2q > 3(n+1)$.

It follows from Theorem 2 that π is also epimorphic if $2q > 3(n+1)$ and either $q \geqq 7$ when $f(M^n) \subset \operatorname{int} Q^q$ or $q \geqq 8$ when $f^{-1}(\partial Q^q) = \partial M^n$. Then π and β are isomorphisms in this range of dimensions.

References

1. H. Gluck, *Embeddings in the trivial range*, Ann. of Math. **81** (1965), 195–210.
2. A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. **36** (1961), 47–82.

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