

## A GEOMETRIC PROOF OF ROCHLIN'S THEOREM

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0. In 1974 Andrew Casson outlined to us a proof of Rochlin's Theorem (stated below) on the index of a smooth, closed 4-manifold  $M^4$ . His proof involved the Arf invariant of a certain quadratic form defined on the first homology group of a surface in  $M^4$  which is dual to the second Stiefel-Whitney class of  $M^4$ . Our proof was derived from Casson's; it is the same in principle but differs considerably in detail. After this manuscript was written, we discovered that Rochlin had already in 1971 given a short sketch of this proof; it appears in a paper [R<sub>3</sub>] about real algebraic curves in  $RP^2$ .

In addition we obtain (Theorem 2) a "stable" converse to the Kervaire-Milnor nonembedding theorem [K-M], and in §2, by relaxing some orientability assumptions, we prove a new (but unspectacular) nonembedding theorem (Theorem 4) and find an obstruction to approximating unoriented simplicial 3-chains in a 5-manifold by an immersed 3-manifold.

We thank John Morgan for several valuable conversations.

1. Let  $M^4$  be a closed, orientable, PL (hence smooth) 4-manifold. It has an intersection form  $H_2(M; Z)/\text{torsion} \times H_2(M; Z)/\text{torsion} \rightarrow Z$  which is a symmetric, unimodular, integral bilinear form [M<sub>1</sub>]; denote this pairing by  $x \cdot y$  or  $xy$ . Its signature is  $\sigma(M) = \text{index } M$ .

We say that  $\omega \in H_2(M; Z)/\text{torsion}$  is characteristic if its mod 2 reduction  $[\omega]_2$  is Poincaré dual to the second Stiefel-Whitney class  $w_2 \in H^2(M; Z_2)$ . This implies that  $\omega \cdot x = x \cdot x \pmod{2}$  for all  $x \in H_2(M; Z)$ . For this congruence follows from the equality  $[\omega]_2 \cdot y = y \cdot y$  for all  $y \in H_2(M; Z_2)$ , which is dual to the equality  $w_2 \cup \hat{y} = \hat{y} \cup \hat{y}$  ( $\hat{\phantom{y}}$  denotes Poincaré dual) which is a definition of the second Stiefel-Whitney class  $w_2$  of a 4-manifold.

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It is an easy bit of algebra [M-H, p. 24], that  $\omega \cdot \omega = \sigma(M) \pmod{8}$ . Thus if  $\omega_2 = 0$  so that 0 is characteristic, then  $\sigma(M) = 0 \pmod{8}$ . Rochlin improved this by a factor of 2.

**THEOREM [R<sub>2</sub>].** *If  $M$  is closed, orientable, PL and  $\omega_2 = 0$ , then  $\sigma(M) = 0 \pmod{16}$ .*

Rochlin's Theorem is an anomaly in this sense: in dimensions  $4k$ ,  $k > 1$ , there are closed, orientable, almost parallelizable PL (not smooth) manifolds  $P^{4k}$  with  $\sigma(P^{4k}) = 8$  [M<sub>2</sub>]. These PL manifolds are missing in dimension 4, and this accounts for the counterexamples to existence and uniqueness of PL structures on manifolds [K-S], [S<sub>2</sub>]. Rochlin's Theorem is not known for topological 4-manifolds; the existence of such a topological 4-manifold of index 8 is equivalent to proving topological transversality in codimension 4 [S<sub>1</sub>].

Let  $\theta_3$  be the group of homology cobordism classes of homology 3-spheres. Then Rochlin's Theorem provides an epimorphism  $\theta_3 \rightarrow^R Z_2$ . If  $\Sigma^3$  is a representative of  $\theta_3$ , it bounds a PL, parallelizable 4-manifold  $Q^4$ . Then  $\sigma(Q^4)/8 \pmod{2} \in Z_2$  is easily seen to be an invariant of the homology cobordism class of  $\Sigma^3$  by use of Rochlin's Theorem [R<sub>1</sub>].

The usual proof [R<sub>2</sub>], [M-K] of Rochlin's Theorem is homotopy theoretic, requiring: the decomposability of  $Sq^3$ , the calculation of  $J: \pi_3(SO) \rightarrow \pi_3^s$ , Hirzebruch's identity,  $\sigma(M^4) = \frac{1}{2}p_1(\tau_{M^4}) [M^4]$ , and the identification of  $p_1(\tau_{M^4})$  with  $\pm 2\{\text{obstruction to extending over } M \text{ a trivialization of } \tau_{M^4} | (M^4\text{-point})\}$ . Our proof is geometric, except for the use of the isomorphism  $\Omega_4 \rightarrow^\sigma Z$  where  $\Omega_4$  is the oriented bordism group of oriented 4-manifolds.

Here is some motivation for our proof. From now on, all 4-manifolds are closed, orientable and smooth. First consider the generalization [K-M]: if the characteristic element  $\omega \in H_2(M^4; Z)$  is represented by a smooth, imbedded  $S^2$ , then  $\omega \cdot \omega - \sigma(M^4) = 0 \pmod{16}$ .

In general,  $\omega$  is represented by an orientable (if  $H_1(M; Z/2) = 0$ ) surface  $K^2$ , and  $\omega \cdot \omega - \sigma(M^4) = 0 \pmod{8}$ , so it is predictable that there is a  $Z^2$  obstruction associated with surgering  $K^2$  to a 2-sphere. Here is the simplest case where that obstruction occurs.

Let  $Q^4 = CP^2 \# \#^8 \overline{CP}^2$  be complex projective space connected sum eight copies reversed orientation. Let  $\alpha_0$  generate  $H_2(CP^2; Z)$  and let  $\alpha_i$  generate the  $i$ th copy of  $H_2(\overline{CP}^2; Z)$ . Then  $\omega = 3\alpha_0 + \alpha_1 + \dots + \alpha_8$  is characteristic and  $\omega \cdot \omega = 1$ . Since  $\omega \cdot \omega - \sigma(Q^4) = 8$ ,  $\omega$  cannot be represented by a smooth imbedded  $S^2$ . (To see this directly from Rochlin's Theorem, suppose  $S^2$  represents  $\omega$ ; its normal disk bundle  $N$  is the Hopf bundle, since  $\omega \cdot \omega = 1$ , so  $\partial N = S^3$ ; remove  $N$  and sew in  $B^4$ ; the new manifold has  $\omega_2 = 0$  and index  $-8$ , a contradiction.) In this case  $\alpha_0$  is represented by any nonsingular cubic, all of which are tori  $S^1 \times S^1$ .  $\alpha_i$  is represented by  $CP^1 = S^2$ , so  $\omega$  is represented by an  $S^1 \times S^1$ , and we cannot reduce the genus.

We can try to surger  $K$  inside  $M^4$  to get a 2-sphere. Let  $A_1, \dots, A_{2s}$  be imbedded circles representing the generators of a symplectic basis of  $H_1(K; Z) = Z^{2s}$ . To surger some  $A_i$ , we must smoothly imbed a 2-ball  $D_i$  in  $M$  with  $D_i \cap K = A_i$ ,  $D_i$  and  $K$  transverse, and so that the normal vector field  $\nu$  to  $\partial D_i$  which is tangent to  $K$  extends to a normal vector field  $V$  to  $D_i$ . We can then replace the normal 1-disk

bundle to  $A_i$  in  $K$  by the normal 0-sphere bundle (the boundary of the 1-disk bundle determined by the vector field  $V$ ) of  $D_i$ , thereby reducing the genus of  $K$  by 1.

We can always imbed  $D_i$  transversely to  $K$  with  $\partial D_i = A_i$ . The obstruction to extending  $\nu$  over  $D_i$  is an integer  $x \in Z = \pi_1(SO(2))$ . The algebraic intersection of  $\text{int } D_i$  with  $K$ ,  $\int (\text{int } D_i, K)$ , is another integer  $d$ . Also we may spin  $D_i$  once around  $A_i$ , as in Figure 1, changing  $x$  to  $x \pm 1$  and  $d$  to  $d \pm 1$ , so that  $d - x$  is unchanged. By iteration we can make either  $d$  or  $x$  zero.

Let  $S$  be a smooth imbedded 2-sphere in  $M^4$ . Since  $K$  is characteristic,  $K \cdot S = S \cdot S \pmod{2}$ ;  $S \cdot S$  is the Euler class of the normal bundle. Under the connected sum  $D_1 \# S$ ,  $x$  changes to  $x + S \cdot S$  and  $d$  changes to  $d + K \cdot S$ . Thus  $d - x = d + x = d + x + K \cdot S + S \cdot S \pmod{2}$  is a possible  $Z_2$  obstruction to surgery on  $A_i$ . Associating  $d + x \pmod{2}$  to each  $A_i$ , we obtain a quadratic form  $\bar{q}: H_1(K; Z_2) \rightarrow Z_2$ . The Arf invariant of  $\bar{q}$ ,  $\phi(M, K)$ , is shown in Lemmas 3-5 to be an invariant of the pair  $(M, K)$  up to cobordism of such pairs.

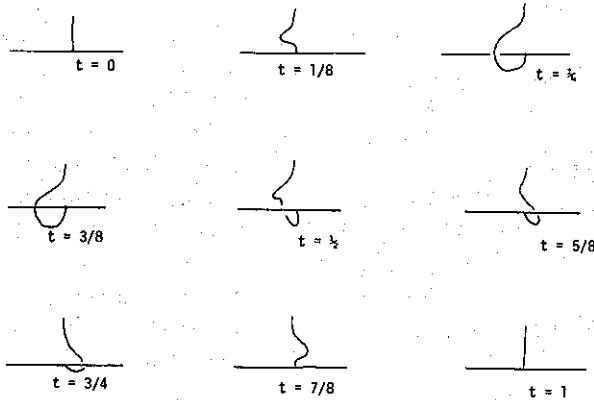


FIGURE 1

This is a movie of  $D_i$  spinning around  $A_i$ . In particular, we choose an interval of  $A_i$ , represented by the time  $t$  axis, so that at a fixed time  $A_i$  is represented by the center dot. The horizontal line is a slice of  $K$ , normal to  $A_i$ . The vertical line represents a collar of  $\partial D_i$  in  $D_i$ . The one point of intersection of  $K$  and  $\text{int } D_i$  occurs at time  $3/8$ .

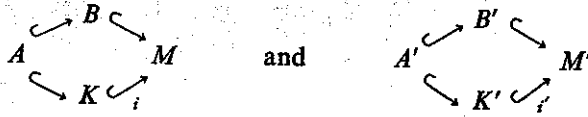
To be precise, let  $\Omega_4^{\text{char}}$  be the group (under disjoint union) of "characteristic" pairs  $(M^4, K^2)$  up to "characteristic" bordism, where  $M$  and  $K$  are closed and oriented and  $[K] \in H_2(M; Z) / \text{torsion}$  is characteristic. Two pairs  $(M, K)$  and  $(M', K')$  are characteristically bordant if there exist an oriented 5-manifold  $\bar{M}$  and an oriented 3-submanifold  $\bar{K}^3$  with  $[\bar{K}]_2$  dual to  $\omega_2(\tau_{\bar{M}})$  and  $\partial(\bar{M}, \bar{K}) = (M, K) \cup -(M', K')$ .

Thus we have  $\phi: \Omega_4^{\text{char}} \rightarrow Z_2$ .

We show (Lemmas 1 and 2) that  $\alpha: \Omega_4^{\text{char}} \rightarrow Z \oplus Z$  is an isomorphism where  $\alpha(M, K) = (\sigma(M), (K \cdot K - \sigma(M))/8)$ , and exhibit generators of  $\Omega_4^{\text{char}}$ . Only here do we need the fact that  $\Omega_4 = Z$ .

Finally we show that the  $Z_2$ -invariant  $\theta: \Omega_4^{\text{char}} \rightarrow Z_2$ , defined by  $\theta(M, K) = ((K \cdot K - \sigma(M))/8)(2)$ , is equal to  $\phi$ , by showing they are equal on the generators of  $\Omega_4^{\text{char}}$  (Lemma 6, Theorem 1).

family of circles,  $A$ , in  $K$ , and an imbedding of an unoriented surface,  $B$ , in  $M$ , which meets  $K$  normally at  $A = \partial B$  and transversally at isolated points of  $K - A$ . (Generally we will suppress inclusions from our notation.) We say that two char-2-ads,



are equivalent (written  $\sim$ ) if  $(M, K)$  and  $(M', K')$  are characteristically bordant, via  $(\bar{M}, \bar{K})$ , and  $A \cup A' \subset \partial(\bar{K})$  bounds an unoriented surface,  $\bar{A} \subset \bar{K}$ .

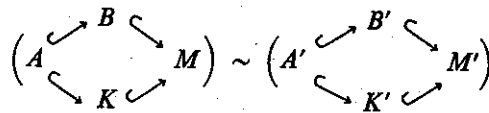
We now define a  $Z_2$  valued invariant,  $q$ , of characteristic 2-ads. Let  $\nu_{B \subset M}$  and  $\nu_{\partial B \subset K}$  denote the normal bundles. We have  $\nu_{B \subset M} |_{\partial B} = \nu_{\partial B \subset K} \oplus \xi$ . Since  $K$  is orientable the first summand is trivial; so  $\xi$  is also trivial (as  $\nu_{\partial B \subset K} \oplus \xi$  extends over  $B$ ). Let  $F$  be a framing  $\nu_{B \subset M} |_{\partial B}$ , restricting to a framing on each factor. Let  $w_2(\nu_{B \subset M} |_{\partial B}, F) = \chi_2 \in H^2(B, \partial B; Z_2)$  be the relative Stiefel-Whitney class.

( $\chi_2$  is the mod 2 reduction of the obstruction in  $H^2(B, \partial B, Z_{\text{twisted}})$  to extending  $\nu_{\partial B \subset K}$  to a section of  $\nu_{B \subset M}$ .) Let  $X = \chi_2[B, \partial]_2 = 0$  or 1. We define

$$q \left( \begin{array}{ccc} & B & \\ \swarrow & & \searrow \\ A & & M \\ \searrow & & \swarrow \\ & K & \end{array} \right) = \# \#_M(\text{int } B, K) + X \pmod{2}.$$

By  $\# \# ( , )$  we mean the number (mod 2) of transverse intersections.

LEMMA 3. If

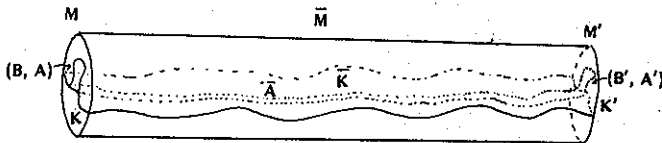


then

$$q \left( \begin{array}{ccc} & B & \\ \swarrow & & \searrow \\ A & & M \\ \searrow & & \swarrow \\ & K & \end{array} \right) = q \left( \begin{array}{ccc} & B' & \\ \swarrow & & \searrow \\ A' & & M' \\ \searrow & & \swarrow \\ & K' & \end{array} \right).$$

PROOF.  $Y = \bar{A} \cup B \cup B'$  is a closed unoriented surface.

Diagram:



Let  $h: Y \hookrightarrow M$  be the inclusion

$$Y \cdot \bar{K} = \# \#_M(B, K) + \# \#_M(\bar{A}, \bar{K}) + \# \#_{M'}(B', K').$$

The middle term is to be interpreted as the obstruction (mod 2) to extending  $\nu_{A \cup A' \subset B \cup B'}$  to a section of  $\nu_{\bar{K} \subset M/\bar{A}}$ .

On the other hand,

$$\begin{aligned} Y \cdot \bar{K} &= h^*(w_2(\tau(\bar{M}))) [Y] = w_2(\tau(\bar{M}) |_Y) [Y] \\ &= w_2(\nu_{Y \subset M}) + w_2(\tau(Y)) + w_1(\nu_{Y \subset M}) \cdot w_1(\tau(Y)). \end{aligned}$$

But  $\bar{M}$  is oriented, so  $0 = w_1(\tau\bar{M}/Y) = w_1(\nu_{Y\subset\bar{M}}) + w_1(\tau(Y))$ , so  $w_1(\nu_{Y\subset\bar{M}}) \cdot w_1(\tau(Y)) = w_1(\tau(Y))^2 = w_2(\tau(Y))$  by the Wu formula, so  $w_2(\tau(\bar{M})|_Y)[Y] = w_2(\nu_{Y\subset\bar{M}})[Y]$ . (Note that if  $Y$  is oriented and  $\bar{M}$  unoriented the preceding assertion is still true.)

Rounding corners,  $\nu_{Y\subset\bar{M}/\bar{A}} \parallel A'$  has a framing,  $\bar{F}$ , which extends  $F$  (restricted to  $A$  this is  $(F, \text{inward normal to } B)$ ). We can use this framing to break  $w_2(\nu_{Y\subset\bar{M}})$  up into three relative Stiefel-Whitney classes

$$w_2(\nu_{Y\subset\bar{M}})[Y] = w_2(\nu_{B\subset M}, F)[B, \partial] + w_2(\nu_{A\subset\bar{M}}, \bar{F})[\bar{A}, \partial] + w_2(\nu_{B'\subset M'}, F')[B', \partial']$$

$\parallel \text{ def}$   $\parallel \text{ def}$   
 $x$   $x'$

$$w_2(\nu_{A\subset\bar{M}}, \bar{F}) = w_2(\nu_{\bar{K}\subset\bar{M}}|_A, \bar{F}_{2,3}) + w_1(\nu_{\bar{K}\subset\bar{M}}, \bar{F}_{2,3}) \cdot w_1(\nu_{A\subset\bar{K}}, F_1)$$

where  $\bar{F}_{2,3}$  (and  $\bar{F}_1$ ) denote the last two (and the first) vectors of  $\bar{F}$ . Let  $w_1(\nu_{\bar{K}\subset\bar{M}}, F_{2,3}) = u$  and  $w_1(\nu_{A\subset\bar{K}}, F_1) = v$ .

$$u + v = w_1(\nu_{\bar{A}\subset\bar{M}}, \bar{F}) = w_1(\tau(\bar{A}), \text{framing of } \tau(A \parallel A'))$$

(because  $\bar{M}$  is orientable). As in the closed case,  $(u + v) \cup x = \text{Sq}^1 x = x^2$  for all  $x \in H^1(\bar{A}, \partial; \mathbb{Z}_2)$ .  $v^2 = (u + v) \cup v = uv + v^2$ , so  $uv = 0$ . Therefore,  $w_2(\nu_{Y\subset\bar{M}})[Y] = + \prod_M(\bar{A}, \bar{K})$ , so

$$\prod_M(B, K) + x = \prod_{M'}(B', K') + x' \pmod{2}. \quad \square$$

*Note.* The above result holds if the hypothesis that  $(\bar{M}; M, M')$  is oriented is replaced by:  $Y$  and  $\bar{K}$  are oriented.

**COROLLARY 1.** *If  $(M, K)$  is a characteristic pair,  $q$  determines a well-defined function,  $\bar{q}: H_1(K; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ .*

**PROOF.** We kill  $H_1(M; \mathbb{Z}_2)$  with a finite number of framed 1-surgeries in the complement of  $K$  (call the trace  $N$ ). To  $\beta \in H_1(K; \mathbb{Z}_2)$  we associate an unoriented surface  $(B, \partial) \subset (M, K)$  with  $[\partial B] = \beta$ . Define

$$\bar{q}(\beta) = q \left( \begin{array}{ccc} & B & \\ \swarrow & & \searrow \\ \partial B & & M \\ \searrow & & \swarrow \\ & K & \end{array} \right).$$

To check that this procedure is well defined let  $\tilde{N}$  be an alternative trace. Let

$$(\bar{M}, \bar{K}) = (N, K \times I) / \bigcup_{M, K \times 0} (\tilde{N}, K \times I), \quad \partial(\bar{M}, \bar{K}) = (M_0, K_0) \cup -(M_1, K_1).$$

Let  $(B_0, \partial)$  and  $(B_1, \partial)$  denote unoriented surfaces in  $(M_0, K_0)$  and  $(M_1, K_1)$  with  $[\partial B_0] = \beta \in H_1(K_0; \mathbb{Z}_2)$ ,  $[\partial B_1] = \beta \in H_1(K_1; \mathbb{Z}_2)$ . There is an unoriented bordism  $\bar{A} \subset \bar{K}$  with  $\partial \bar{A} = \partial B_0 \cup \partial B_1$ , so by definition

$$\left( \begin{array}{ccc} & B_0 & \\ \swarrow & & \searrow \\ \partial B_0 & & M_0 \\ \searrow & & \swarrow \\ & K_0 & \end{array} \right) \sim \left( \begin{array}{ccc} & B_1 & \\ \swarrow & & \searrow \\ \partial B_1 & & M_1 \\ \searrow & & \swarrow \\ & K_1 & \end{array} \right).$$

By Lemma 3,  $\bar{q}$  is well defined.

**LEMMA 4.**  $\bar{q}: H_1(K; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$  is quadratic, i.e.  $q(\varepsilon + \delta) - q(\varepsilon) - q(\delta) = \varepsilon \cdot \delta$  for all  $\varepsilon, \delta \in H_1(K; \mathbb{Z}_2)$ .

PROOF. Let  $A_\varepsilon$  and  $A_\delta$  represent  $\varepsilon$  and  $\delta$ . Let  $B_\varepsilon$  and  $B_\delta$  be as before. Suppose  $A_\varepsilon$  and  $A_\delta$  intersect transversally at one point  $p$ . Let  $A_\gamma$  be the connected sum (representing  $\varepsilon + \delta$ ) as in Figure 2. Piecewise linearly, we get  $B_\gamma$  from  $B_\varepsilon \cup B_\delta \cup T_1 \cup T_2$  where  $T_1$  and  $T_2$  are two curved triangles as shaded in Figure 2; note that  $\partial B_\gamma = A_\gamma$ . The normal vector fields on  $B_\varepsilon$  and  $B_\delta$  must be extended to the new part of  $A_\gamma$  as drawn. Consider the boundary (a circle) of a neighborhood of  $B$  in  $B_\gamma$  and push it off itself using the vector field. The two circles link, indicating that the obstruction to extending the vector field over the neighborhood is one. We have verified that  $\chi_{\varepsilon+\delta} - \chi_\varepsilon - \chi_\delta = \varepsilon \cdot \delta = 1$ .

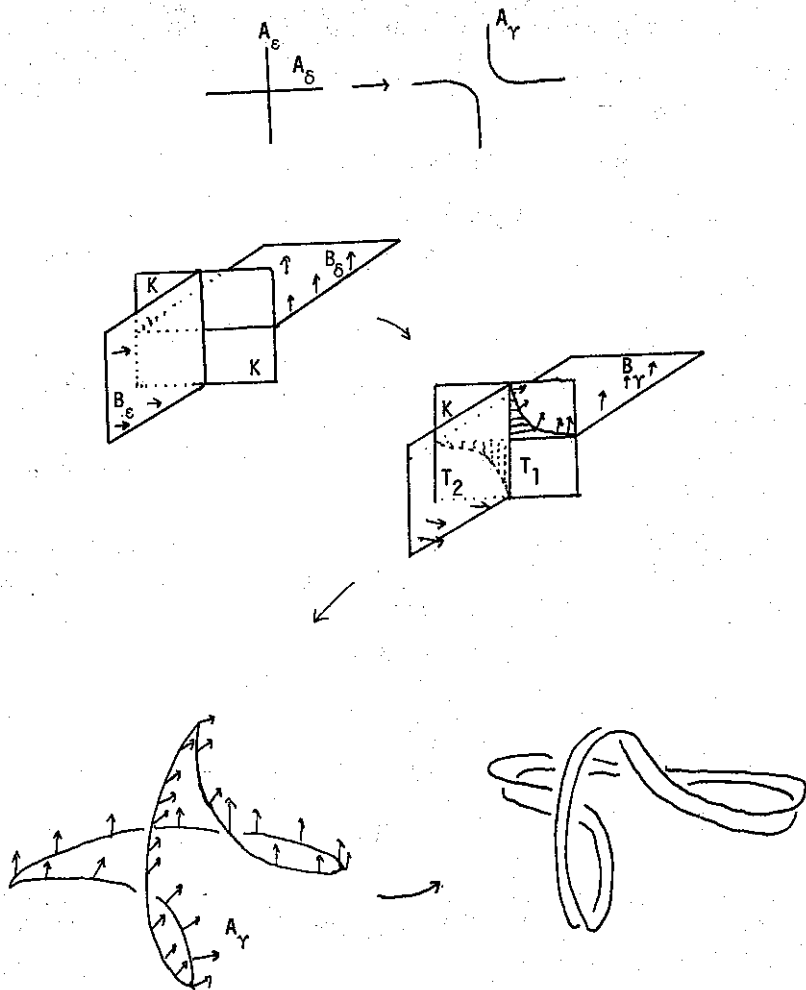


FIGURE 2

If  $\text{int } B_\varepsilon \cap \text{int } B_\delta \neq \emptyset$ , then we may push these double points of  $B_\gamma$  off the boundary, adding two points to  $\text{int } B_\gamma \cap K$ . Thus  $\prod(B_\varepsilon, K) + \prod(B_\delta, K) = \prod(B_\gamma, K) \pmod{2}$ . We have shown in this special case that  $q(\varepsilon + \delta) - q(\varepsilon) - q(\delta) = \varepsilon \cdot \delta$ ; the other (easier) cases are left to the reader.  $\square$

Let  $\phi(M, K)$  be the Arf invariant of  $\bar{q}: H_1(K; Z_2) \rightarrow Z_2$ . (See the Appendix of [R-S] for a short presentation of the Arf invariant.)

LEMMA 5.  $\phi$  determines a well-defined homomorphism:  $\Omega_4^{\text{char}} \rightarrow \phi Z_2$ .

PROOF. Assume that  $(M, K)$  and  $(M', K')$  are characteristically bordant via  $(\bar{M}, \bar{K})$ . For the usual reason

$$\begin{aligned} \dim(V = \text{Ker}(H_1(K \cup K', Z_2) \rightarrow H_1(\bar{K}; Z_2))) \\ = \frac{1}{2} \dim(H_1(\partial \bar{K}; Z_2)). \end{aligned}$$

The intersection pairing on  $H_1(K \cup K'; Z_2)$  is identically zero when restricted to  $V$ .

$\bar{q}(K \cup K') = \bar{q}(K) \oplus \bar{q}(K')$ . If viewed properly, Lemma 3 implies that  $\bar{q}(K \cup K')|_V$  is identically zero. (If  $A \in V$  and  $\partial A = A$ , then  $A \in H_2(\bar{K}, K \cup K'; Z_2)$ ; one should regard  $\partial(\bar{M}, \bar{K}, A)$  as  $(M \cup -M', K \cup -K', A) \cup (\phi, \phi, \phi)$ .) Hence  $\phi(\bar{q}(K \cup K')) = 0$ , so  $\phi(\bar{q}(K)) = \phi(\bar{q}(K'))$ .  $\square$

LEMMA 6.  $\phi(CP^2, CP^1) = 0$  and  $\phi(CP^2 \# \overline{CP^2}, 3\gamma \# \overline{CP^1}) = 1$ .

PROOF.  $CP^1$  is  $S^2$ , so  $H_1$  is zero and thus  $\phi(CP^2, CP^1) = 0$ .

$\phi(CP^2 \# \overline{CP^2}, 3\gamma \# \overline{CP^1}) = \phi(CP^2, 3\gamma)$ . In  $CP^2$ ,  $3\gamma$  is represented by any cubic; it is convenient to pick  $x^3 = y^2z$ . In the coordinate charts  $x = 1$  or  $y = 1$ , the solution is nonsingular, but for  $z = 1$ ,  $y^2 = x^3$  is the cone on the (2, 3)-torus knot (= trefoil knot). So  $3\gamma$  is represented by a smooth 2-sphere except for the cone point. If  $B^4$  is centered at the cone point, replace the cone in  $B^4$  by the Seifert surface (Figure 3) of the trefoil knot in  $S^3 = \partial B^4$ , obtaining a 2-torus  $T^2$  as a representative of  $3\gamma$ . The circles  $A_1, A_2$  in the Seifert surface generate  $H_1(T^2; Z)$ . Each  $A_i$  is a trivial knot and bounds a 2-ball  $B_i$  in  $B^4$ . The obstruction to extending  $\nu_{A_i; T^2}$  to a section of  $\nu_{B_i; B^4}$  is the linking number of  $A_i$  and  $s(A_i)$  in  $S^3$ , where  $s$  is a nonzero section of  $\nu_{A_i; T^2}$ .  $L(A_i, s(A_i)) = \pm 1$ . Since  $\int (\text{int } B_i, T^2) = 0$ , it follows that  $q(A_i) = 1$ . Thus  $\phi(CP^2, T^2) = 1$  (see Appendix [R-S]).  $\square$

Let  $\theta: \Omega_4^{\text{char}} \rightarrow Z_2$  be the homomorphism given by  $\theta(M, K) = (K \cdot K - \sigma(M))/8 \pmod{2}$ .

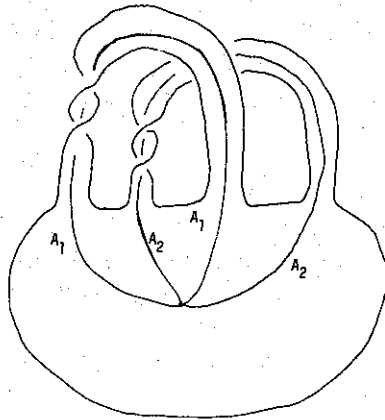


FIGURE 3

**THEOREM 1.**  $\theta(M, K) = \phi(M, K)$ .

**PROOF.** Since both  $\theta$  and  $\phi$  are homomorphisms (Lemma 5), it is sufficient to check the equality on the generators of  $\Omega_4^{\text{char}}$ ,  $(CP^2, CP^1)$  and  $(CP^2 \# \overline{CP^2}, 3\gamma \# \overline{CP^1})$ . But we have seen (Lemmas 2 and 6), that  $\theta$  and  $\phi$  are both zero on the first generator, as on the second.  $\square$

**COROLLARY 2 [K-M, THEOREM 1]).** *If  $(M, K)$  is a characteristic pair, and  $K$  is a 2-sphere, then  $\theta(M, K) = 0$ .*

**COROLLARY 3 (ROCHLIN'S THEOREM [R<sub>2</sub>]).** *If  $(M, \phi)$  is a characteristic pair, then  $\theta(M, \phi) = 0$ .*

**PROOFS.**  $H_1(K, Z_2) = 0$  for  $K = S^2$  or  $\emptyset$ , therefore  $\phi(M, K) = 0$ . By Theorem 1,  $\theta(M, K) = 0$ .

*Note.* Suppose  $K \hookrightarrow M$  is a PL imbedding with nonlocally flat points  $p_1, \dots, p_r$  at which  $K$  is the cone on knots  $S_1, \dots, S_r$ . Let  $\text{Arf}(S_i)$  be the Arf invariant of  $S_i$  (see [R<sub>2</sub>]). Then if we define

$$\phi(M, K) = \phi(\bar{q}(H_1(K; Z_2))) + \sum_{i=1}^r \text{Arf}(S_i) \pmod{2},$$

we may still conclude that  $\theta(M, K) = \phi(M, K)$ .

We now show that "stably"  $\phi$  is the only obstruction to surgering  $K$  to a 2-sphere. Let  $M_s = M \#_s(S^2 \times S^2)$ , and let  $j_s$  be the composition

$$H_2(M; Z) \xrightarrow{\text{inc}_*^{-1}} H_2(M - D^4; Z) \xrightarrow{\text{inc}_*} H_2(M_s; Z).$$

**THEOREM 2.** *Suppose  $\pi_1(M^4) = 0$ . Then  $\phi(M, K) = 0$  iff for some  $s, \exists (M_s, K')$  such that  $K'$  is a 2-sphere and  $i_*[K'] = j(i_*[K])$ .*

*Note.* Larry Taylor has independently obtained this result.

**PROOF.** The if direction follows from Theorem 1 [K-M] and our Theorem 1.

The argument for the only if direction will be quite liberal with copies of  $S^2 \times S^2$ . Since  $\phi(M, K) = 0$ , there is a subspace  $V \subset H_1(K; Z_2)$  such that (1)  $\dim(V) = \frac{1}{2} \dim(H_1(K; Z_2))$ , (2)  $v_1 \cdot v_2 = 0$  for  $v_1, v_2 \in V$ , and (3)  $q(v) = 0$  for  $v \in V$ . Let  $A_1, \dots, A_p$  be circles disjointly imbedded in  $K$  representing a basis for  $V$ . Let  $\bar{A}_1, \dots, \bar{A}_p$  denote copies of  $A_1, \dots, A_p$  pushed off in a normal direction to  $K$  so that the linking numbers  $L(K, \bar{A}_i) = 0$ . Since  $K$  is characteristic,  $\tau(M) | \bar{A}_i$  is canonically framed. Since  $\pi_1(M) = 0$ , framed surgery on  $\{\bar{A}_1, \dots, \bar{A}_p\}$  replaces  $M$  with  $M_p$ . In  $M_p$ , there are disjointly imbedded 2-disks,  $B_1, \dots, B_p$  with  $\partial B_i = A_i$  and  $B_i \cap K = A_i$ . There is an Euler class obstruction  $\chi_i \in H^2(B_i, \partial; Z)$  to extending  $\nu_{A_i \hookrightarrow K}$  to a section of  $\nu_{B_i \hookrightarrow M_p}$ , but  $0 = q[A_i] \equiv \chi_i[B_i, \partial] \pmod{2}$ .

Consider the diagonal imbedding  $\Delta: S^2 \hookrightarrow S^2 \times S^2$ .  $\chi(\nu_{S^2 \times S^2 \times S^2})[S^2] = 2 \cdot \chi_i[B_i, \partial]$  may be altered by  $\pm 2$  by taking a connected sum of pairs  $(M_p, B_i) \# (S^2 \times S^2, \pm \Delta(S^2))$ . In this way it is possible to alter  $M_p$  and  $B_i$  so that  $\chi_i[B_i, \partial]$  becomes zero and  $M_p$  becomes  $M_{p+r}$ .  $\chi_i$  is the only obstruction to ambient surgery on  $K$  along  $B_i$ . The result of surgery on  $B_1, \dots, B_p$  is a smooth submanifold,  $K'$ , with  $H_1(K'; Z_2) = 0$ ; so  $K'$  is a 2-sphere. To verify  $i_*[K'] = j(i_*[K])$ , recall that  $L(K, \bar{A}_i) = 0$ .  $\square$

*Note.* A similar argument shows: If  $\pi_1(M^4) = 0$  and  $\xi \in H_2(M^4; Z)$ , then for some  $s, j_s(\xi)$  is represented by an imbedded torus.



2. In §1,  $M, \bar{M}, K, \bar{K}$  were taken to be oriented, while  $B$  and  $Y$  were unoriented (and possibly unorientable). Orienting  $M, \bar{M}, K, \bar{K}$  was convenient in that it enabled us to calculate the corresponding bordism group,  $\Omega_4^{\text{char}}$ . We chose  $B$  (and  $Y$ ) unoriented because we were defining a quadratic form on  $H_1(K; Z_2)$  and orientations would be superfluous.

In the following application we unorient  $K$ .

**THEOREM 3.** *Although  $(CP^2, \gamma) - (CP^2, 3\gamma) = \partial(CP^2 \times I, \text{unoriented simplicial 3-chain})$ ,  $(CP^2, \gamma) - (CP^2, 3\gamma)$  cannot be written as  $\partial(CP \times I, \text{immersed unoriented 3-manifold})$ .*

**PROOF.** This is an exercise from the proof of Lemma 3.  $\phi(CP^2, \gamma) \neq \phi(CP^2, 3\gamma)$ , but a manifold bordism (even an immersed one)  $\bar{K}$  from  $\gamma$  to  $3\gamma$  would force  $\phi$  to assume equal values at each end.  $\square$

**REMARK 1.** There is an old and elegant procedure for approximating a  $Z_2$ -simplicial-cycle of dimension 2 in a triangulated 3-manifold by an unoriented, triangulated submanifold. This procedure generalizes for 2-dimensional cycles in any manifold, and for  $(n - 1)$ -dimensional cycles in any  $n$ -manifold. The first open case would be: Can you approximate a 3-dimensional  $Z_2$ -simplicial-chain in a triangulated 5-manifold by an unoriented, triangulated submanifold? Theorem 3 provides an example (in the relative case) where the answer is no.

**REMARK 2.** A key calculation occurs in the proof of Lemma 3 in which we show

$$h^*w_2(\tau(\bar{M}))[Y] = w_2(\nu_{Y \subset h\bar{M}})[Y].$$

We observe that this equality still holds if we replace the assumption (1)  $\bar{M}$  is oriented and  $Y$  is unoriented, with (2)  $\bar{M}$  is unoriented and  $Y$  is oriented.

Suppose  $M, M'$  are unoriented with  $K \subset M$  and  $K' \subset M'$  oriented surfaces dual to  $w_2(\tau(M))$  and  $w_2(\tau(M'))$ , respectively, and that  $\text{image}(H_1(K; Z)) \subset H_1(M; Z)$  and  $\text{image}(H_1(K'; Z)) \subset H_1(M'; Z)$  are zero. Then if  $\bar{M}$  is unoriented and  $\bar{K}$  is oriented, with  $\bar{K}$  dual to  $w_2(\tau(\bar{M}))$ ,  $\bar{q}(H_1(K; Z_2))$  and  $\bar{q}(H_1(K'; Z_2))$  will be defined and

$$\phi(M, K) = \text{Arf}(\bar{q}(H_1(K; Z_2))) = \text{Arf}(\bar{q}(H_1(K'; Z_2))) = \phi(M, K')$$

This is because, with the above assumptions, we will be able to choose  $B$  and  $\bar{A}$  (see notation preceding Lemma 3) to be oriented compatibly so that  $Y$  also is oriented. Now the proof of Lemma 5 goes through using our second set of orientability assumptions. This leads to the following nonimbedding theorem.

**THEOREM 4.** *Let  $M$  be an orientable PL 4-manifold and let  $\alpha \in H_2(M; Z)$  satisfy  $(\alpha \cdot \alpha - \sigma(M))/8 \equiv 1 \pmod{2}$ . Let  $N$  be an unorientable PL 4-manifold with  $w_2(\tau(N)) = 0$  (or it is sufficient to assume  $w_2(\tau(N))$  is represented by a smoothly imbedded oriented surface  $A_2 \subset^{\text{inc}} N$  with  $\text{inc}_*(H_1(A; Z)) = 0 \in H_1(N; Z)$  and  $\phi(N, A) = 0$ ; note that  $\phi$  is defined for the pair despite the fact that  $N$  is unoriented). Then  $\alpha' = \text{image}(\alpha) \in H_2(M \# N; Z)$  is not represented by a smooth imbedding  $S^2 \subset^h M \# N$ .*

**LEMMA 7.** *A  $Z$ -oriented simplicial cycle,  $\beta^3$ , of dimension 3 in an unoriented triangulated 5-manifold may be approximated by a  $Z$ -oriented imbedded, triangulated submanifold.*

PROOF. By a modification in a spindle neighborhood of the 2-simplexes of  $\beta^3$  (see Figure 4), we may assume  $\beta^3$  is a manifold away from its 1 skeleton.

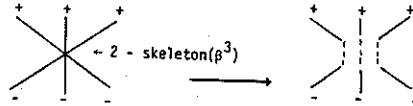


FIGURE 4

Let  $V$  be a tubular neighborhood of the 1-skeleton  $\partial V = j(S^1 \times S^3) \# K(S^1 \times S^3)$  ( $S^1 \times S^3$  denotes the twisted product).  $\partial V \cap \beta^3 \stackrel{\text{def}}{=} L$  is an oriented surface imbedded in  $\partial V$  (oriented, because  $L$  is imbedded with trivial normal bundle in the  $(\beta^3 - 1)$ -skeleton). We will show that there is an oriented 3-manifold  $\bar{L} \subset V$  with  $\partial \bar{L} = L$ .

There are  $j + k$  normal 4-disks  $(D^4, \partial)_i \subset (V, \partial)$  such that  $V_j$  "cut" along  $\bigcup_i D^4_i$  is a closed 4-disk,  $X$ . For each  $i$ , there are 2-copies of  $D^4_i$ ,  $D^4_{i,1}$  and  $D^4_{i,2}$  included in  $\partial X$ . We may assume  $L$  meets each  $D_i$  in a link  $L_i$ . Because  $L$  was oriented, the associated links  $L_{i,1} \subset \partial D^4_{i,1}$  and  $L_{i,2} \subset \partial D^4_{i,2}$  are oriented oppositely (comparing orientations by regluing  $D^4_{i,1}$  and  $D^4_{i,2}$  to form  $D^4_i$ ). Let  $J_i \subset D^4_i$  be an orientable surface with  $\partial J_i = L_i$ . Again let  $J_{i,1} \subset D^4_{i,1}$  and  $J_{i,2} \subset D^4_{i,2}$  be the corresponding copies of  $J_i$  in  $\partial X$  oriented so that  $\partial(J_{i,1}) = -L_{i,1}$  and  $\partial(J_{i,2}) = -L_{i,2}$ .

$$W \stackrel{\text{def}}{=} \left( L - \bigcup_i L_i \right) \cup_i J_{i,1} \cup_i J_{i,2} \subset \partial X$$

is an oriented surface. Let  $Z \subset X$  be an oriented 3-manifold with  $\partial Z = W$ . Because the orientations on  $L_{i,1}$  and  $L_{i,2}$  are opposite, so are the orientations on  $J_{i,1}$  and  $J_{i,2}$ . As a result, if we reglue  $X$  to form  $V$ , the image of  $Z$  (which we will call  $\bar{L}$ ) is an orientable 3-manifold contained in  $V$  with  $\partial \bar{L} = L$ .

Now we can approximate  $\beta^3$  by  $(\beta^3 - V) \cup -\bar{L}$ , an oriented submanifold.  $\square$

PROOF OF THEOREM 4. We only consider the case:  $w_2(\tau(N)) = 0$ . (As an example  $N$  might be  $S^1 \times S^3$ .) By Theorem 1,  $\alpha$  is represented by a smoothly imbedded surface  $K \subset M$  with  $\phi(M, K) = 1$ . If we consider  $K \subset M \# N$  representing  $\alpha'$ ,  $\phi(M \# N, K)$  is defined and equal to 1. If  $\alpha'$  were represented by a smooth imbedding,  $S^2 \subset M \# N$ ,  $\phi(M \# N, S^2)$  would be defined and equal 0.

But by a relative form of Lemma 7, there is a smooth oriented 3-manifold,  $T$ , with  $\partial T = K - S^2$  and a smooth imbedding

$$(T; K, S^2) \xrightarrow{j} (M \# N \times I, N \# N \times 0, M \# N \times 1)$$

restricting to the imbeddings  $i' \times 0$  and  $h \times 1$  on the boundary.

Remark 2 shows that the existence of  $j: T \subset M \# N \times I$  implies  $\phi(M \# N, K) = \phi(M \# N, S^2)$ , contradicting the above.  $\square$

REFERENCES

[K-M] M. A. Kervaire and J. Milnor, *On 2-spheres in 4-manifolds*, Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 1651-1657. MR 24 #A2968.

- [K-S] R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*, Bull. Amer. Math. Soc. **75** (1969), 742-749. MR **39** #3500.
- [M-K] J. Milnor and M. A. Kervaire, *Bernoulli numbers, homotopy groups, and a theorem of Rohlin*, Proc. Internat. Cong. Math. (Edinburgh, 1958), Cambridge Univ. Press, London and New York, 1960, pp. 454-458. MR **22** #12531.
- [M-H] J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Springer-Verlag, Berlin and New York, 1973.
- [M<sub>1</sub>] J. Milnor, *On simply connected 4-manifolds*, Internat. Sympos. on Algebraic Topology, Univ. Nacional Autonoma, Mexico City, 1958, pp. 122-128. MR **21** #2240.
- [M<sub>2</sub>] ———, *Differential topology*, Lectures on Modern Math., vol. 2, Wiley, New York, 1964, pp. 165-185. MR **31** #2731.
- [R<sub>1</sub>] R. A. Robertello, *An invariant of knot cobordism*, Comm. Pure Appl. Math. **18** (1965), 543-555. MR **32** #447.
- [R<sub>2</sub>] V. A. Rochlin, *New results in the theory of 4-dimensional manifolds*, Dokl. Akad. Nauk SSSR **84** (1952), 221-224. (Russian)
- [R<sub>3</sub>] ———, *Proof of Gudkov's hypothesis*, Functional Anal. Appl. **6** (1972), 136-138.
- [R-S] C. P. Rourke and D. P. Sullivan, *On the Kervaire obstruction*, Ann. of Math. (2) **94** (1971), 397-413. MR **46** #4546.
- [S<sub>1</sub>] M. G. Scharlemann, *Three codimension four transversality theories* (to appear).
- [S<sub>2</sub>] L. C. Siebenmann, *Disruption of low-dimensional handlebody theory by Rohlin's theorem*, Topology of Manifolds, Markham, Chicago, Ill., 1970, pp. 57-76. MR **42** #6836.
- [W<sub>1</sub>] C. T. C. Wall, *On simply connected 4-manifolds*, J. London Math. Soc. **39** (1964), 141-149. MR **29** #627.
- [W<sub>2</sub>] ———, *On the orthogonal groups of unimodular quadratic forms*, Math. Ann. **147** (1962), 328-338.
- [W<sub>3</sub>] ———, *Diffeomorphisms of 4-manifolds*, J. London Math. Soc. **39** (1964), 131-140. MR **29** #626.
- [H-H] F. Hirzebruch and H. Hopf, *Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten*, Math. Ann. **136** (1956), 156-172. MR **20** #7272.

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