SEMI-CLASSICAL ESTIMATES ON THE SCATTERING DETERMINANT

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ABSTRACT. We present a unifying framework for the study of Breit-Wigner formulæ, trace formulæ for resonances and asymptotics for resonances of bottles. Our approach is based on semi-classical estimates on the scattering determinant and on some complex function theory.

1. INTRODUCTION

The purpose of this paper is to present a semi-classical estimate on the scattering determinant and its applications. We work in the technically simplest setting of compactly supported perturbations of $-h^2\Delta$ on $\mathbb{R}^n$, and concentrate on presenting a complex analytic framework for a general study of Breit-Wigner formulæ, trace formulæ for resonances, and asymptotics for resonances of bottles. This allows us to make the paper essentially self-contained.

The scattering matrix constitutes a mathematical model for the data obtained in a scattering experiment or a chemical reaction. Resonances model states which live for certain times but eventually decay – the real part of a resonance gives the rest energy of the state and its imaginary part the rate of its decay. A basic intuition connects resonances and scattering matrices via the time delay operator or the Breit-Wigner approximation: the long living states should contribute peaks in the derivatives of expressions obtained from the scattering matrix (i.e. expressions which at least in principle are obtained from scattering data). Mathematically this connection is expressed most simply through the fact that the resonances are the poles of the meromorphic continuation of the resolvent. We refer to [37] for a basic introduction to the theory of resonances and for references.

The scattering determinant, that is the determinant of the scattering matrix is a natural mathematical object to study. It is closely related to the *scattering phase* which replaces the counting function of eigenvalues for problems on non-compact domains – see [16] for an introduction and references. The connection between the asymptotics of the scattering phase and resonances was first explored by Melrose [15] who proved the Weyl law for the scattering phase using bounds on the resonances. The further connections between resonances and the scattering phase were then investigated by Guillopé and the authors [11],[18],[35] and the present paper is a semi-classical continuation of these works. We are however using, rather than proving, asymptotics of the scattering phase, as established in the generality we consider by Christiansen [6] and Bruneau and the first author [4], who followed among other things, the ideas of Robert [20].

Related problems have been recently studied by Bony [2] and Bony-Sjöstrand [3] without a direct appeal to scattering theory, but following Sjöstrand's work on local trace formulæ [22],[23]. That approach allows obtaining some of the applications directly and in greater generality. For instance, it is shown in [2] that for a large class of perturbations, if $\lambda > 0$ is a non-critical energy level, and $Ch < \delta < 1/C$, then we have

$$\# \{ z \; : \; z \in \operatorname{Res}(P(h)) \, , \; |z - \lambda| < \delta \} = \mathcal{O}(\delta) h^{-n} \, ,$$

where $\operatorname{Res}(P(h))$ denotes the set of resonances. This provides a fine upper bound on resonances in small sets, generalizing [18, Proposition 2] and Lemma 6.1 below.

The basic estimate on the scattering determinant which follows directly from adapting the proofs in the classical case [18],[34] is:

$$|\det S(z,h)| \leq e^{Ch^{-n}}\,, \quad \operatorname{Im} z \geq 0\,, \quad z \in \Omega \Subset \{\operatorname{Re} z > 0\}\,, \tag{1.1}$$

where $S(z,h)$ is the scattering matrix and where $\operatorname{Im} z > 0$ is the "physical half plane" (that is the half plane where $S(z,h)$ is holomorphic). It is interesting and useful that the constant in (1.1) depends only on the size of the support of the perturbation *not* on its properties.

The difficulty in using (1.1) lies in the need for a lower bound

$$\forall\, 0 < h \leq h_0 \ \exists z_0 = z_0(h) \in \Omega\,,\ \operatorname{Im} z_0 > \delta, \quad |\det S(z_0,h)| \geq e^{-Ch^{-n}}\,. \tag{1.2}$$

Here z_0 clearly can depend on h but $\delta > 0$ is fixed.

When we can find z_0's such that (1.2) holds with $\Omega = (a,b) + i(-c,c)\,,\ \ 0 < a < b\,,\ c > 0$ we can factorize $\det S(z,h)$:

$$\begin{gathered}\det S(z,h) = e^{g(z,h)} \frac{\overline{P(\bar z,h)}}{P(z,h)}\,, \quad |g(z,h)| \leq C(N(h) + h^{-n}) + C, \ \ z \in \Omega\,,\\ P(z,h) = \prod_{w \in \operatorname{Res}\,(P(h)) \cap \Omega_\epsilon} (z - w)\,, \quad \Omega_\epsilon = \Omega + D(0,\epsilon)\,,\\ N(h) = \#\left(\operatorname{Res}\,(P(h)) \cap \Omega_\epsilon\right)\,,\end{gathered} \tag{1.3}$$

where we denoted the set of resonances of $P(h)$ by Res $P(h)$.

In particular, this shows that we have an improved estimates $|\det S(z,h)| \leq C\exp(C \operatorname{Im} z h^{-n})$, $\operatorname{Im} z \geq 0$. The factorization is essentially equivalent, via the Birman-Krein formula, to the local trace formula of Sjöstrand [22],[23], just as the earlier global formulæ of Bardos-Guillot-Ralston, Melrose and Sjöstrand-Zworski, were equivalent to global factorization of the scattering determinant [11],[35] – see Sect.5.

Finer analysis under stronger spectral assumptions leads to factorization in sets of size h and that gives for $0 < \delta < h/C$ the semi-classical Breit-Wigner formula:

$$\begin{gathered}\sigma(\lambda+\delta,h) - \sigma(\lambda-\delta,h) = \sum_{\substack{|z-\lambda|<h\\ z\in \operatorname{Res}\,(P(h))}} \omega_{\mathbb{C}_-}(z,[\lambda-\delta,\lambda+\delta]) + \mathcal{O}(\delta)h^{-n}\,,\\ \sigma(\lambda,h) = \frac{1}{2\pi i}\log\det S(\lambda,h)\,, \quad \sigma(0,h) = 0\,,\\ \omega_{\mathbb{C}_-}(z,E) = -\frac{1}{\pi}\int_E \frac{\operatorname{Im} z}{|z-t|^2}dt\,, \quad E \subset \mathbb{R} = \partial\mathbb{C}_-\,,\end{gathered}$$

which generalizes the classical formula from [18] where references on earlier rigorous work on the Breit-Wigner approximation can also be found. As in [18], the Breit-Wigner formula can be used to relate the distribution of resonances close to the real axis to the properties of the scattering phase, and the applications given in [18, Sect.6] can be adapted to the semi-classical setting.

When we exploit the fact that the constant in (1.1) does not depend on the perturbation (only on the radius of its support) we obtain uniform bounds on the number of resonances away from the real axis, and asymptotics of resonances for bottles, improving, in our setting, results of Sjöstrand [23] and including earlier results of Vodev [33] – see Sect.7.

We now discuss the condition (1.2). In the generality we work in, that is without considering special nature of the perturbation, we can only obtain (1.2) when $n \geq 5$. In that case it follows from a strange observation on a resonances free region close to 0 (Proposition 2.3).

For all n we can obtain a weaker estimate (Lemma 4.5), in which the constants depend on the perturbation – that estimate is sufficient for all applications except for the study of fully semi-classical bottles (Theorem 4).

When the dependence on h is homogeneous, $P(h) = h^2 P$ (that is, we work in the high energy régime), (1.2) always holds but h_0 there depends on the perturbation (while it does not when $n \geq 5$). It is an interesting problem if (1.2) holds in lower dimensions.

Finally, we should stress that the assumption on the support of the perturbation was used *only* in the proof of the estimate (1.1) (see Lemma 4.3), and in the proof of the strong version of (1.2) for dimensions greater than 4 (see Lemma 4.6). The remaining arguments are either purely complex-analytic or depend on asymptotics of the scattering phase known in great generality. We expect that using the results of Gérard-Martinez [8] on the meromorphic continuation of the scattering matrix, (1.1) can be proved for short range perturbations dilation analytic near infinity, and for relative scattering matrices for long range perturbations.

Except for the definition of the scattering matrix, the Birman-Krein formula and the asymptotics of the scattering phase, which are all quoted from other works, the paper is essentially self-contained. We denote by C a large constant the value of which may change from line to line.
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2. Review of scattering theory

We start by recalling the framework of "black box" scattering from [24], adapted now to the semi-classical setting. This framework allows a general treatment of resonance and scattering phenomena without going into the particular nature of the perturbation in a compact set.

Thus we consider a complex Hilbert space $\mathcal{H}$ with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)),$$

where $R_0 > 0$ is fixed and $B(x, R) = \{y \in \mathbb{R}^n : |x - y| < R\}$. We assume that $P(h)$, $0 < h \leq 1$, is a family of self-adjoint operators, $P(h) : \mathcal{H} \longrightarrow \mathcal{H}$, with domain $\mathcal{D} \subset \mathcal{H}$, satisfying the following conditions:

$$\begin{gathered} 1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0, R_0)), \\ 1\!\!1_{\mathbb{R}^n \setminus B(0,R_0)} P(h) = -h^2 \Delta|_{\mathbb{R}^n \setminus B(0,R_0)}, \\ (P(h) + i)^{-1} \text{ is compact}, \\ P(h) \geq -C, \ C \geq 0 \,. \end{gathered}$$

For convenience[1] we will also add the *reality* condition:

$$\overline{Pu} = P\bar{u}\,,$$

which is satisfied in interesting situations.

Under the above conditions, it is known that the resolvent $R(z, h) = (P(h) - z)^{-1} : \mathcal{H} \longrightarrow \mathcal{D}$ continues meromorphically from $\{z : \operatorname{Im} z > 0\}$, through $(0, \infty)$, to the double cover of $\mathbb{C}$ when n is odd, and to the logarithmic plane Λ, when n is even (see the proof of Proposition 4.1 for a direct argument). The first sheet, where $R(z, h)$ is meromorphic on $\mathcal{H}$ (with poles corresponding to eigenvalues) is called the *physical plane.* This continuation is as an operator from $\mathcal{H}_{\text{comp}}$ to $\mathcal{D}_{\text{loc}}$, and the poles are of finite rank.

[1]in Proposition 2.2 only, where it also could be avoided

The poles are called *resonances* of $P(h)$. We will denote the set of resonances by Res $(P(h))$, and will always include them according to their multiplicity, $m_R(z,h)$, which for $z \neq 0$ is defined as

$$m_R(z,h) = \operatorname{rank} \int_{\gamma_\epsilon(z)} R(w,h)dw\,, \quad \gamma_\epsilon(z) = \{z + \epsilon e^{it} \,:\, 0 \leq t \leq 2\pi\}\,, \quad 0 < \epsilon \ll 1\,,$$

see [24] and [35] for a discussion of this. We remark that we *include* the *point spectrum* of $P(h)$, denoted by $\sigma(P(h))$, in the set Res $(P(h))$. Strictly speaking, resonances have non-zero imaginary parts and a distinction could be made.

In order to guarantee a polynomial bound on the counting function of resonances, we need a spectral condition on $P(h)$. It is formulated in terms of a reference operator constructed from $P(h)$: let

$$\mathcal{H}^\sharp = \mathcal{H}_{R_0} \oplus L^2(\mathbb{T}^n_{R_1} \setminus B(0,R_0))\,, \quad \mathbb{T}^n_{R_1} = \mathbb{R}^n/(R_1\mathbb{Z}^n)\,, \ R_1 \gg R_0\,,$$

and define $P^\sharp(h)$ by replacing $-h^2\Delta_{\mathbb{R}^n}$ by $-h^2\Delta_{\mathbb{T}^n_{R_1}}$ in the definition of $P(h)$ (see [24] and [22]). The assumptions on $P(h)$ imply that $P^\sharp(h)$ has discrete spectrum and we assume that if $N(P^\sharp(h),\lambda)$ is the number of eigenvalues of $P^\sharp(h)$ in $(-\infty,\lambda]$ then

$$N(P^\sharp(h),[-C,\lambda]) = \mathcal{O}\left(\left(\frac{\lambda}{h^2}\right)^{n^\sharp/2}\right)\,, \text{ for } \lambda \geq 1 \tag{2.1}$$

for some number $n^\sharp \geq n$. As was observed in [24], this assumption does not depend on R_1, only on $P(h)$.

The scattering matrix for a "black box" perturbation is defined just as in the usual obstacle or potential scattering (see [16], [6] and references given there). We recall the stationary definition here: for any $\lambda > 0$ and a function $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, there exists $u \in \mathcal{D}_{\text{loc}}$, such that for $|x| > R_0$

$$(P-\lambda)u = 0\,, \quad u(x) = |x|^{-\frac{n-1}{2}}\left(e^{-\frac{i\sqrt{\lambda}|x|}{h}}f(x/|x|) + e^{\frac{i\sqrt{\lambda}|x|}{h}}g(x/|x|) + \mathcal{O}(1/|x|)\right)\,, \tag{2.2}$$

where $g \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$. By Rellich's Uniqueness Theorem (see for instance [38, Sect.3]), u is unique up to a *compactly supported* eigenfunction $\tilde{u} \in \mathcal{D}_{\text{comp}}$, $(P-\lambda)\tilde{u} = 0$. From the black box assumptions we know that the set of such λ's is discrete, and the compact support of the eigenfunctions $\tilde{u}$, makes them irrelevant in our study of scattering.

The function f can be considered as the incoming data, and g as the outgoing data. This is consistent with our notion of the outgoing resolvent, $R(z,h)$, which is bounded on $\mathcal{H}$ for $\operatorname{Im} z > 0$: the outgoing term $\exp(i\sqrt{z}|x|/h)$ is bounded in L^2 for $\operatorname{Im} z > 0$.

The absolute scattering matrix relates the two data:

$$\widetilde{S}(\lambda,h) \,:\, f \longmapsto g\,,$$

and we denote by $\widetilde{S}_0(\lambda,h)$ the free scattering matrix corresponding to $P = -h^2\Delta$. It is essentially given by the antipodal map:

$$\widetilde{S}_0(\lambda,h)f(\omega) = i^{1-n}f(-\omega)\,, \quad \lambda > 0$$

(see the proof of Proposition 2.1 below). We then define the standard (relative) scattering matrix as

$$S(\lambda,h) = \widetilde{S}_0(\lambda,h)^{-1}\widetilde{S}(\lambda,h)\,.$$

It has the form (see (2.5) below):

$$S(\lambda,h) = I + A(\lambda,h), \quad A(\lambda,h) \in \mathcal{C}^\infty(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})\,.$$

Under our assumptions, it continues meromorphically in λ to the double cover of $\mathbb{C}$ (Riemann surface for $z = w^2$) for n odd and to the logarithmic plane, Λ when n is even. It is holomorphic in $\operatorname{Im} z > 0$, $\operatorname{Re} z > 0$ and the poles of its continuation correspond to resonances of $P(h)$ (see Proposition 2.2 below). We recall also the crucial unitarity

$$S(z,h)^{-1} = S(\bar{z},h)^* . \tag{2.3}$$

It follows from the pairing formula recalled in the proof of Proposition 2.1.

We now present one of many possible representations of $A(z,h)$ in terms of the resolvent (see [17, Sect.2] and [36, Sect.3]) and its proof contains the proof of the general statements about $S(z,h)$ made above.

Proposition 2.1. *For $\phi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n)$ let us denote by*

$$\mathbb{E}_\pm^\phi(z,h) : L^2(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1}) \tag{2.4}$$

the operator with the kernel $\phi(x)\exp(\pm i\sqrt{z}\langle x,\omega\rangle/h)$, with $\sqrt{z}$ positive on the real axis. Let us choose $\chi_i \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n)$, $i = 1,2,3$, such that $\chi_i \equiv 1$ near $B(0,R_0)$, and $\chi_{i+1} \equiv 1$ on $\operatorname{supp}\chi_i$, $i = 1,2$.

Then for $\operatorname{Im} z > 0$, $\operatorname{Re} z > 0$ we have

$$A(z,h) = c_n h^{-n} z^{\frac{n-2}{2}} \mathbb{E}_+^{\chi_3}(z,h)[h^2\Delta,\chi_1]R(z,h)[h^2\Delta,\chi_2]{}^t\mathbb{E}_-^{\chi_3}(z,h),\ c_n = i\pi(2\pi)^{-n} , \tag{2.5}$$

where ${}^t\mathbb{E}$ denotes the transpose of $\mathbb{E}$.

We remark that the transpose is defined using the Schwartz kernel: ${}^t\mathbb{E}(x,\omega) = \mathbb{E}(\omega,x)$.

Proof. We give a direct proof in the spirit of [30] and use the pairing formula: if $\lambda > 0$ and

$$(P-\lambda)u_i = f_i \in \mathcal{H} , \quad f_i|_{\mathbb{R}^n\setminus B(0,R_0)} \in \mathcal{S},$$
$$u_i(x) = |x|^{-\frac{n-1}{2}}\left(e^{-\frac{i\sqrt{\lambda}|x|}{h}}a_i^-(x/|x|) + e^{\frac{i\sqrt{\lambda}|x|}{h}}a_i^+(x/|x|) + \mathcal{O}(1/|x|)\right) , \quad |x| \longrightarrow \infty ,$$

then

$$\langle u_1, f_2\rangle_{\mathcal{H}} - \langle f_1, u_2\rangle_{\mathcal{H}} = 2ih\sqrt{\lambda}\left(\langle a_1^-, a_2^-\rangle_{L^2(\mathbb{S}^{n-1})} - \langle a_1^+, a_2^+\rangle_{L^2(\mathbb{S}^{n-1})}\right) .$$

Let us introduce the operators $\mathbb{E}_\pm(\lambda,h)$ with Schwartz kernels $\exp(\pm i\sqrt{\lambda}\langle x,\omega\rangle/h)$ and assume that $\lambda > 0$ is not an eigenvalue of P. For $g_1, g_2 \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ let us put

$$u_1 = (1-\chi_1){}^t\mathbb{E}_-(\lambda,h)g_1,$$
$$u_2 = \left((1-\chi_2){}^t\mathbb{E}_-(\lambda,h) - R(\lambda,h)[h^2\Delta,\chi_2]{}^t\mathbb{E}_-(\lambda,h)\right)g_2$$

so that $(P-\lambda)u_2 = 0$ and

$$(P-\lambda)u_1 = [h^2\Delta,\chi_1]{}^t\mathbb{E}_-(\lambda,h)g_1 .$$

A stationary phase argument now gives

$$a_1^- = \alpha_n g_1 ,$$
$$a_1^+ = \alpha_n i^{1-n} g_1(-\bullet) ,$$
$$\alpha_n = \lambda^{-\frac{1}{4}(n-1)} h^{\frac{1}{2}(n-1)} e^{\frac{i}{4}\pi(n-1)} (2\pi)^{\frac{1}{2}(n-1)}.$$

For u_2 we note that since $R(\lambda,h)$ is the outgoing resolvent, the only incoming contribution comes from the free term $(1-\chi_1)^t\mathbb{E}_-(\lambda,h)g_2$ (that $R(\lambda,h)$ has not incoming term is seen, for instance, from the properties of the free resolvent and (4.2) below). Hence

$$a_2^- = \alpha_n g_2 ,$$
$$a_2^+ = \alpha_n i^{1-n} S g_2(-\bullet) .$$

Using the fact that $(1-\chi_2)[h^2\Delta,\chi_1]=0$, and the pairing formula above we see that

$$\begin{aligned}
&\langle g_1, \mathbb{E}_+(\lambda,h)[h^2\Delta,\chi_1]R(\lambda,h)[h^2\Delta,\chi_2]^t\mathbb{E}_-(\lambda,h))g_2\rangle_{L^2(\mathbb{S}^{n-1})}\\
&= -\langle [h^2\Delta,\chi_1]^t\mathbb{E}_-(\lambda,h)g_1, (1-\chi_2)^t\mathbb{E}_-(\lambda,h)g_2 - R(\lambda,h)[h^2\Delta,\chi_2]^t\mathbb{E}_-(\lambda,h)g_2\rangle_{\mathcal{H}}\\
&= \langle u_1,(P-\lambda)u_2\rangle_{\mathcal{H}} - \langle (P-\lambda)u_1,u_2\rangle_{\mathcal{H}}\\
&= 2i\lambda^{-\frac{1}{2}(n-2)}h^n(2\pi)^{n-1}\langle g_1,(I-S(\lambda,h))g_2\rangle_{L^2(\mathbb{S}^{n-1})},
\end{aligned}$$

The general result follows from analytic continuation – in fact, we proved here that the scattering matrix has an analytic continuation, once that of $R(z,h)$ is established. □

Remark. It is interesting to note that the representation (2.5) does not depend on the cut-off functions, and that we can reverse the condition $\chi_2\equiv 1$ on the support of χ_1 to $\chi_1\equiv 1$ on the support of χ_2. Both facts follow directly from the properties of the scattering matrix but here we propose a direct argument based on the standard properties of "quantum flux". Suppose that χ_2 is equal to one on the supports of functions χ_1, $\tilde\chi_1$, which are equal to 1 near $B(0,R_0)$. We claim that

$$\mathbb{E}_+^{\chi_3}(z,h)[h^2\Delta,\chi_2]R(z,h)[h^2\Delta,\chi_1-\tilde\chi_1]^t\mathbb{E}_-^{\chi_3}(z,h)\equiv 0\,.$$

This will follow from showing that

$$(-h^2\Delta - z)v_j = 0\,,\ j=1,2 \implies \langle R(z,h)[h^2\Delta,\chi_1-\tilde\chi_1]v_1,[h^2\Delta,\chi_2]v_2\rangle_{\mathcal{H}} = 0\,,$$

which is clear as the left hand side is equal to

$$\langle R(z,h)(-P(\chi_1-\tilde\chi_1)-(\chi_1-\tilde\chi_1)h^2\Delta)v_1,[h^2\Delta,\chi_2]v_2\rangle_{\mathcal{H}} = -\langle(\chi_1-\tilde\chi_1)v_1,[h^2\Delta,\chi_2]v_2\rangle_{\mathcal{H}}=0\,,$$

since $(\chi_1-\tilde\chi_1)[h^2\Delta,\chi_2]=0$. Similarly, if $\chi_1\equiv 1$ on the support of $\tilde\chi_1$, and $\tilde\chi_1\equiv 1$ near $B(0,R_0)$, then

$$\mathbb{E}_+^{\chi_3}(z,h)[h^2\Delta,\chi_2-\tilde\chi_1]R(z,h)[h^2\Delta,\chi_1]^t\mathbb{E}_-^{\chi_3}(z,h)\equiv 0\,,$$

which shows that we can switch the conditions on χ_1 and χ_2. Yet another argument of the same type shows that

$$\mathbb{E}_+^{\chi_3}(z,h)[h^2\Delta,\chi_2]R_0(z,h)[h^2\Delta,\chi_1]^t\mathbb{E}_-^{\chi_3}(z,h)\equiv 0\,,\quad R_0(z,h)=(-h^2\Delta-z)^{-1}\,.$$

In the next proposition we list two well known facts:

Proposition 2.2. *If we define the multiplicity of a pole or a zero of* $\det S(z,h)$ *as*

$$m_S(z,h) = -\frac{1}{2\pi i}\operatorname{tr}\int_{\gamma_\epsilon(z)} S(w,h)^{-1}\frac{d}{dw}S(w,h)dw\,, \tag{2.6}$$

$\gamma_\epsilon(z)=\{z+\epsilon e^{it}\ :\ 0\le t\le 2\pi\}\,,\quad 0<\epsilon\ll 1\,,$ *then*

- $\det S(w,h) = (w-z)^{-m_S(z,h)}g_z(w)$, *for* w *near* z, *with* $g_z(z)\neq 0$,
- $m_S(z,h) = m_R(z,h) - m_R(\bar z,h)$, $\operatorname{Re} z>0$, *where one of* z, $\bar z$, *is in the physical, and one in the non-physical half-plane.*

In particular, the non-negative eigenvalues of $P(h)$ do not contribute to the poles of the scattering matrix. We outline the proof for the reader's convenience:

Proof. The first part is a direct application of a classical result of Gohberg and Sigal [10]. To see the second part we will use the continuity properties of the multiplicities and the generic simplicity of resonances (see [13]): this makes the argument considerably simpler. By continuity property we mean the fact that for any w_0 and $\epsilon>0$, $\sum_{|w-w_0|<\epsilon} m_\bullet(w)$ is constant for sufficiently small perturbations, which follows from the definition of multiplicities using integrals.

Consequently we can assume that $m_R(w,h)\le 1$ as the general statement follows from a deformation to the generic case. Suppose then that $-\pi/2<\arg w_0<0$, that is, that w_0 is in the first sheet of the non-physical plane, and that $m_R(w_0, h) = 1$. The proof of the meromorphic continuation (see the derivation of (4.2) below) shows that in this case

$$R(w,h) = \frac{A}{w - w_0} + B(w)\,,$$

where $B(w)$ is holomorphic in w near w_0. The reality of P implies that $R(w,h)$ is symmetric (with respect to the indefinite form $\langle \bullet, \bar{\bullet} \rangle_{\mathcal{H}}$) and consequently $A = \phi \otimes \phi$, $Au = \langle \phi, \bar{u} \rangle_{\mathcal{H}} \phi$. Another look at the structure of the resolvent (see (4.2), and for a more detailed discussion [38, Lemma 1]) shows that $\phi = R_0(w_0,h)g$, where $g \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and $R_0(z,h)$ is the free resolvent. Proposition 2.1 shows that

$$S(w,h) = \frac{A_1}{w - w_0} + B_1(w)\,, \quad A_1 = c_n z^{\frac{n-2}{2}} h^{-n} \mathbb{E}_+(w_0,h)g \otimes \mathbb{E}_-(w_0,h)g\,. \tag{2.7}$$

In fact, all that needs to be checked is that

$$\mathbb{E}_\mp(z,h)g = \mathbb{E}_\mp(z,h)[h^2\Delta, \chi]R_0(z,h)g\,, \quad g \in \mathcal{C}^\infty_c(\mathbb{R}^n)\,, \quad (1-\chi)g = 0\,,$$

and that follows from integration by parts: for $z \in (0, \infty)$ and $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$,

$$\begin{aligned} \langle\langle [h^2\Delta, \chi]R_0(z,h)g, {}^t\mathbb{E}_\pm(z,h)f \rangle_{\mathcal{H}} &= -\langle (-h^2\Delta - z)\chi R_0(z,h)g, {}^t\mathbb{E}_\pm(z,h)f\rangle_{\mathcal{H}} + \langle g, {}^t\mathbb{E}_\pm(z,h)f\rangle_{\mathcal{H}} \\ &= \langle g, {}^t\mathbb{E}_\pm(z,h)f\rangle_{\mathcal{H}}\,. \end{aligned}$$

The essentially standard Rellich's Uniqueness Theorem type argument (see [38, Sect.3]) shows that for $\arg z \neq 2\pi k$, $k = 0, 1$, $\mathbb{E}_\pm(w_0,h)g \neq 0$. We can then find invertible operators, $F_k(w)$, $k = 1, 2$, holomorphic near w_0, such that

$$S(w,h) = F_1(w)\left(\frac{P_1}{w - w_0} + P_0(w)\right) F_2(w)\,, \quad P_1^2 = P_1\,,$$

with $P_0(w)$, holomorphic near w_0. As shown in [10], the operators $F_k(w)$ make no contribution to the integral in (2.6), and consequently we can assume that $S(w,h)$ is given by the expression in the middle. The representation (2.7) shows that

$$d_{S^{-1}}(w_0) \stackrel{\text{def}}{=} \dim\{\psi \in L^2(\mathbb{S}^{n-1}) \;:\; S^{-1}(z,h)\psi = \mathcal{O}(|z - w_0|^k)\} \leq 1\,,$$

and that the only power k which can occur is $k = 1$. In fact using the projection P_1 we can construct an element of the kernel and hence $d_{S^{-1}}(w_0) = m_R(w_0,h)$. On the other hand, for $k \geq 1$ we have

$$d_S(w_0) \stackrel{\text{def}}{=} \dim\{\psi \in L^2(\mathbb{S}^{n-1}) \;:\; S(z,h)\psi = \mathcal{O}(|z - w_0|^k)\} = 0\,,$$

since the equality (2.3) implies that $S^{-1}(z,h)$ is continuous at w_0.

If we apply [10, Theorem 2.1] in this situation, we obtain that

$$m_R(w_0,h) = d_{S^{-1}}(w_0) - d_S(w_0) = -\frac{1}{2\pi i} \operatorname{tr} \int_{\gamma_\epsilon(w_0)} S(w,h)^{-1} \frac{d}{dw} S(w,h)\, dw$$

and that proves the second part of the proposition for $\operatorname{Im} z < 0$, as then $m_R(\bar{z},h) = 0$. For $\operatorname{Im} z > 0$, we use (2.3), which shows than that $m_S(z,h) = -m_S(\bar{z},h) = -m_R(\bar{z},h)$. As now $m_R(z,h) = 0$, we obtain our formula. □

Remark. We could avoid the results of [13] which strictly speaking do not apply to the whole logarithmic plane when the dimension is even (but apply in the region considered here), and used instead the direct argument of [11, Sect.2] which is based on [10].

Proposition 2.1 has the following strange consequence which perhaps has been observed before:

Proposition 2.3. *Suppose that $n \geq 5$. Then $S(z,h)$ is holomorphic in $\mathcal{O}_n(h,R_0) = h^2 R_0^{-2}\mathcal{O}_n$,*

$$\mathcal{O}_n = \{re^{i\theta}\,;\, r^{n-4} \leq \alpha_n \sin^2\theta\,,\ 0 < r \leq 1\}$$

where α_n is a constant depending on the dimension. Moreover,

$$\frac{1}{2} \leq \left|\det S(z,h)\right| \leq 2\,, \quad z \in \mathcal{O}_n(h,R_0)\,.$$

We recall that it is well known that if $n \geq 5$ and 0 is a pole of the resolvent, than it is an eigenvalue. The proposition shows that this phenomenon of absence of resonances propagates to a set near zero.

Proof. We can take $R_0 = 1$ as the general result follows from scaling. To show that $S(z,h)$ is holomorphic in $\mathcal{O}_n(h,1)\cap\{\operatorname{Im} z < 0\}$, we will show that $S(z,h)$ is invertible in $\mathcal{O}_n(h,1)\cap\{\operatorname{Im} z > 0\}$. That is done by showing that $\|A(z,h)\|$ is small there. In fact, for $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$, $\chi \equiv 1$ near $B(0,1)$,

$$\begin{aligned}
\|[h^2\Delta,\chi]^t\mathbb{E}^\phi_\pm(z,h)\|_{L^2(\mathbb{S}^{n-1})\to L^2(\mathbb{R}^n)} &\leq C_\chi(h^2+|z|)e^{C_\chi|z|^{\frac{1}{2}}/h}\,,\\
\|\mathbb{E}^\phi_\pm(z,h)[h^2\Delta,\chi]\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{S}^{n-1})} &\leq C_\chi(h^2+|z|)e^{C_\chi|z|^{\frac{1}{2}}/h}\,,\\
\|(1-\chi)R(z,h)(1-\chi)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} &\leq \frac{C}{\operatorname{Im} z}\,,\ \operatorname{Im} z > 0\,.
\end{aligned}$$

Hence

$$\|A(z,h)\| \leq C\frac{|z|}{\operatorname{Im} z}e^{C|z|^{\frac{1}{2}}/h}\left((h^{-2}|z|)^{\frac{n-4}{2}} + (h^{-2}|z|)^{\frac{n}{2}}\right)\,,\quad \operatorname{Im} z > 0\,,$$

where the constants depend on the cut-off functions used and the dimension. By choosing α_n in the definition of $\mathcal{O}_n$ small enough we can make $\|A(z,h)\|$ small in $\mathcal{O}_n(h,1)$.

To estimate the determinant we observe that

$$e^{-\|(I+A(z,h))^{-1}\|\|A(z,h)\|_{\rm tr}} \leq |\det S(z,h)| \leq e^{\|A(z,h)\|_{\rm tr}}\,,\quad \operatorname{Im} z > 0\,.$$

Since $\|A(z,h)\|_{\rm tr} \leq C\|(I-\Delta_{\mathbb{S}^{n-1}})^{\frac{n+1}{2}}A(z,h)\|$, the determinant estimate follows from the previous argument by observing that

$$\|(I-\Delta_{\mathbb{S}^{n-1}})^{\frac{n+1}{2}}\mathbb{E}^\phi_\pm(z,h)[h^2\Delta,\chi]\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{S}^{n-1})} \leq C(1+(h^{-2}|z|)^{\frac{n+1}{2}})(|z|+h^2)\,,\ |z| \leq h^2\,.$$

□

The standard object of study in scattering theory is the *scattering phase* which is defined as

$$\sigma(\lambda,h) = \frac{1}{2\pi i}\log s(\lambda,h)\,, \tag{2.8}$$

with some choice of the logarithm, for instance, $\sigma(0,h) = 0$. It is related to the *spectral shift function* which is defined using normalized traces of functions of $P(h)$. To present this relation we introduce a normalized trace: for $g \in \mathcal{S}(\mathbb{R})$ we let

$$\begin{aligned}
\widetilde{\operatorname{tr}} g(P(h)) = & \operatorname{tr}_{\mathcal{H}}\left(g(P(h)) - (1-\chi)g(-h^2\Delta)(1-\chi)\right)\\
& -\ \operatorname{tr}_{L^2(\mathbb{R}^n)}\left(g(-h^2\Delta) - (1-\chi)g(-h^2\Delta)(1-\chi)\right)\,,
\end{aligned} \tag{2.9}$$

where $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$, $\chi \equiv 1$ on $B(0,R_0+a)\,,\ a > 0$.

The Birman-Krein formula then takes the following well known form

$$\widetilde{\operatorname{tr}} g(P(h)) = -\int \frac{dg}{d\lambda}(\lambda)\sigma(\lambda,h)d\lambda + \sum_{\lambda\in\sigma(P(h))} g(\lambda)\,,\quad g \in \mathcal{S}((0,\infty))\,, \tag{2.10}$$

and for the adaptation of the standard proof to the black box case we refer to [6]. By using the assumption (2.1) and the representation of the scattering phase (see [4, Theorem 3]) we have, for every $J \Subset \mathbb{R}^+$,

$$|\sigma(\lambda, h)| \leq C(J)h^{-n^\sharp}, \quad \lambda \in J, \ 0 < h \leq h_0(J). \tag{2.11}$$

If we define

$$N^\sharp(\lambda, h) = \# \, \sigma(P^\sharp(h)) \cap (0, \lambda] - \# \, \sigma(-h^2 \Delta_{\mathbb{T}^n_{R_1}}) \cap (0, \lambda],$$

then, as shown recently by Bruneau and the first author [4, Theorem 3], for $E > 0$ and $\mu > 0$ we have

$$\sigma(E+\mu, h) - \sigma(E, h) + \sum_{\lambda \in \sigma(P(h))} \delta_\lambda((E, E+\mu]) = N^\sharp(E+\mu, h) - N^\sharp(E, h) + \mathcal{O}(h^{-n^\sharp+1}). \tag{2.12}$$

In particular, in the interesting situation when $n = n^\sharp$,

$$N^\sharp(E+\mu, h) - N^\sharp(E, h)) = W(E, \mu)h^{-n} + \mathcal{O}(h^{-n+1}), \tag{2.13}$$

and $\sigma(P(h)) \cap (0, \infty) = \emptyset$, we have

$$\sigma(E+\mu, h) - \sigma(E, h) = W(E, \mu)h^{-n} + \mathcal{O}(h^{-n+1}), \tag{2.14}$$

where the Weyl term $W(E, \mu)$ is assumed to be smooth in μ as is the case for spectral asymptotics near non-degenerate energy levels (see for instance [7, Sect.11]).

3. Some complex analysis

In the aspects of scattering theory studied here we apply the following principle of complex analysis: if a holomorphic function is not identically zero then, at most points, it is bounded from below by a constant times the reciprocal of its upper bound *provided* we have control on the lower bound of the function at one point.

This follows from a precise statement for the disc:

Lemma 3.1. *Suppose $f(z)$ is holomorphic in the disc $|z| \leq r$ and that $f(0) \neq 0$. Suppose that the number of zeros of $f(z)$ in $|z| \leq r$ is equal to N. Then for any $\theta \in (0,1)$ we have*

$$\min_{|z|=\rho} \log|f(z)| > -\left(\frac{r+\rho}{r-\rho} \max_{|z|=r} \log|f(z)| + N \log\frac{1}{\theta}\right) + \frac{2r}{r-\rho} \log|f(0)|, \tag{3.1}$$

for $\rho \in (0, r) \setminus \cup_{k=1}^K (\rho_k - \delta_k, \rho_k + \delta_k)$, $0 < \delta_k < \rho$, $\sum_{k=1}^K \delta_k \leq 6\theta r$.

The proof follows from the classical lemma of Cartan (see for instance [12, Lemma 6.17]) and the Poisson-Jensen formula (see [12, Lemma 6.18]). We recall that N can be estimated using Jensen's formula by

$$\left(\log(1 + \frac{\epsilon}{r})\right)^{-1} \left(\max_{|z|=r+\epsilon} \log|f(z)| - \log|f(0)|\right), \ \epsilon > 0.$$

For future use we will recall here Cartan's beautiful estimate:

Given arbitrary numbers $z_m \in \mathbb{C}$, $m = 1, \ldots, M$, for any $\eta > 0$ there exists a set, $\bigcup_{l=1}^L D(a_l, r_l)$, formed by the union of $L \leq M$ discs, $D(a_l, r_l)$, centered at some points $a_l \in \mathbb{C}$, such that $\sum_{l=1}^L r_l < 2e\eta$ and

$$\prod_{m=1}^M |z - z_m| > \eta^M, \quad z \in \mathbb{C} \setminus \bigcup_{l=1}^L D(a_l, r_l). \tag{3.2}$$

Lemma 3.1 is then a consequence of this, and of the Carathéodory or Harnack inequalities (see the proof of Proposition 4.2 for a direct application of a similar argument).

We will also need a result in the case of a cone for which we quote [5, Theorem 56]:

Lemma 3.2. *Suppose that f is holomorphic in $\{z \; : \; 0 < \arg z < \pi/k + \epsilon\}$, $\epsilon > 0$ and that $\log |f(z)| \leq B_1 |z|^k$, $\log |f(z_0)| \geq -B_2$ with $0 < \arg z_0 < \pi/k$. Then for any $\delta > 0$,*

$$\begin{gathered}\log |f(re^{i\theta})| > -C_\delta r^k \, , \;\; r > r_0 \, , \;\; \theta \in (0, \theta_0) \setminus \Sigma(r) \, , \;\; |\Sigma(r)| < \delta \, , \\ C_\delta = C_\delta(\epsilon, z_0, B_1, B_2) \, , \quad r_0 = r_0(\epsilon, z_0, B_1, B_2) \, . \end{gathered} \tag{3.3}$$

We remark that this estimate also follows from the estimates obtained more directly by Sjöstrand [23, Sect.7].

We recall also the standard Carleman inequality which in a similar context was already used in [18] (see for instance [29, 3.7]):

Lemma 3.3. *Let $f(z)$ be holomorphic in $|z - \lambda| \leq R$, $\operatorname{Im} z \geq 0$, with $\lambda \in \mathbb{R}$. Let z_j denote the zeros of $f(z)$ and let $0 < \rho < r < R$ be such that no zeros of $f(z)$ lie on $|z - \lambda| = \rho$ and $|z - \lambda| = r$, and on the real axis. Then for $0 < \delta < 1 - \frac{\rho}{r}$ we have*

$$\begin{aligned} \sum_{\rho < |z_j - \lambda| < (1-\delta) r} \frac{\operatorname{Im} z_j}{|z_j - \lambda|^2} \leq \frac{1}{\delta} \Bigg(& \frac{1}{\pi r} \int_0^\pi \log |f(\lambda + re^{i\theta})| \sin \theta d\theta \\ & - \frac{1}{\pi \rho} \int_0^\pi \log |f(\lambda + \rho e^{i\theta})| \sin \theta d\theta \\ & + \frac{1}{2\pi} \int_\rho^r \left(\frac{1}{y^2} - \frac{1}{r^2} \right) \log |f(\lambda + y) f(\lambda - y)| dy \Bigg) \, . \end{aligned} \tag{3.4}$$

For future reference we recall the following lemma already used in [19][2]

Lemma 3.4. *Suppose that u is harmonic in $D(0,1)$, and that*

$$|u(z)| \leq \frac{K}{|\operatorname{Im} z|} \, , \quad u(z) = -u(\bar{z}) \, , \quad z \in D(0,1) \, .$$

Then, for every $0 < \epsilon < 1$, there exists $C = C(\epsilon)$ such that

$$|u(z)| \leq CK |\operatorname{Im} z| \, , \quad z \in D(0, 1 - \epsilon) \, .$$

Proof. We use the Poisson formula and the symmetry (and we can assume that the hypotheses hold in a slightly bigger disc):

$$\begin{aligned} u(re^{i\theta}) = & \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2) u(e^{i\varphi})}{1 - 2r\cos(\theta - \varphi) + r^2} d\varphi \\ = & \frac{1}{\pi} \int_0^\pi \frac{8(1 - r^2) r \sin \theta \sin \varphi u(e^{i\varphi})}{(1 - 2r\cos(\theta - \varphi) + r^2)(1 - 2r\cos(\theta + \varphi) + r^2)} d\varphi \, . \end{aligned}$$

Since we know that

$$|u(e^{i\varphi})| \leq K / \sin \varphi \, , \quad 0 \leq \varphi \leq \pi \, ,$$

we conclude that for $r < 1 - \epsilon$,

$$|u(z) / \operatorname{Im} z| \leq 8 \epsilon^{-4} K \, .$$

□

Finally, we present a version of a semi-classical maximum principle related to [27, Lemma 2]

[2]This lemma was pointed out to us by W.K. Hayman.

Lemma 3.5. *Suppose that $F(z,h)$ is holomorphic in $z \in \Omega_0(h)$, continuous in $\overline{\Omega}_0(h)$, $\Omega_\epsilon(h) = (a+\epsilon, b-\epsilon) + i(0, h^M)$, $\epsilon \geq 0$, and that $\log|F(z,h)| \leq Ch^{-K}$ for $z \in \Omega_0(h)$. If $2M > K$ then*

$$\log|F(z,h)| \leq M_0(h) + M_1(h) + \mathcal{O}(1)\,, \quad z \in \Omega_\epsilon(h)\,, \quad 0 < h \leq h(\epsilon)\,,$$
$$M_0(h) = \max_{\overline{\Omega}_0(h)\cap\{\operatorname{Im} z = 0\}} \log^+|F(z,h)|\,, \quad M_1(h) = \max_{\overline{\Omega}_0(h)\cap\{\operatorname{Im} z = h^M\}} \log^+|F(z,h)|\,,$$

where $\log^+ x = \max(0, \log x)$.

Proof. For $\epsilon > 0$ let us introduce,

$$f_\epsilon(z,h) = \frac{h^{-L/2}}{\sqrt{\pi}} \int_{a+\epsilon/2}^{b-\epsilon/2} \exp(-h^{-L}(x-z)^2)dx\,, \quad L < 2M\,,$$

so that $|f_\epsilon(z,h)| \leq \epsilon$ on $\Omega_0(h)$, $|f_\epsilon(z,h)| \geq 1/2$ on $\Omega_\epsilon(h)$, if $h \leq h(\epsilon)$, and $|f_\epsilon(z,h)| \leq C\exp(-C_\epsilon h^{-L})$, on $\Omega_0(h) \setminus \Omega_{\epsilon/4}(h)$. We then apply the maximum principle to the subharmonic function

$$\log|G(z,h)| = \log|F(z,h)| + \log|f_\epsilon(z,h)| - M_0(h) - M_1(h) - 1\,.$$

If we choose $L > K$ then, on $\partial\Omega_0(h)$ we have $\log|G(z,h)| \leq 0$ and on $\Omega_\epsilon(h)$ we get $\log|f_\epsilon(z,h)| = \mathcal{O}(1)$. □

4. Estimates on the scattering determinant

We will give a self-contained discussion of the estimates for the number of resonances, the cut-off resolvent and the scattering determinant in the setting of semi-classical compactly supported black box perturbations. Our presentation comes largely from [28, Sect.4] and it is based on the works of Melrose [14], Sjöstrand [22],[24], Vodev [31], and the second author [34],[36]. We also adapt the methods of [18] to the semi-classical setting to obtain the estimate on the scattering determinant (Lemma 4.3), and its factorization (Proposition 4.4).

We start with a polynomial bound on the number or resonances:

Proposition 4.1. *If $n^\sharp$ is as in (2.1) and $\Omega \Subset \{z : \operatorname{Re} z > 0\}$ is a pre-compact neighborhood of $E \in \mathbb{R}_+$, then*

$$\#\{z : z \in \operatorname{Res}(P(h)) \cap \Omega\} = \mathcal{O}(h^{-n^\sharp})\,, \ \Omega \Subset \mathbb{C}. \tag{4.1}$$

Remark. This proposition can be improved by replacing $h^{-n^\sharp}$ by a more precise bound on the number of eigenvalues of the reference operator, $\Phi(h^{-2})$ – see [22]. For a large class of majorants Φ, the proof given here can be improved following [31]. Consequently we can replace $h^{-n^\sharp}$ by $\Phi(Ch^{-2})$ in all subsequent estimates, which we avoid for the sake of clarity. The most interesting case is of course $n^\sharp = n$ and a nice case where $\Phi(t) = t^{n^\sharp/2}$, $n^\sharp > n$ is given by finite volume hyperbolic quotients [24].

Proof. Let $R_0(z,h)$ be the meromorphic continuation of the free resolvent $(-h^2\Delta - z)^{-1}$ from $\operatorname{Im} z > 0$ to $\widetilde{\Omega}$, $\overline{\Omega} \Subset \widetilde{\Omega} \Subset \mathbb{C}$. Let us also consider the following cut-off functions $\chi_i \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $i = 0,1,2$, $\chi_0 \equiv 1$ near $\overline{B}(0,R_0)$, $\chi_i \equiv 1$ near $\operatorname{supp}\chi_{i-1}$ and $\chi \equiv 1$ near $\operatorname{supp}(\chi_2)$. We then define

$$Q_0 = Q_0(z,h) = (1-\chi_0)R_0(z,h)(1-\chi_1)\,,$$
$$Q_1 = Q_1(z_0,h) = \chi_2 R(z_0,h)\chi_1\,, \ \operatorname{Im} z_0 > 0\,,$$

so that

$$\begin{aligned}
&(P(h)-z)(Q_0(z,h)+Q_1(z_0,h)) = I + K_0(z,h) + K_1(z_0,z,h),\\
&K_0(z,h) = -[h^2\Delta,\chi_0]R_0(z,h)(1-\chi_1),\\
&K_1(z_0,z,h) = -[h^2\Delta,\chi_2]R(z_0,h)\chi_1 + \chi_2(z_0-z)R(z_0,h)\chi_1\,.
\end{aligned}$$

We now put

$$K = K(z_0,z,h) = K_0(z,h)\chi + K_1(z_0,z,h)\chi$$

which is a compact operator $\mathcal{H} \to \mathcal{H}$ and the norm of $K(z_0,z_0,h)$ is $\mathcal{O}(h)$. Hence $(I + K(z_0,z_0,h))^{-1}$ exists for h small enough and consequently (via analytic Fredholm theory) $(I + K(z_0,z,h))^{-1}$ is meromorphic in z (under our assumptions, on the Riemann surface of $z = w^2$ for n odd and $z = e^w$ for n even). Hence

$$R(z,h)\chi = (Q_0(z,h)\chi + Q_1(z_0,h)\chi)(I + K(z_0,z,h))^{-1}\,, \tag{4.2}$$

and we have essentially reviewed the proof of the meromorphic continuation of the resolvent from [24].

We now introduce

$$f(z,h) = \det(I + K^{n^\sharp+1}(z,h))\,,$$

where $n^\sharp$ is as in (2.1) and, as we will see below, the choice of the power justifies the existence of the determinant. By Weyl inequalities (see for instance [9, Chapter II, Corollary 3.1]), $|f(z,h)| \leq M(h)$, $z \in \widetilde{\Omega}$, where

$$M(h) = \sup_{z\in\widetilde{\Omega}} \det(I + K^*K)^{\frac{n^\sharp+1}{2}})\,.$$

To estimate $M(h)$ we need to estimate the eigenvalues of $(K^*K)^{\frac{1}{2}}$, that is the characteristic values $\mu_j(K)$ of K. The standard properties of characteristic values (see [9, Chapter II]) show that it is enough to estimate the characteristic values of various summands.

We start by proving that

$$\begin{aligned}
&\mu_j([h^2\Delta,\chi_2]R(z_0,h)\chi_1) \leq Ch\left(\frac{j}{h}\right)^{-\frac{1}{n^\sharp}}\,,\\
&\mu_j(\chi_2 R(z_0,h)\chi_1) \leq C\left(\frac{j}{h}\right)^{-\frac{2}{n^\sharp}}\,.
\end{aligned}$$

In fact, for all N, M,

$$\chi_2 R(z_0,h)\chi_1 - \chi_2(P^\sharp(h)-z_0)^{-1}\chi_1 = \mathcal{O}(h^N) \;:\; \mathcal{H} \longrightarrow \mathcal{D}^M\,,$$

(see the proof of [24, Proposition 5.4]). From this the estimates follow from the estimates on the characteristic values of $\chi_2(P^\sharp(h)-z_0)^{-1}\chi_1$ which in turn follow from (2.1).

Greater difficulty lies in estimating the $K_0\chi$ term, where we encounter exponential growth. We start by observing that for $\operatorname{Im} z \geq 0$,

$$\mu_j([h^2\Delta,\chi_2]\chi R_0(z,h)\chi) \leq Ch\left(\frac{j}{h}\right)^{-\frac{1}{n^\sharp}}$$

(see, for instance, [34, Lemma 4]). For $\operatorname{Im} z < 0$ we write

$$\chi R_0(z,h)\chi = \chi(-h^2\Delta - z)^{-1}\chi + \chi(R_0(z,h) - (-h^2\Delta - z)^{-1})\chi\,,$$

where $R_0(z,h)$ is the meromorphic continuation of the resolvent from $\operatorname{Im} z > 0$ and $(-h^2\Delta - z)^{-1}$ is the resolvent, holomorphic on L^2 for $z \in \mathbb{C}\setminus\mathbb{R}_+$. This reduces the problem to estimating the characteristic values of $\chi(R_0(z,h)-(-h^2\Delta-z)^{-1})\chi$. We rewrite this operator using the standard representation of the spectral projection (see for instance the proof of [34, Lemma 1]):

$$\chi(R_0(z,h)-(-h^2\Delta-z)^{-1})\chi = \tilde{c}_n h^{-n} z^{\frac{n-2}{2}}\, {}^t\mathbb{E}_+^\chi(z,h)\mathbb{E}_-^\chi(z,h)\,,$$

where $\mathbb{E}_\pm^\chi$ are as in (2.4). Hence,

$$\mu_j(\chi(R_0(z,h)-(-h^2\Delta-z)^{-1})\chi) \leq |\tilde{c}_n||z|^{\frac{n-2}{2}} h^{-n} \|{}^t\mathbb{E}_+^\chi(z,h)\|\mu_j(\mathbb{E}_-^\chi(z,h))$$

and we estimate

$$\begin{aligned}\mu_j(\mathbb{E}_-^\chi(z,h)) &= \mu_j((I-\Delta_{\mathbb{S}^{n-1}})^{-k}(I-\Delta_{\mathbb{S}^{n-1}})^k\mathbb{E}_-^\chi(z,h)) \\ &\leq \mu_j((I-\Delta_{\mathbb{S}^{n-1}})^{-k})\|(I-\Delta_{\mathbb{S}^{n-1}})^k\mathbb{E}_-^\chi(z,h)\| \\ &\leq C^k j^{-\frac{2k}{n-1}}(2k)!\exp(C/h) \\ &\leq C\exp(Ch^{-1}-j^{\frac{1}{n-1}}/C)\,,\end{aligned} \tag{4.3}$$

where we used the Cauchy inequalities and then optimized in k. By summing up the contributions from different terms in K, we obtain the following estimate on the determinant

$$M(h) = \mathcal{O}(e^{Ch^{-n^\sharp}})\,. \tag{4.4}$$

Since $I+K(z_0,z_0,h)^{n^\sharp+1}$ can be inverted by Neumann series, and since the same estimates hold for

$$(I+K(z_0,z_0,h)^{n^\sharp+1})^{-1} = I-K(z_0,z_0,h)^{n^\sharp+1}(I+K(z_0,z_0,h)^{n^\sharp+1})^{-1}\,,$$

we can estimate $f(z_0,h)^{-1}$ so we get

$$|f(z_0,h)| > e^{-Ch^{-n^\sharp}}\,. \tag{4.5}$$

Let us now put $\Omega_0 = D(z_0,r)$, $\overline{\Omega}_0 \subset \widetilde{\Omega}$, $|\operatorname{Im} z_0| < r < \operatorname{Re} z_0$ (choosing z_0 appropriately for that). Let $N(h)$ be the number of zeros, $w_j(h)$, of $f(z,h)$ in $\overline{D(z_0,r+\epsilon)} \subset D(z_0,r+2\epsilon) \subset \widetilde{\Omega}$. Then by the Jensen inequality

$$\begin{aligned}N(h) &\leq C_\epsilon(\max_{D(z_0,r+2\epsilon)}\log|f(z,h)|-\log|f(z_0,h)|) \\ &\leq C_\epsilon(\log M(h)-\log|f(z_0,h)|) \leq Ch^{-n^\sharp}\,.\end{aligned} \tag{4.6}$$

By Lemma 3.1, we can cover Ω by discs centered at $\tilde{z}$ at which (4.5) holds with z_0 replaced by $\tilde{z}$. Hence by repeating the argument we obtain (4.1). $\square$

The next result holds in greater generality (see [27, Lemma 1] and references given there), but we will give a direct argument following directly from the proof of Proposition 4.1:

Proposition 4.2. *If Ω is as in* (4.1)*, then for $0 < h \leq h_0$ we have*

$$\begin{gathered}\|\chi R(z,h)\chi\|_{\mathcal{H}\to\mathcal{H}} \leq C_\Omega\exp(C_\Omega h^{-n^\sharp}\log(1/F(h)))\,, \\ z \in \Omega \setminus \bigcup_{z_j(h)\in\operatorname{Res}P(h)} D(z_j(h),F(h))\,,\quad 0 < F(h) \ll 1\,,\end{gathered}$$

where $R(z,h)$ is the meromorphically continued resolvent, $z_j(h)$ are the resonances of $P(h)$ and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\chi \equiv 1$ near $B(0,R_0)$.

Proof. To estimate the resolvent we now use, with the notation of the proof of Proposition 4.1 the following inequality

$$\begin{aligned}\|\chi R(z,h)\chi\| &\leq (\|Q_0\chi\| + \|Q_1\chi\|)\|(I + K(z_0,z,h))^{-1}\| \\ &\leq (\|Q_0\chi\| + \|Q_1\chi\|)\frac{\det(I + (K^*K)^{\frac{n^\sharp+1}{2}})}{|\det(I + K^{n^\sharp+1})|} \leq e^{Ch}M(h)|f(z,h)|^{-1}\,.\end{aligned}$$

Here in the second inequality we have used [9, Chapter V, Theorem 5.1]. Hence the problem is reduced to lower bounds on $|f(z,h)|$. We could apply Lemma 3.1 but instead we trade the quality of the lower bound for an explicit characterization of the exceptional set.

Thus, again with the same notation as in the proof of Proposition 4.1, we write

$$f(z,h) = e^{g(z,h)}\prod_{j=1}^{N(h)}(z - w_j(h))\,, \quad z \in D(z_0,r)\,,$$

where $g(z,h)$ is holomorphic in $D(z_0,r)$ and $w_j(z)$ are the zeros of $f(z,h)$ in $D(z_0,r+\epsilon)$. Next using the estimate (3.2) for $\prod_{j=1}^{N(h)}(z - w_j(h))$ with some $\eta_0 > 0$, the bound (4.6) for $N(h)$, the estimate (4.4) and the maximum principle for the harmonic function $\mathrm{Re}\, g(z,h)$, we deduce an upper bound $\mathrm{Re}\, g(z,h) \leq Ch^{-n^\sharp}$, $z \in D(z_0,r)$. Carathéodory's inequality (see for instance [29, 5.5]) gives

$$\max_{|z-z_0|=\rho}|g(z,h)| \leq \frac{2\rho}{r-\rho}\max_{|z-z_0|=r}\mathrm{Re}\, g(z,h) + \frac{r+\rho}{r-\rho}|g(z_0,h)|\,, \quad r > \rho\,. \tag{4.7}$$

Taking $0 < \mathrm{Im}\, z_0 < \mathrm{Re}\, z_0$, $z_0 \notin \bigcup_{j=1}^{N(h)} D(w_j(h), F(h))$, we get $\log|f(z_0,h)| > -Ch^{-n^\sharp}$ and

$$\log|\prod_{j=1}^{N(h)}(z_0 - w_j(h))| \geq N(h)\log F(h) \geq -Ch^{-n^\sharp}\log\frac{1}{F(h)}\,,$$

which yields

$$|\,\mathrm{Re}\, g(z_0,h)| \leq Ch^{-n^\sharp}\log(1/F(h))\,.$$

We can choose appropriately $\mathrm{Im}\, g(z_0,h)$ so that $|g(z_0,h)| \leq Ch^{-n^\sharp}\log(1/F(h))$, and that gives the lower bound

$$\log|f(z,h)| \geq -Ch^{-n^\sharp}\log(1/F(h))) \ \text{for}\ z \in D(z_0,\rho) \setminus \bigcup_{j=1}^{N(h)} D(w_j(h), F(h))\,.$$

Now recall that the resonances $z_j(h)$ are included in the set of zeros of $f(z,h)$, so applying the maximum principle for the operator-valued holomorphic function $\chi R(z,h)\chi$, outside the discs centered at $z_j(h)$, we obtain the conclusion of the proposition for $z \in D(z_0,\rho) \setminus \bigcup_{z_j(h)\in \mathrm{Res} P(h)} D(z_j(h), F(h))$. Covering Ω by discs and using the successive lower bounds for $|f(z,h)|$ gives the result for general domains. □

We now give the crucial estimate on the scattering determinant. It generalizes the estimate given in [34, Proposition 2, (14)]. Its interest comes from its *universality*: it does not depend in any way on the structure of the perturbation, only on the size of its support:

Lemma 4.3. *If* $s(z,h) = \det S(z,h)$ *and* Ω *is as in* (4.1)*, then*

$$|s(z,h)| \leq Ce^{Ch^{-n}}\,, \quad z \in \Omega \cap \overline{\mathbb{C}}_+\,, \quad C = C(R_0,\Omega)\,. \tag{4.8}$$

Proof. This is an almost immediate consequence of Proposition 2.1, the estimates (4.3), the resolvent estimate of Proposition 4.2.

As before, we use Weyl inequalities to have

$$|s(z,h)| \leq \prod_{j=1}^{\infty}(1+\mu_j(A(z,h)))\,.$$

For $\operatorname{Im} z > h^M$, M fixed, we have that $R(z,h) = \mathcal{O}(h^{-M}) : \mathcal{H} \to \mathcal{H}$, and the equation, $(1-\chi_1)(-h^2\Delta - z)R(z,h) = 1 - \chi_1$, then gives

$$\|[h^2\Delta, \chi_2]R(z,h)[h^2\Delta, \chi_1]\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = \mathcal{O}(h^{-M+2})\,.$$

Hence,

$$\mu_j(A(z,h)) \leq C_n(\Omega)h^{-n-M+2}\|\mathbb{E}_+^{\chi_3}(z,h)\|\mu_j(\mathbb{E}_-^{\chi_3}(z,h))\,.$$

We now use the estimate (4.3) to obtain

$$\mu_j(A(z,h)) \leq \exp(C_M h^{-1} - j^{\frac{1}{n-1}}/C)\,.$$

Consequently, the product of $1+\mu_j(A(z,h))$ over $j \geq Ch^{-n+1}$ is bounded by $\exp(Ch^{-1})$, which implies that

(4.9)
$$|s(z,h)| \leq e^{Ch^{-1}}(1+\|A(z,h)\|)^{Ch^{-n+1}} \leq e^{C(\operatorname{Im}\sqrt{z}h^{-n}+(n+M-2)\log(1/h)h^{-n+1})}\,, \quad \operatorname{Im} z > h^M\,,$$

where, as in the proof of Proposition 2.3, we estimated $\|A(z,h)\|$ by $Ch^{-n-M+2}\exp(C\operatorname{Im}\sqrt{z}/h)$.

Since $|s(z,h)| = 1$ for $z \in \mathbb{R}$, we can apply a version of the three lines theorem given in Lemma 3.5 to conclude the proof. For that we need some weak estimate valid everywhere and we claim that

$$|s(z,h)| \leq e^{Ch^{-(n^\sharp+1)n}}\,, \quad \operatorname{Im} z \geq 0\,, \quad z \in \Omega_\epsilon = \Omega + D(0,\epsilon)\,. \tag{4.10}$$

In fact, Proposition 4.2 shows that for every $x \in \Omega_\epsilon \cap \mathbb{R}$ we can find $x' \in \Omega_{2\epsilon} \cap \mathbb{R}$ such that $|x - x'| < \epsilon$ and for $z \in x' + i[0, h^M]$ and $0 < h \leq h(\epsilon)$ we have

$$\|\chi R(z,h)\chi\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(e^{h^{-n^\sharp-1}})\,.$$

Hence

$$\|[h^2\Delta, \chi_2]R(z,h)[h^2\Delta, \chi_1]\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = \mathcal{O}(e^{h^{-n^\sharp-1}})\,.$$

Consequently, by the same argument as above,

$$\mu_j(A(z,h)) \leq \exp(Ch^{-n^\sharp-1} - j^{\frac{1}{n-1}}/C)\,,$$

which proves (4.10) and concludes the proof of the proposition. □

As recalled in Sect.2, the poles of the scattering determinant are given by the poles of the resolvent, away from the real axis. That, and the unitarity (2.3), immediately imply a factorization of the scattering determinant. The issue is the estimate on the non-vanishing term in that factorization and this is addressed in

Proposition 4.4. *Let $s(z,h) = \det S(z,h)$ be the scattering determinant. Then*

$$s(z,h) = e^{g(z,h)} \frac{\overline{P(\bar z,h)}}{P(z,h)}, \quad |g(z,h)| \leq \begin{cases} C_\epsilon h^{-n^\sharp}, & n \geq 1 \\ C_\epsilon (N(h) + h^{-n}), & n \geq 5 \end{cases}, \quad z \in R,$$

$$P(z,h) = \prod_{w \in \mathrm{Res}\,(P(h)) \cap R_\epsilon} (z - w), \tag{4.11}$$

$$N(h) = \#\Big(\mathrm{Res}\,(P(h)) \cap R_\epsilon\Big),$$

where $g(z,h)$ is holomorphic in R_ϵ and

$$R = (a,b) + i(-c,c), \quad 0 < a < b, \ 0 < c, \quad R_\epsilon = R + D(0,\epsilon).$$

In particular for $n = n^\sharp$ we have an improved estimate

$$|s(z,h)| \leq C e^{C \,\mathrm{Im}\, z\, h^{-n}}, \quad z \in R \cap \overline{\mathbb{C}}_+ . \tag{4.12}$$

To obtain this proposition we need the following

Lemma 4.5. *For any $\Omega = [a,b] + i(0,c)$, $0 < a < b$, $c > 0$, there exist $\delta > 0$ and C, such that for any $0 < h \leq h_0$, there exists $z_0 = z_0(h)$, which satisfies*

$$\log |s(z_0,h)| \geq -C h^{-n^\sharp}, \quad z_0 \in \Omega, \quad \mathrm{Im}\, z_0 > \delta, \tag{4.13}$$

The constant C in (4.13) *depends on $P(h)$.*

Proof. If we factorize $s(z,h)$ as in (4.11), then Cartan's lemma (3.2) and the bound on the number of resonances (4.1) show that we need to find z_0 for which $\mathrm{Re}\, g(z_0,h) \geq -C h^{-n^\sharp}$.

We normalize $g(z,h)$ by assuming that

$$|g(\tilde a, h)| \leq 2\pi, \quad a < \tilde a < b,$$

and note that Lemma 4.3 and Cartan's lemma imply that

$$\mathrm{Re}\, g(z,h) \leq C h^{-n^\sharp}, \quad z \in \overline{\Omega} \cap \{\mathrm{Im}\, z \geq 0\}.$$

We claim that

$$|\,\mathrm{Im}\, g(z,h)| \leq C h^{-n^\sharp}, \quad z \in R \cap \mathbb{R}.$$

In fact, for λ real we have

$$\mathrm{Im}\, g(\lambda,h) - \mathrm{Im}\, g(\tilde a,h) = 2\pi(\sigma(\lambda,h) - \sigma(\tilde a,h)) + 2 \sum_{w \in \mathrm{Res}\,(P(h)) \cap R_\epsilon} \int_{\tilde a}^{\lambda} \frac{\mathrm{Im}\, w}{|w - t|^2} dt.$$

Using (4.1) and the estimate

$$\int_{\tilde a}^{\lambda} \frac{y}{y^2 + (x-t)^2} dt \leq \pi,$$

we see that the second term on the right hand side is $\mathcal{O}(h^{-n^\sharp})$. The first term satisfies the same estimate in view of (2.11).

If we put $f_h(z) = g(z,h) h^{n^\sharp}$, then we know that

$$\overline{f_h(\bar z)} = -f_h(z), \ z \in R, \quad |f_h(z)| \leq C_1, \ z \in R \cap \mathbb{R}, \quad \mathrm{Re}\, f_h(z) \leq C, \ z \in R \cap \{\mathrm{Im}\, z \geq 0\},$$

and we want to show that

$$\exists\, \delta > 0, \ C_2 > 0, \ \forall\, 0 < h \leq h_0, \ \exists\, z_0 = z_0(h) \in R, \quad \mathrm{Im}\, z_0 > \delta, \ \mathrm{Re}\, f_h(z_0) \geq -C_2. \tag{4.14}$$

If not, we would have a sequence of holomorphic functions g_N such that

$$\overline{g_N(\bar z)} = -g_N(z)\,, \ z\in R\,, \quad |g_N(z)|\le C_1\,, \ z\in R\cap\mathbb{R}\,,$$
$$\operatorname{Re} g_N \le -N\,, \quad \operatorname{Im} z > 1/N\,, \quad \operatorname{Re} g_N \le C\,, \quad \operatorname{Im} z\ge 0\,.$$

The Poisson formula applied as in the proof of Lemma 3.4 shows that

$$\operatorname{Re} g_N \le -N\operatorname{Im} z/C\,, \quad z\in D(\tilde a,\rho)\,, \quad \operatorname{Im} z\ge 0\,,$$

for some ρ and C independent of N. Since $\operatorname{Re} g_N|_{\mathbb{R}}=0$ we conclude that $\partial_{\operatorname{Im} z}\operatorname{Re} g_N|_{\mathbb{R}}\le -N/C$. From Cauchy-Riemman equations we now get

$$\partial_{\operatorname{Re} z}\operatorname{Im} g_N(z)\ge N/C\,, \quad z\in D(\tilde a,\rho)\cap\mathbb{R}\,,$$

and that contradicts the uniform boundedness of g_N on $R\cap\mathbb{R}$. Hence (4.14) holds and the lemma is proved. ☐

When the dimension is large enough we obtain the following stronger result:

Lemma 4.6. *Suppose that $n\ge 5$. For any $\Omega=[a,b]+i(0,c)$, $0<a<b$, $c>0$, there exist $\delta>0$ and C, such that for any $0<h\le h_0$, there exists $z_0=z_0(h)$, which satisfies*

$$\log|s(z_0,h)|\ge -Ch^{-n}\,, \quad z_0\in\Omega\,, \quad \operatorname{Im} z_0>\delta\,. \tag{4.15}$$

The constant C depends only on Ω and the support of the perturbation, $B(0,R_0)$.

Proof. We first make the following observation based on the proof of Lemma 4.3: fix any $H>0$, then for any $0<h\le H$, we have, for any compact set $\Omega_1\Subset\mathbb{C}$,

$$|s(z,h)|\le Ce^{Ch^{-n}} \quad z\in\Omega_1\cap\{\operatorname{Im} z>\min(1/C,h^M)\}\,, \tag{4.16}$$

where the constants depend *only* on M, H, and R_0. Put $P_\rho(h)=\rho^{-1}P(\sqrt{\rho}h)$, $\rho>0$. Then $P(h)|_{\mathbb{R}^n\setminus B(0,R_0)}=P_\rho(h)|_{\mathbb{R}^n\setminus B(0,R_0)}$ and $P_\rho(h)$ satisfies the black box assumptions of Sect.2 (without uniformity with respect to ρ). If $s_\rho(z,h)$ is the scattering determinant corresponding to $P_\rho(h)$, then we have the following relation:

$$s(w\rho,h)=s_\rho(w,h/\sqrt{\rho})\,.$$

We can now apply (4.16) to s_ρ and that gives

$$|s_\rho(w,h/\sqrt{\rho})|\le C\exp(Ch^{-n}\rho^{\frac n2})\,, \quad w\in\Omega_1\Subset\mathbb{C}, \quad \operatorname{Im} w>(h/\sqrt{\rho})^M\,.$$

By scaling, using $\rho\sim|z|$, this implies that

$$|s(z,h)|\le C\exp(Ch^{-n}|z|^{\frac n2})\,, \quad \operatorname{Im} z>h^2/C, \ \operatorname{Re} z>0\,,$$

if we take $M>2$. We now put $f_h(w)=s(h^2w,h)$, which in view of the previous estimate satisfies

$$\log|f_h(w)|\le C|w|^{\frac n2}+C\,, \quad \operatorname{Im} w>1/C\,, \quad \operatorname{Re} w>0\,,$$

uniformly with respect to h. Proposition 2.3 shows that there exist many $\tilde w$'s, $|\tilde w|\le 1$, $\operatorname{Im}\tilde w>1/C$, such that

$$\log|f_h(\tilde w)|\ge -C$$

holds with a constant independent of h. We can now apply Lemma 3.2 and conclude that

$$\log|f_h(re^{i\theta})|>-Cr^{n/2}\,, \ r>r_0\,, \ \theta\in(0,\theta_0)\setminus\Sigma(r,h)\,, \ |\Sigma(r,h)|<\delta_0\,.$$

This implies the existence of $z_0(h)=h^2w_0(h)$, $\operatorname{Im} w_0(h)>\delta/h^2$, $|w_0(h)|\le C/h^2$, such that $z_0(h)$ satisfies the conditions in (4.15), and

$$\log|s(z_0(h),h)|=\log|f_h(w_0(h))|>-C|w_0(h)|^{\frac n2}>-Ch^{-n}\,.$$

☐

Proof of Proposition 4.4: Since we clearly have a factorization given in (4.11), the only thing to check is the estimate on $g(z,h)$. The slight difference with the standard arguments lies in having estimates on $|s(z,h)|$ for $\operatorname{Im} z \geq 0$ only. The unitarity implies however that $g(z,h) = -\overline{g(\bar{z},h)}$, and hence we only need to estimate g for $\operatorname{Im} z \geq 0$. In that region, the bound (4.8), an application of Cartan's lemma (3.2), and the maximum principle give

$$\operatorname{Re} g(z,h) \leq C_1(h^{-n} + N(h)) \,, \ \operatorname{Im} z \geq 0 \,, \quad z \in R_{\epsilon/2} \,.$$

Lemmas 4.5 and 4.6, and the trivial bound

$$\frac{|z-\bar{w}|}{|z-w|} \leq 1 \,, \quad \operatorname{Im} z \geq 0 \,, \quad \operatorname{Im} w \leq 0 \,, \tag{4.17}$$

give an existence of $z_0 = z_0(h) \in R$, $\operatorname{Im} z_0 \geq \delta > 0$, such that

$$\operatorname{Re} g(z_0,h) \geq \begin{cases} -C_2 h^{-n^\sharp} & n \geq 1 \\ -C_2 h^{-n} & n \geq 5 \end{cases} .$$

When $n \geq 5$, Harnack's inequality, applied to the harmonic function $G(z,h) = 2C_1(h^{-n} + N(h)) - \operatorname{Re} g(z,h)$, positive for $\operatorname{Im} z \geq 0$, $z \in R_{\epsilon/2}$, shows that

$$|\operatorname{Re} g(z,h)| \leq \frac{1}{\rho} C(N(h) + h^{-n}) \,, \ \ z \in R_{\epsilon/4}, \ \ \operatorname{Im} z > \rho \,. \tag{4.18}$$

In fact, if $0 < \rho < \operatorname{Im} z_0$ is such that $D(z_0, \operatorname{Im} z_0 - \rho) \subset R_\epsilon$, we have

$$\max_{z \in D(z_0, \operatorname{Im} z_0 - \rho)} G(z,h) \leq \frac{2|z_0|}{\rho} G(z_0,h) \leq \frac{2|z_0|}{\rho} \left((2C_1 + C_2) h^{-n} + 2C_1 N(h) \right) \,.$$

Using this inequality with different ρ and z_0, we get the bound (4.18) for all $z \in R$, $\operatorname{Im} z > \rho$.

In view of (4.18), we can apply Lemma 3.4 to $u(z,h) = (h^{-n} + N(h))^{-1} \operatorname{Re} g(z,h)$ and deduce the estimate

$$|\operatorname{Re} g(z,h)| \leq C(h^{-n} + N(h)) |\operatorname{Im} z|, \ \ z \in R_{\epsilon/4} \tag{4.19}$$

which combined with the Carathéodory inequality gives the bound

$$|g(z,h)| \leq C(h^{-n} + N(h)), \ \ z \in R \,. \tag{4.20}$$

Recalling (4.17), it also gives (4.12). We proceed similarly for lower dimensions. $\square$

5. Local trace formula for resonances

As an application of the results of Sect.4 we present a proof of a slight improvement of Sjöstrand's local trace formula in the setting of semi-classical compactly supported perturbations. We stress that it depends only on the upper bound on the number of resonances (4.1), the factorization of the scattering determinant (4.11), and on the Birman-Krein formula (2.10). It is essentially a localized version of the arguments of [11] and [35].

Theorem 1. *Suppose that $P(h)$ satisfies the assumptions of Sect.2. Let Ω, $\overline{\Omega} \Subset \{\operatorname{Re} z > 0\}$, be an open, simply connected set such that $\Omega \cap \mathbb{R}$ is connected. Suppose that f is holomorphic on a neighborhood of Ω and that that $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfies*

$$\psi(\lambda) = \begin{cases} 0, & d(\Omega \cap \mathbb{R}, \lambda) > 2\epsilon, \\ 1, & d(\Omega \cap \mathbb{R}, \lambda) < \epsilon, \end{cases}$$

where $\epsilon > 0$ is sufficiently small. Then

$$\begin{aligned}\widetilde{\mathrm{tr}}(\psi f)(P(h)) &= \sum_{z \in \mathrm{Res}\,(P(h)) \cap \Omega} f(z) + E_{\Omega,f,\psi}(h)\,, \\ |E_{\Omega,f,\psi}(h)| &\leq M(\psi,\Omega) h^{-n^\sharp} \sup\{|f(z)| \;:\; 0 < d(z,\Omega) < 2\epsilon\,,\ \mathrm{Im}\, z \leq 0\}\,,\end{aligned} \tag{5.1}$$

where $\widetilde{\mathrm{tr}}$ *is defined in* (2.9) *and* $n^\sharp$ *is as in* (4.1).

Remark. We note that unlike in [22],[23] we only estimate the function f in the lower half plane to control the error $E_{\Omega,f,\psi}(h)$.

Proof. The Birman-Krein formula recalled in Sect.2 shows that

$$\widetilde{\mathrm{tr}}(\psi f)(P(h)) = \int (\psi f)(\lambda) \frac{d\sigma}{d\lambda}(\lambda, h) d\lambda + \sum_{\lambda \in \sigma(P(h))} (\psi f)(\lambda)\,. \tag{5.2}$$

Let $\tilde{\psi} \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{C})$ be an almost analytic extension of ψ satisfying

$$\mathrm{supp}\, \bar{\partial}_z \tilde{\psi} \subset \{z \;:\; \epsilon < d(z,\Omega) < 2\epsilon\}\,,$$

which can certainly be arranged. We note that this implies that $\tilde{\psi} \equiv 1$ on Ω. An application of Green's formula gives

$$\widetilde{\mathrm{tr}}(\psi f)(P(h)) = \sum_{z \in \mathrm{Res}\,(P(h))} (\tilde{\psi} f)(z) + \frac{1}{\pi} \int\!\!\int_{\mathbb{C}_-} \bar{\partial}_z \tilde{\psi}(z) f(z) \frac{\partial_z s(z,h)}{s(z,h)} \mathcal{L}(dz)\,,$$

where we used the definition of the scattering phase $\sigma(\lambda, h)$ given in (2.8), and where $\mathcal{L}(dz)$ denotes the Lebesgue measure on $\mathbb{C}$. Notice that if $\lambda \in \sigma(P(h))$, then $\lambda \in \mathrm{Res}\,(P(h))$ so the eigenvalues are included in the first term. On the other hand, $\partial_z s(z,h)/s(z,h)$ is regular on $\Omega \cap \mathbb{R}$ which justifies the application of Green's formula.

We first note that the properties of $\tilde{\psi}$ and Proposition 4.1 show that

$$\sum_{z \in \mathrm{Res}\,(P(h))} (\tilde{\psi} f)(z) = \sum_{z \in \mathrm{Res}\,(P(h))} f(z) + \mathcal{O}(h^{-n^\sharp}) \sup\{|f(z)| \;:\; 0 < d(z,\Omega) < 2\epsilon\,,\ \mathrm{Im}\, z \leq 0\}\,.$$

Using the elementary inequality

$$\begin{aligned}\int\!\!\int_{\Omega_1} \frac{1}{|z-w|} \mathcal{L}(dz) &\leq \int\!\!\int_{D(w,\rho)} \frac{1}{|z-w|} \mathcal{L}(dz) + \int\!\!\int_{\Omega_1 \setminus D(w,\rho)} \frac{1}{|z-w|} \mathcal{L}(dz) \\ &\leq 2\pi\rho + \frac{1}{\rho}|\Omega_1| \leq 2\sqrt{2\pi|\Omega_1|}\,, \quad \rho = (|\Omega_1|/(2\pi))^{\frac{1}{2}}\,,\end{aligned} \tag{5.3}$$

(4.1) and (4.11) conclude the proof, as, with $\Omega \subset R$,

$$\left| \frac{s'(z,h)}{s(z,h)} \right| \leq |g'(z,h)| + \sum_{w \in \mathrm{Res}\,(P(h)) \cap R_\epsilon} \left(\frac{1}{|z-w|} + \frac{1}{|z-\bar{w}|} \right)\,.$$

□

6. Breit-Wigner approximation

We now establish the semi-classical version of the Breit-Wigner approximation and throughout this section we assume that $n = n^\sharp$. Again, it is a purely complex-analytic consequence of the estimate on the scattering determinant, and of the existence of a good remainder in the Weyl law for the scattering phase. It generalizes the large energy result of [18].

Theorem 2. *Suppose that $\sigma(P(h)) \cap (0,\infty) = \emptyset$, and that the spectral condition* (2.14) *holds for E in a neighbourhood of $\lambda > 0$ and for μ sufficiently small. Then for any $0 < \delta < h/C$ we have*

$$\sigma(\lambda+\delta,h) - \sigma(\lambda-\delta,h) = \sum_{\substack{|z-\lambda|<h \\ z\in \mathrm{Res}\,(P(h))}} \omega_{\mathbb{C}_-}(z,[\lambda-\delta,\lambda+\delta]) + \mathcal{O}(\delta)h^{-n}\,, \tag{6.1}$$

where

$$\omega_{\mathbb{C}_-}(z,I) = -\frac{1}{\pi}\int_I \frac{\mathrm{Im}\, z}{|z-t|^2}dt\,, \quad I \subset \mathbb{R} = \partial\mathbb{C}_-\,.$$

Remark. The assumption $\sigma(P(h)) \cap (0,\infty) = \emptyset$ is natural when we have a compactly supported black-box perturbation and $n = n^\sharp$, as it is satisfied in all reasonable situations. The arguments below could be modified to include the case of embedded eigenvalues, using (2.12) and (2.13).

To prove the theorem we start with a lemma which is the semi-classical version of [18, Proposition 2]. In yet greater generality and by a different method, the result was recently proved by Bony [2].

Lemma 6.1. *Under the assumptions of Theorem 2 we have, for $\lambda > 0$, and $h/C < \delta < 1/C$*

$$\#\,\{z \ : \ z \in \mathrm{Res}\,(P(h))\,,\ |z-\lambda| < \delta\} = \mathcal{O}(\delta h^{-n})\,. \tag{6.2}$$

Proof. We recall that the spectral assumption (2.14) implies that the scattering phase satisfies

$$\sigma(\lambda+2\delta,h) - \sigma(\lambda-2\delta,h) = \mathcal{O}(\delta h^{-n})\,.$$

As in [18, Proposition 1] we now show that

$$\sigma(\lambda+2\delta,h) - \sigma(\lambda-2\delta,h) \geq \frac{1}{2}\#\,\{z \in \mathrm{Res}\,(P(h)) \ : \ |z-\lambda| < \delta\} - \mathcal{O}(\delta h^{-n})\,, \tag{6.3}$$

which then implies the lemma.

To see (6.3), we apply (4.11) with R centered at λ, so that

$$\begin{aligned} |\sigma(\lambda+2\delta,h) - \sigma(\lambda-2\delta,h)| &= \left| \frac{1}{2\pi i}\int_{\lambda-2\delta}^{\lambda+2\delta} \frac{s'(t,h)}{s(t,h)}dt \right| \\ &= \frac{1}{2\pi}\left| \int_{\lambda-2\delta}^{\lambda+2\delta} \left(g'(t,h) - \sum_{z\in \mathrm{Res}\,(P(h))\cap R_\epsilon} \frac{2i\,\mathrm{Im}\, z}{|z-t|^2} \right) dt \right| \\ &\geq \frac{1}{\pi}\int_{\lambda-2\delta}^{\lambda+2\delta} \sum_{z\in \mathrm{Res}\,(P(h))\cap R_\epsilon} \frac{|\,\mathrm{Im}\, z|}{|z-t|^2} - \mathcal{O}(\delta h^{-n}) \\ &\geq \frac{1}{2}\#\,\{z \in \mathrm{Res}\,(P(h)) \ : \ |z-\lambda| < \delta\} - \mathcal{O}(\delta h^{-n})\,, \end{aligned} \tag{6.4}$$

since for $0 < y < \delta$ and $|x-\lambda| < \delta$ we have

$$\int_{\lambda-2\delta}^{\lambda+2\delta} \frac{y}{(x-t)^2+y^2}dt \geq \int_{-\delta/y}^{\delta/y} \frac{1}{1+r^2}dr \geq \frac{\pi}{2}\,.$$

□

We need one more lemma which is a h-local version of Proposition 4.5:

Lemma 6.2. *Let* $\Omega(h) = \{z : |z-\lambda| \leq C_1 h\}$, $\lambda > 0$, *and, for* $|z-\lambda| < C_2 h$, $0 < C_2 < C_1$, *put*

$$s(z,h) = e^{g_\lambda(z,h)} \frac{\overline{P_\lambda(\bar{z},h)}}{P_\lambda(z,h)}, \quad P_\lambda(z,h) = \prod_{w \in \mathrm{Res}\,(P(h)) \cap \Omega(h)} (z-w)\,.$$

Then under the assumptions of Theorem 2 we can choose g_λ *so that*

$$|g_\lambda(z,h)| \leq C h^{-n+1}, \ |z-\lambda| \leq C_2 h\,.$$

Proof. We will use the factorization in Proposition 4.4 in the domain $R_\epsilon = \Omega = (\lambda/2, 3\lambda/2) + i(-c,c)$, $c > 0$ and we denote by $g(z,\lambda)$ the corresponding holomorphic function and recall that $P(z,h) = \prod_{w \in \mathrm{Res}\,(P(h)) \cap \Omega}(z-w)$. Comparing the expressions for $s(z,h)$, we see that

$$g_\lambda(z,h) = g(z,h) + \log \frac{\overline{P(\bar{z},h)} P_\lambda(z,h)}{\overline{P_\lambda(\bar{z},h)} P(z,h)}\,,$$

and we need to show that the second term on the right hand side is bounded by Ch^{-n+1} for $|z-\lambda| < C_2 h$. In fact, the real part of the first term is bounded by $C|\operatorname{Im} z| h^{-n} = \mathcal{O}(h^{-n+1})$ because of (4.19) and by Carathéodory inequality we conclude that this term is $\mathcal{O}(h^{-n+1})$.

Now we will show that

$$\begin{aligned} &\left| \frac{d}{dz} \log \frac{\overline{P(\bar{z},h)} P_\lambda(z,h)}{P(z,h)\overline{P_\lambda(\bar{z},h)}} \right| = \\ (6.5) \qquad &\left| \sum_{\substack{w \in \mathrm{Res}\,(P(h)) \cap \Omega \\ |w-\lambda| > C_1 h}} \left(\frac{1}{z-\bar{w}} - \frac{1}{z-w} \right) \right| \leq C h^{-n}\,, \end{aligned}$$

for $|z-\lambda| < C_2 h$, from which the needed estimate follows by integration and a choice of the branch of logarithm.

To see (6.5), we proceed as in [19] and rewrite the expression to be estimated as

$$(6.6) \qquad \sum_{\substack{w \in \mathrm{Res}\,(P(h)) \cap \Omega \\ |w-\lambda| > C_1 h}} \left(\frac{2|\operatorname{Im} w|}{|\operatorname{Re} z - w|^2} + \int_0^{\operatorname{Im} z} \left(\frac{1}{((\operatorname{Re} z + iy) - w)^2} - \frac{1}{((\operatorname{Re} z + iy) - \bar{w})^2} \right) dy \right).$$

The sum of the integrated terms is harmless as

$$\begin{aligned} \sum_{\substack{w \in \mathrm{Res}\,(P(h)) \cap \Omega \\ |w-\lambda| > C_1 h}} \frac{1}{|z-w|^2} &\leq \sum_{\substack{w \in \mathrm{Res}\,(P(h)) \cap \Omega \\ |w-z| > (C_1 - C_2) h}} \frac{1}{|z-w|^2} \\ &\leq \sum_{k=1}^{C \log(1/h)} \sum_{C_3 2^k h \leq |z-w| < C_3 2^{k+1} h} \frac{1}{(C_3 2^k h)^2} \\ &\leq C \sum_{k=1}^{C\log(1/h)} \frac{(2^{k+1}h) h^{-n}}{(2^k h)^2} \\ &\leq 2C h^{-n-1} \sum_{k=1}^{\infty} \frac{1}{2^k} \leq \widetilde{C} h^{-n-1}\,, \end{aligned}$$

by Lemma 6.1. Since $|\operatorname{Im} z| < C_2 h$, an integration in y adds an additional multiple of h, giving the desired bound $\mathcal{O}(h^{-n})$.

The first term in (6.6) is estimated using Carleman inequality (see Lemma 3.3):

$$\sum_{\substack{w\in \mathrm{Res}\,(P(h))\cap\Omega \\ |w-\lambda|>C_1h}} \frac{|\operatorname{Im} w|}{|\operatorname{Re} z - w|^2} \leq C\left(\frac{1}{r}\int_0^\pi \log|s(\lambda+re^{i\theta},h)|\sin\theta d\theta \right.$$
$$\left. -\frac{1}{h}\int_0^\pi \log|s(\lambda+C_1he^{i\theta},h)|\sin\theta d\theta\right)\,,$$

where we used the fact that $|s(z,h)|=1$ for z real and $r>0$ is chosen so that $\Omega\subset\{w\in\mathbb{C}: |w-\lambda|<r\}$. By Lemma 4.3 the first integral is bounded from above by Ch^{-n}. To estimate the absolute value of the second integral, we rewrite it as follows. We put $\Omega_{\lambda,h}=\{z:\operatorname{Im} z\geq 0\,, |z-\lambda|\leq C_1h\}$, define $\Gamma_{\lambda,h}$, as its boundary, denote by $\mathcal{L}(dz)$ the Lebesgue measure on $\mathbb{C}$ and use Green's formula:

$$\begin{aligned}\frac{C_1}{h}\int_0^\pi \log|s(\lambda+C_1he^{i\theta},h)|\sin\theta d\theta &= -\frac{1}{h^2}\operatorname{Re}\int_{\Gamma_{\lambda,h}}\log|s(z,h)|dz \\ &= -\frac{1}{h^2}\operatorname{Re}\int\int_{\Omega_{\lambda,h}} 2i\,\bar\partial_z\log|s(z,h)|\mathcal{L}(dz) \\ &= \frac{1}{h^2}\operatorname{Re}\int\int_{\Omega_{\lambda,h}} i\,\partial_z\log s(z,h)\mathcal{L}(dz)\,.\end{aligned}$$

The integrand in this last integral can be rewritten as

$$(6.7)\qquad \frac{i}{h^2}\left(g'(z,h) - \sum_{\substack{w\in \mathrm{Res}\,(P(h))\cap\Omega \\ |w-\lambda|\leq C_1h}}\left(\frac{1}{z-w}-\frac{1}{z-\bar w}\right) - \sum_{\substack{w\in \mathrm{Res}\,(P(h))\cap\Omega \\ |w-\lambda|> C_1h}}\left(\frac{1}{z-w}-\frac{1}{z-\bar w}\right)\right)\,.$$

The integral of the first term is estimated by

$$\frac{1}{h^2}\int\int_{\Omega_{\lambda,h}}|g'(z,h)|\mathcal{L}(dz)\leq Ch^{-n-2}|\Omega_{\lambda,h}|\leq \widetilde{C}h^{-n}\,,$$

and that of the second one by

$$\begin{aligned}&\frac{1}{h^2}\left|\int\int_{\Omega_{\lambda,h}}\sum_{\substack{w\in \mathrm{Res}\,(P(h))\cap\Omega \\ |w-\lambda|\leq C_1h}}\left(\frac{1}{z-w}-\frac{1}{z-\bar w}\right)\mathcal{L}(dz)\right| \\ &\leq \frac{1}{h^2}\int\int_{\Omega_{\lambda,h}}\sum_{\substack{w\in \mathrm{Res}\,(P(h))\cap\Omega \\ |w-\lambda|\leq C_1h}}\Big(\frac{1}{|z-w|}+\frac{1}{|z-\bar w|}\Big)\mathcal{L}(dz) \\ &\leq \frac{C}{h^2}|\Omega_{\lambda,h}|^{\frac12}h^{-n+1}\leq Ch^{-n}\,,\end{aligned}$$

where we used (5.3) and Lemma 6.1.

It remains to estimate the integral of the last term in (6.7) (the sum over $|w-\lambda|>C_1h$). That term is exactly the left hand side of (6.5), and we rewrite it again as in (6.6)[3] The second

[3]There is something seemingly circular about this argument: we are estimating the left hand side of (6.5) by its integral! The gain comes precisely from that integration.

term in (6.6) is treated the same way as before, and the first term is estimated using (6.4):

$$\frac{1}{h^2}\int\int_{\Omega_{\lambda,h}} \sum_{\substack{w\in \mathrm{Res}\,(P(h))\cap\Omega \\ |w-\lambda|>C_1 h}} \frac{|\operatorname{Im} w|}{|\operatorname{Re} z - w|^2}\mathcal{L}(dz) \leq \frac{C}{h}\int_{\lambda-Ch}^{\lambda+Ch} \sum_{\substack{w\in \mathrm{Res}\,(P(h))\cap\Omega \\ |w-\lambda|>C_1 h}} \frac{|\operatorname{Im} w|}{|t-w|^2}dt$$
$$\leq \frac{C}{h}Ch^{-n+1} \leq \widetilde{C}h^{-n}\,,$$

and this estimate completes the proof of the lemma. $\square$

Proof of Theorem 2: In the notations of (4.11) and (6.1), and for $0<\delta<h/C$ we get

$$\begin{aligned}\sigma(\lambda+\delta,h)-\sigma(\lambda-\delta,h) &= \frac{1}{2\pi i}\int_{\lambda-\delta}^{\lambda+\delta}\frac{s'(t,h)}{s(t,h)}dt \\ &= \frac{1}{2\pi i}\int_{\lambda-\delta}^{\lambda+\delta}\left(g'_\lambda(t,h) - \sum_{\substack{z\in \mathrm{Res}\,(P(h))\cap R_\epsilon \\ |z-\lambda|<h}}\frac{2i\operatorname{Im} z}{|z-t|^2}\right)dt \\ &= \sum_{\substack{z\in \mathrm{Res}\,(P(h)) \\ |z-\lambda|<h}} \omega_{\mathbb{C}_-}(z,[\lambda-\delta,\lambda+\delta]) + \mathcal{O}(\delta)h^{-n}\,,\end{aligned}$$

which is the statement of the theorem. $\square$

By using Lemma 6.2 in place of Proposition 4.4 we obtain, under our assumptions, a slightly stronger version of the h-local trace formula of Bony and Sjöstrand [3][4]:

Theorem 3. *Let $\Omega \Subset \mathbb{C}$ be an open, simply connected neighbourhood of* 0*, such that $\Omega\cap\mathbb{R}$ is connected. Let $\chi\in\mathcal{C}_c^\infty(\mathbb{R},[0,1])$ satisfy*

$$\chi(x) = \begin{cases} 0, & d(\Omega\cap\mathbb{R},x)>2\epsilon, \\ 1, & d(\Omega\cap\mathbb{R},x)<\epsilon, \end{cases}$$

and let f be holomorphic in a neighbourhood of $\Omega_h = \lambda + h\Omega$, $\lambda\in\Omega\cap\mathbb{R}$. Then, under the assumptions of Theorem 2 we have

$$\widetilde{\mathrm{tr}}\left(\chi\left(\frac{P(h)-\lambda}{h}\right)f(P(h))\right) = \sum_{z\in \mathrm{Res}\,(P(h))\cap\Omega_h} f(z) + E_{\Omega,\lambda,f,\chi}(h)\,,$$
$$|E_{\Omega,\lambda,f,\chi}(h)| \leq M(\chi,\Omega,\lambda)h^{-n+1}\sup\left\{|f(z)|\;:\;0<d(z,\Omega_h)<2\epsilon h\,,\ \operatorname{Im} z\leq 0\right\},$$

where $\widetilde{\mathrm{tr}}$ is defined in (2.9).

Proof. We follow the proof of Theorem 1 using Lemmas 6.1 and 6.2:

$$\widetilde{\mathrm{tr}}\left(\chi\left(\frac{P(h)-\lambda}{h}\right)f(P(h))\right) =$$
$$\sum_{z\in \mathrm{Res}\,(P(h))}\tilde{\chi}((z-\lambda)/h)f(z) + \frac{1}{\pi h}\int\int_{\mathbb{C}_-}(\bar{\partial}_z\tilde{\chi})((z-\lambda)/h)f(z)\frac{\partial_z s(z,h)}{s(z,h)}\mathcal{L}(dz)\,,$$

where $\tilde{\chi}$ is an almost analytic extension of χ. As in the proof of Theorem 1, the first term, modulo the desired error, gives us the sum over resonances, while the second term is estimated

[4]That this formula is implicit in the Breit-Wigner approximation was suggested to us by J. Sjöstrand.

using

$$\frac{1}{\pi h}\int\int_{\mathbb{C}_-}|(\bar\partial_z\tilde{)}\chi((z-\lambda)/h)||g'_\lambda(z,h)|\mathcal{L}(dz)\leq\frac{1}{h}C|\Omega_h+D(0,2\epsilon h)|\max_{\Omega_h+D(0,2\epsilon h)}|g'_\lambda|\leq Ch^{-n+1}\,,$$

where by Cauchy's inequality,

$$\max_{\Omega_h+D(0,2\epsilon h)}|g'_\lambda|\leq C\max_{|z-\lambda|\leq Ch}|g_\lambda|/h=\mathcal{O}(h^{-n})\,,$$

and

$$\frac{1}{\pi h}\int\int_{\mathbb{C}_-}|(\bar\partial_z\tilde\chi)((z-\lambda)/h)|\sum_{w\in\mathrm{Res}\,(P(h))\cap(\Omega_h+D(0,2\epsilon h))}\left(\frac{1}{|z-w|}+\frac{1}{|z-\bar w|}\right)\mathcal{L}(dz)$$
$$\leq\frac{1}{h}Ch^{-n+1}|\Omega_h+D(0,2\epsilon h)|^{\frac{1}{2}}\leq\widetilde{C}h^{-n+1}\,,$$

by (5.3) and Lemma 6.1. □

Remark. It is quite likely that by reversing the argument in the proof of Theorem 3, one can deduce the Breit-Wigner approximation from the h-local trace formula of Bony and Sjöstrand [3].

7. Resonances for bottles

In the same spirit as in Sect.5, we now discuss resonances for "bottles", that is for for black box perturbations, depending on a parameter which does not change the size of the black box and keeps the Laplacian outside fixed.

In the classical case ($P(h)=h^2P$) but for a much more general class of operators, the result was proved by Sjöstrand [22],[23] by a method which did not involve scattering theory. In our approach, we exploit the fact that the constant in (4.8) depends *only* on the size of the black box, not on its "inside". For the sake of clarity we assume in this section that $\sigma(P(h))\cap(0,\infty)=\emptyset$. That is not essential, as in the general case, we add the spectral contribution to the Birman-Krein formula (2.10).

We start with a purely complex-analytic result (which for simplicity we formulate only in the context of scattering):

Proposition 7.1. *For $\gamma>0$, let $\Omega_\gamma=(a-\gamma,b+\gamma)-i(0,c)$, $0<a<b$, $0<c$, and let $z_0=z_0(h)$, $\bar z_0\in\Omega_0$ satisfy $\operatorname{Im}z_0(h)>2\delta>0$, with $0<\delta<1$ fixed. Suppose that $\psi^\pm_\epsilon\in\mathcal{C}^\infty_c(\mathbb{R};[0,1])$ have the properties that $\psi^\pm_\epsilon\equiv1$ in $\overline{\Omega}_{\pm\epsilon}\cap\mathbb{R}$ and $\operatorname{supp}\psi^\pm_\epsilon\subset\overline{\Omega}_{\epsilon\pm\epsilon}\cap\mathbb{R}$. Then we have*

$$\#\,\mathrm{Res}\,(P(h))\cap\Omega_{2\epsilon}\cap\{\operatorname{Im}z<-\delta\}\leq C_1h^{-n}-C_2\log|s(z_0(h),h)|\,,\tag{7.1}$$

and

$$\int\psi^-_\epsilon(\lambda)\frac{d\sigma}{d\lambda}(\lambda,h)d\lambda-E^-_\epsilon(h)\leq\#\left(\mathrm{Res}\,(P(h))\cap\Omega_0\right)\leq\int\psi^+_\epsilon(\lambda)\frac{d\sigma}{d\lambda}(\lambda,h)d\lambda+E^+_\epsilon(h)\,,\tag{7.2}$$

where, in the notation of (4.1),

$$|E^\pm_\epsilon(h)|\;\leq\;A_0(\sqrt{\delta}+\epsilon)\Big(\#\,\mathrm{Res}\,(P(h))\cap\Omega_{3\epsilon}\setminus\Omega_{-\epsilon}\Big)+A_1h^{-n}-A_2\log|s(z_0(h),h)|\,,$$

with the constants $A_0=A_0(R_0,\Omega_0,\epsilon)$, $A_i=A_i(R_0,\Omega_0,\epsilon,\delta)$, $i=1,2$, which do not depend on $P(h)$.

Proof. We first observe that Lemma 3.1, Jensen's inequality, and (4.8) imply (7.1), and that there exist z's satisfying

$$\log|s(z,h)| \geq C_1 h^{-n} - C_2 \log|s(z_0(h),h)|\,, \quad z \in \Omega_{2\epsilon} \cap \{\operatorname{Im} z > \delta/2\}\,,$$

for any $\delta > 0$. The factorization argument, as in the proof of (4.11), now shows that for $z \in \Omega_{2\epsilon}$, $|\operatorname{Im} z| > \delta$, we have

$$s(z,h) = e^{g_\delta(z,h)} \frac{\overline{P_\delta(\bar{z},h)}}{P_\delta(z,h)}\,, \quad P_\delta(z,h) = \prod_{\substack{w \in \operatorname{Res}\,(P(h)) \cap \Omega_{3\epsilon} \\ \operatorname{Im} w < -\delta/2}} (z-w)\,, \tag{7.3}$$

with

$$|g_\delta'(z,h)| \leq C_3 h^{-n} - C_4 \log|s(z_0(h),h)|\,, \quad z \in \Omega_{2\epsilon} \cap \{|\operatorname{Im} z| > \delta\}\,,$$

where the new constants again depend only on R_0 as far as the dependence on $P(h)$ is concerned.

We now proceed as in the proof of Theorem 1: let $\tilde{\psi}_\epsilon^\pm \in \mathcal{C}_c^\infty(\mathbb{C};[0,1])$ be an almost analytic extension of $\psi_\epsilon^\pm$ satisfying

$$\operatorname{supp} \bar{\partial}_z \tilde{\psi}_\epsilon^\pm \subset \Omega_{\epsilon\pm\epsilon} \setminus \Omega_{\pm\epsilon}\,.$$

Green's formula then gives

$$\begin{aligned}
\int \psi_\epsilon^\pm(\lambda) \frac{d\sigma}{d\lambda}(\lambda,h) d\lambda &= \sum_{z \in \operatorname{Res}\,(P(h))} \tilde{\psi}_\epsilon^\pm(z) + \frac{1}{\pi} \int\int_{\mathbb{C}_-} \bar{\partial}_z \tilde{\psi}_\epsilon^\pm(z) \frac{\partial_z s(z,h)}{s(z,h)} \mathcal{L}(dz) \\
&= \sharp(\operatorname{Res}\,(P(h)) \cap \Omega_0) + \sum_{z \in \operatorname{Res}\,(P(h)) \setminus \Omega_0} \tilde{\psi}_\epsilon^\pm(z) \\
&\quad + \sum_{z \in (\operatorname{Res}\,(P(h)) \cap \Omega_0)} \Big(\tilde{\psi}_\epsilon^\pm(z) - 1\Big) + \frac{1}{\pi} \int\int_{\mathbb{C}_-} \bar{\partial}_z \tilde{\psi}_\epsilon^\pm(z) \frac{\partial_z s(z,h)}{s(z,h)} \mathcal{L}(dz)\,,
\end{aligned}$$

and if we call the sum of the last three terms on the right hand side $E_\epsilon^\pm(h)$, then (7.2) holds and we need to estimate $E_\epsilon^\pm(h)$. We first use (5.3) and deduce from (4.11) (just as in the proof of Theorem 1) that

$$\left| \int\int_{-\delta \leq \operatorname{Im} z \leq 0} \bar{\partial}_z \tilde{\psi}_\epsilon^\pm(z) \frac{\partial_z s(z,h)}{s(z,h)} \mathcal{L}(dz) \right| \leq C_0 \sqrt{\delta} \max\Big(h^{-n}, \#\,(\operatorname{Res}\,(P(h)) \cap (\Omega_{3\epsilon} \setminus \Omega_{-\epsilon}))\Big)\,.$$

For $\operatorname{Im} z < -\delta$ we use the improved factorization (7.3) which, again as in the proof of Theorem 1, gives

$$\left| \int\int_{\operatorname{Im} z < -\delta} \bar{\partial}_z \tilde{\psi}_\epsilon^\pm(z) \frac{\partial_z s(z,h)}{s(z,h)} \mathcal{L}(dz) \right| \leq C_5 h^{-n} - C_6 \log|s(z_0(h),h)|\,,$$

where now C_i, $i = 5, 6$, depend on δ, z_0, R_0, ϵ and the domain Ω_0. Next, applying the estimate (7.1), we obtain

$$\left| \sum_{z \in (\operatorname{Res}\,(P(h)) \setminus \Omega_0)} \tilde{\psi}_\epsilon^\pm(z) \right| \leq C_7 h^{-n} - C_8 \log|s(z_0(h),h)| + C_9(\delta+\epsilon)\sharp\Big(\operatorname{Res}\,(P(h)) \cap (\Omega_{2\epsilon} \setminus \Omega_{-\epsilon})\Big)\,.$$

We estimate in a similar way the term involving $(\tilde{\psi}_\epsilon^\pm - 1)$ and this completes the proof. □

Lemma 4.6 allows us to estimate $\log|s(z,h)|$ from below in a way independent of the perturbation, and hence we can apply Proposition 7.1 to obtain

Theorem 4. *Suppose that $P(h)$ satisfies the assumptions of Sect.2. and that $n \geq 5$. Let*

$$N_\delta([a,b],h) = \#\mathrm{Res}\,(P(h)) \cap \{z \;:\; a \leq \mathrm{Re}\, z \leq b\,,\; \delta \leq |\,\mathrm{Im}\, z| \leq c\}$$

with $c > 0$ fixed, $0 < a < b$. Then for any $\delta > 0$ we have

$$N_\delta([a,b],h) \leq C(R_0, \delta, a, b) h^{-n}\,, \quad 0 < h \leq h_0(R_0, \delta, a, b)\,.$$

If $N^\sharp([a,b],h) = \#\Big(\sigma(P^\sharp(h)) \cap [a,b]\Big)$ is the counting function for the reference operator, then for any $\epsilon > 0$ we have

$$N^\sharp([a+\epsilon, b-\epsilon],h) - E_-(h) \leq N_0([a,b],h) \leq N^\sharp([a-\epsilon, b+\epsilon],h) + E_+(h)\,,$$
$$0 \leq E_\pm(h) \leq C\epsilon h^{-n^\sharp} + C(R_0,\epsilon)h^{-n} + C(\epsilon, P)h^{-n^\sharp+1}\,.$$

Remark. The theorem is stated in a weaker form than actually available: if we use the optimal version of Proposition 4.1 discussed in the remark following it, we can replace $h^{-n^\sharp}$ by a better bound in the estimates on $E_\pm(h)$.

Proof. When we apply (2.12) and (7.2) we only need to check that

$$\begin{aligned}\int \psi_\epsilon^\pm(\lambda)\frac{d}{d\lambda}(\sigma(\lambda,h) - \sigma(a_\pm,h))d\lambda &= -\int \frac{d}{d\lambda}\psi_\epsilon^\pm(\lambda)(\sigma(\lambda,h) - \sigma(a_\pm,h))d\lambda \\ &= -\int \frac{d}{d\lambda}\psi_\epsilon^\pm(\lambda)\Big[N^\sharp([a_\pm,\lambda],h) + \mathcal{O}_P(h^{-n^\sharp+1})\Big]d\lambda \\ &\begin{cases} \geq N^\sharp([a+\epsilon,b-\epsilon],h) - \mathcal{O}_{\epsilon,P}(h^{-n^\sharp+1}), & - \\ \leq N^\sharp([a-2\epsilon,b+2\epsilon],h) + \mathcal{O}_{\epsilon,P}(h^{-n^\sharp+1}), & + \end{cases}\end{aligned}\,,$$

with $a_+ = a - 2\epsilon$, $a_- = a$. An application of Proposition 4.1 to estimate $\#$ Res $(P(h)) \cap (\Omega_{3\epsilon} \setminus \Omega_{-\epsilon})$ completes the proof (we take $\delta = \epsilon^2$ and we change ϵ in the estimate involving $N^\sharp([a-2\epsilon, b+2\epsilon],h)$. ☐

With this in place we immediately obtain Sjöstrand's bottle theorem [23] for compactly supported perturbations:

Theorem 5. *Suppose that P satisfies the assumptions of Sect.2 with $h = 1$. Let*

$$N_\delta(r) = \#\{z \in \mathrm{Res}(P) \;:\; 1 \leq |z| \leq r\,, -\pi/2 < \arg(z) < -\delta\}.$$

Then for $\delta > 0$ we have

$$N_\delta(r) \leq C(\delta, R_0) r^n\,, \quad r \geq r_0(\delta, P)\,, \tag{7.4}$$

where $C(\delta, R_0)$ does not depend on P.

For any $\epsilon > 0$, and $r \geq r_1(\epsilon, P)$, we have

$$\begin{aligned} N^\sharp((1-\epsilon)r) - E_-(r) \leq N_0(r) \leq N^\sharp((1+\epsilon)r) + E_+(r)\,, \\ 0 \leq E_\pm(r) \leq C_0\epsilon r^{n^\sharp} + C_1(R_0,\epsilon)r^n + C_2(\epsilon,P)r^{n^\sharp-1}\,. \end{aligned} \tag{7.5}$$

where, as indicated, the constants C_0 and $C_1(R_0,\epsilon)$ in the error terms do not depend on P, and where $N^\sharp(r) = \sharp\Big(\sigma(P^\sharp) \cap [1, r^2]\Big)$ is the normalized counting function of eigenvalues of the reference operator $P^\sharp$. When $n \geq 5$ then $r_0(\delta, P)$, and $r_1(\epsilon, P)$ depend only on R_0.

Proof. This is a straightforward application of Theorem 4. We only comment on the case of $n < 5$. In that case, we can apply the proof of Lemma 4.6 to obtain a desired lower bound on the scattering determinant since we always have $\log|s(z)| > -C_P$ at some z, $\operatorname{Im} z > 0$, $|z| \leq C$. We refer to [19] for more details. $\square$

To illustrate the theorem we conclude with two examples which are implicit in [23]:

Example 7.1. Let $P = -\Delta_g$ be a metric perturbation of the Laplacian which satisfies

$$\operatorname{vol}_g(B(0, R_0)) \gg R_0^n .$$

Then the number of resonances in any conic neighbourhood of the real axis is comparable to r^n, if r is sufficiently large. In fact, a scaling argument shows that the constants depending on R_0 in (7.5) are all bounded by CR_0^n. This generalizes the estimate given in [25, Example 3].

Example 7.2. Suppose that $N^\sharp(r) \sim Cr^p \log^q r$ where $p+q > n$. Such examples can be obtained by considering hypoelliptic operators – see [26, Example 5.1] and references given there. Here we use a stronger version of Theorem 5 as discussed in the remark following the statement of Theorem 4. We then obtain that

$$N_0(r) = Cr^p \log^q r(1 + o(1)) ,$$

which was first proved by Vodev [33].

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