CONTROL THEORY AND HIGH ENERGY EIGENFUNCTIONS

NICOLAS BURQ AND MACIEJ ZWORSKI

1. Introduction

In this talk we illustrate the "black box" point of view [7] in control theory by applying it to the study of eigenfunctions for billiards which have rectangular components: they include the Bunimovich billiard, the Sinai billiard, and the recently popular pseudointegrable billiards – see [8], [15].

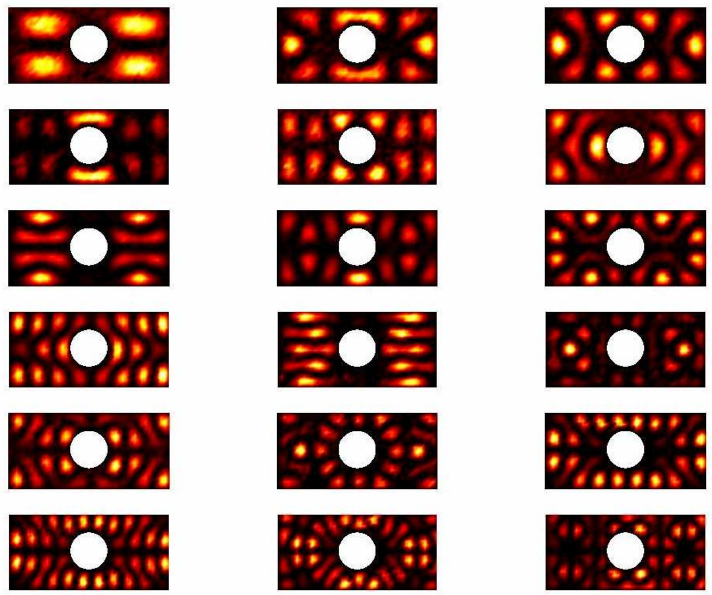

Figure 1. Experimental images of eigenfunctions in a Sinai billiard microwave cavity – see `http://sagar.physics.neu.edu`. We see that there is always a non-vanishing presence near the boundary of the obstacle as predicted by Theorem 2 below.

By a partially rectangular billiard we mean a connected planar domain, Ω, with a piecewise smooth boundary, which contains a rectangle, $R \subset \Omega$, such that if we decompose the boundary of R, into pairs of parallel segments, $\partial R = \Gamma_1 \cup \Gamma_2$, then $\Gamma_i \subset \partial\Omega$, for at least one i. Motivated by the general theory of [7] we have used elementary methods [8] to show that for such domains the eigenfunctions of the Dirichlet, Neumann, or periodic Laplacian, cannot concentrate in the rectangle, away from the remaining two sides of the rectangle – see Theorem 1 below.

A combination of this elementary result with the now standard, but highly non-elementary, propagation results of Melrose-Sjöstrand [16] and Bardos-Lebeau-Rauch [2], gives improved results in some interesting situations. That was already indicated, in a special case, in [7, Theorem 3′] but here we give an independent and more general presentation. For the motivation coming from *quantum chaos* we suggest [10],[17],[8], and references given there (see also Sect.5 below). A more complete treatment of eigenfunctions for partially rectangular billiards, including a discussion of some pseudointegrable cases, will be given by Marzuola in [15].

ACKNOWLEDGMENTS. This work was supported in part by the National Science Foundation under the grant DMS-0200732. We would also like to thank Srinivas Sridhar for letting us use the experimental images shown in Fig.1.

2. PRELIMINARIES

In this section we will recall the basic control result [4],[7] for rectagles, and the propagation results [16],[2],[5],[6] for billiards. Since in the specific application presented in Sect.4 we only use propagation away from the boundary only that, easier, case will be reviewed.

The following result [4] is related to some earlier control results of Haraux [12] and Jaffard [13][1]

Proposition 2.1. *Let Δ be the Dirichlet, Neumann, or periodic Laplace operator on the rectangle $R = [0,1]_x \times [0,a]_y$. Then for any open non-empty $\omega \subset R$ of the form $\omega = \omega_x \times [0,a]_y$, there exists C such that for any solutions of*

$$(\Delta - z)u = f \quad \text{on } R, \ u\restriction_{\partial R} = 0 \tag{2.1}$$

we have

$$\|u\|^2_{L^2(R)} \leq C\left(\|f\|^2_{H^{-1}([0,1]_x;L^2([0,a]_y))} + \|u\restriction_\omega\|^2_{L^2(\omega)}\right) \tag{2.2}$$

Proof. We will consider the Dirichlet case (the proof is the same in the other two cases) and decompose u, f in terms of the basis of $L^2([0,a])$ formed by the Dirichlet eigenfunctions $e_k(y) = \sqrt{2/a}\sin(2k\pi y/a)$,

$$u(x,y) = \sum_k e_k(y)u_k(x), \qquad f(x,y) = \sum_k e_k(y)f_k(x) \tag{2.3}$$

we get for u_k, f_k the equation

$$\left(\Delta_x - \left(z + (2k\pi/a)^2\right)\right)u_k = f_k, \qquad u_k(0) = u_k(1) = 0 \tag{2.4}$$

We now claim that

$$\|u_k\|^2_{L^2([0,1]_x)} \leq C\left(\|f_k\|^2_{H^{-1}([0,1]_x)} + \|u_k\restriction_{\omega_x}\|^2_{L^2(\omega)}\right) \tag{2.5}$$

from which, by summing the squares in k, we get (2.2).

To see (2.5) we can use the propagation result below in dimension one, but in this case an elementary calculation is easily available – see [8]. □

To state the propagation theorem in the form sufficient for our applications we follow [5] and introduce microlocal defect measures.

[1]We remark that as noted in [4] the result holds for any product manifold $M = M_x \times M_y$, and the proof is essentially the same.

Consider for $a(x,\xi) \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^{2d})$ and $\varphi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^d)$ equal to 1 near the x-projection of the support of a. To the symbol a we associate the family of operators $\mathrm{Op}_\varphi(a)(x,hD_x)$ defined by

$$\mathrm{Op}_\varphi(a)(x,hD_x)f = \frac{1}{(2\pi)^d}\int e^{ix\cdot\xi}a(x,h\xi)\widehat{\varphi f}(\xi)d\xi \tag{2.6}$$

By the symbolic calculus the operator $\mathrm{Op}_\varphi(a)(x,hD_x)$ is, modulo operators bounded in L^2 by $\mathcal{O}(h^\infty)$, independent of the choice of the function φ. To simplify notation we drop writing φ.

Let us now consider a Riemannian manifold without boundary, M. By partitions of unity we can define semi-classical pseudo-differential operators $a(x,hD_x)$ associated to symbols $a(x,\xi) \in \mathcal{C}^\infty_{\rm c}(T^*M)$

Now we consider a sequence (u_n) bounded in $L^2(M)$. satisfying

$$(-h_n^2\Delta - 1)u_n = 0 \tag{2.7}$$

Using (2.7), as in [11] (see also [5]) we can prove the following

Proposition 2.2. *There exist a subsequence (n_k) and a positive Radon measure on T^*M, μ (a semi-classical measure for the sequence (u_n)), such that for any $a \in \mathcal{C}^\infty_{\rm c}(T^*M)$*

$$\lim_{k\to+\infty}(Op(a)(x,h_{n_k}D_x)f_{n_k},f_{n_k})_{L^2(M} = \langle\mu, a(x,\xi)\rangle \tag{2.8}$$

Furthermore this measure satisfies

1. *The support of μ is included in the characteristic manifold:*

$$\Sigma \stackrel{\rm def}{=} \{(x,\xi)\in T^*M; p(x,\xi) = \|\xi\|_x = 1\} \tag{2.9}$$

   where $\|\cdot\|_x$ is the norm for the metric at the point x

2. *The measure μ is invariant by the bicharacteristic flow (the flow of the Hamilton vector field of p):*

$$H_p\mu = 0 \tag{2.10}$$

3. *For any $\varphi \in \mathcal{C}^\infty_{\rm c}(M)$,*

$$\lim_{n\to+\infty}\|\varphi u_n\|^2 = \langle\mu, |\varphi|^2\rangle \tag{2.11}$$

The two first properties above are weak forms of the elliptic regularity and propagation of singularities results whereas the last one states that there is no loss of L^2-mass at infinity in the ξ variable.

3. Partially rectangular billiards

The following theorem is an easy conseauence of Proposition 2.1:

Theorem 1. *Let Ω be a partially rectangular billiard with the rectangular part $R \subset \Omega$, $\partial R = \Gamma_1 \cup \Gamma_2$, a decomposition into parallel components satisfying $\Gamma_2 \subset \partial\Omega$. Let Δ be the Dirichlet or Neumann Laplacian on Ω. Then for* any *neighbourhood of Γ_1 in Ω, V, there exists C such that*

$$-\Delta u = \lambda u \implies \int_V |u(x)|^2dx \geq \frac{1}{C}\int_R |u(x)|^2dx\,, \tag{3.1}$$

that is, no eigenfuction can concentrate in R and away from Γ_1.

Proof. Let us take x, y as the coordinates on the stadium, so that x parametrizes $\Gamma_2 \subset \partial\Omega$ and y, Γ_1,

$$R = [0,1]_x \times [0,a]_y \,.$$

Let $\chi \in \mathcal{C}_c^\infty((0,1))$ be equal to 1 on $[\varepsilon, 1-\varepsilon]$. Then $\chi(x)u(x,y)$ is solution of

$$(\Delta - z)\chi u = [\Delta, \chi] u \text{ in } R \tag{3.2}$$

with the boundary conditions satisfied on ∂R. Applying Proposition 2.1, we get

$$\|\chi u\|_{L^2(R)} \leq C \left\| [\Delta, \chi] u \|_{H_x^{-1}; L_y^2} + \| u\restriction_{\omega_\varepsilon} \|_{L^2(\omega_\varepsilon)} \right) \leq C' \| u\restriction_{\omega_\varepsilon} \|_{L^2(\omega_\varepsilon)} \,, \tag{3.3}$$

where ω_ε is a neighbourhood of the support of $\nabla\chi$. Since a neighbourhood of Γ_1 in Ω has to contain ω_ε for some ε, (3.1) follows. □

4. Applications

In [7] and [8] we used Proposition 2.1 to prove that in the case of the Bunimovich billiard shown in Fig.2 the states have nonvanishing density near the vertical boundaries of the rectangle. That follows from Theorem 1 which shows that we have to have positive density in the wings of the billiard, and the propagation result (in the boundary case) based on the fact that any diagonal controls a disc geometrically (see [7, Sect.6.1]; in fact we can use other control regions as shown in Fig.2). Here we consider another case which accidentally generalizes a control theory result of Jaffard [13].

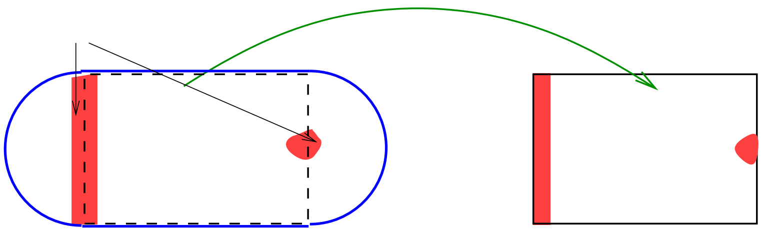

Figure 2. Control regions in which eigenfunctions have positive mass and the rectangular part for the Bunimovich stadium.

The Sinai billiard (see Fig.1) is defined by removing a strictly convex open set, $\mathcal{O}$, with a $\mathcal{C}^\infty$ boundary, from a flat torus, $\mathbb{T}^2 \stackrel{\text{def}}{=} \mathbb{S}^1 \times \mathbb{S}^1$:

$$S \stackrel{\text{def}}{=} \mathbb{T}^2 \setminus \mathcal{O} \,.$$

Taking circles with different lengths might also possible but for simplicity we will restrict our attention to a square torus.

Theorem 2. *Let V be any open neighbourhood of the convex boundary, $\partial\mathcal{O}$, in a Sinai billiard, S. If Δ is the Dirichlet or Neumann Laplace operator on S then there exists a constant, $C = C(V)$, such that*

$$-\Delta u = \lambda u \implies \int_V |u(x)|^2 dx \geq \frac{1}{C} \int_S |u(x)|^2 dx \,. \tag{4.1}$$

Proof. Suppose that the result is not true, that is, there exists a sequence of eigenfunctions u_n, $\|u_n\| = 1$, with the corresponding eigenvalues $\lambda_n \to \infty$, such that $\int_V |u_n(x)|^2 dx \to 0$. We first observe that the only directions in the support of the corresponding semi-classical defect measure, μ, have to be rational: the projection of a trajectory with an irrational direction is dense on the torus and hence has to encounter the obstacle $\partial\mathcal{O}$ (and consequently V). The propagation result recalled in Proposition 2.2 gives a contradiction (remark that we apply this result as long as the trajectory does not encounter the obstacle and consequently we need only the *interior* propagation).

Hence let us assume that there exists a rational direction in the support of the measure which then contains the periodic trajectory in that direction. As shown in Fig. 3 we can find a maximal rectangular neighbourhood of the projection of that trajectory which avoids the obstacle: the sides parallel to the projection correspond to Γ_1 in Theorem 1.

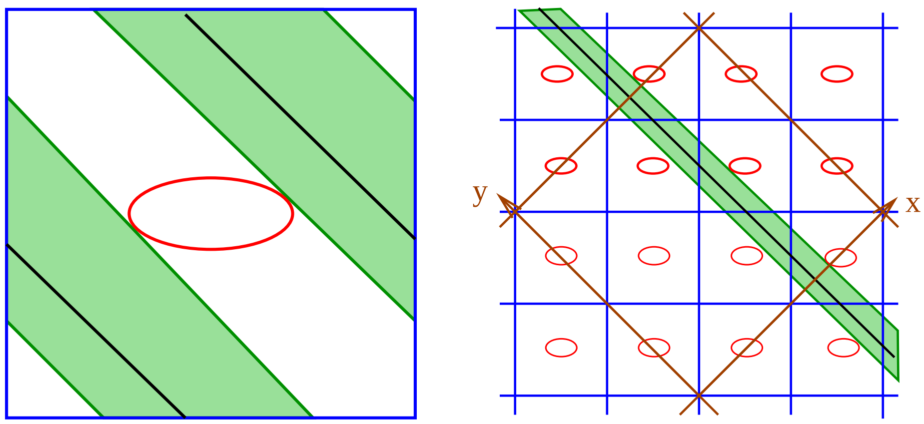

FIGURE 3. A maximal rectangle in a rational direction, avoiding the obstacle. On the right an explicit realization as a flat rectangle.

The rectangle can be described as $R = [0,a]_x \times [0,b]_y$ with the the y coordinate parametrizing the trajectory. Let u be an eigenfunction in our sequence and let $\chi = \chi(x)$ be a smooth function, supported in $(0,a)$ and equal to one outside of a small neighbourhood of the endpoints. Then $\chi(x)u(x,y)$ is a function on R satisfying periodicity condition. Let E_ξ be a microlocal projection onto a neighbourhood of the $R\times\{\xi\}\subset T^*R$, the semi-classical sense with $h = 1/\sqrt{\lambda}$. Let Δ_R is the (periodic) Laplacian on R. Using Fourier decomposition we can arrange that $[\Delta_R, E_\xi] = 0$. Hence,

$$(-\Delta_R - \lambda)E_\xi\chi u = [\Delta_R, E_\xi\chi]\tilde{\chi}u = E_\xi[\Delta_R,\chi]\widetilde{E}_\xi\tilde{\chi}u + \mathcal{O}(\lambda^{-\infty}), \quad \|u\| = 1,$$

where $\tilde{\chi}$ has the same properties as χ and is equal to one on the support of χ, and similarly for $\widetilde{E}_\xi$. As in the proof of Theorem 1 and using that E_ξ is continuous on $H_x^{-1}; L_y^2$, we now see that

$$\|E_\xi\chi u\| \leq C\int_\omega |\widetilde{E}_\xi\tilde{\chi}u|^2 + \mathcal{O}(\lambda^{-\infty}), \tag{4.2}$$

where ω is a neighbourhood of $\nabla\chi$ (we are using here the calculus of semi-classical pseudo-differential operators). Since the semi-classical defect measure of $E_\xi\chi u$ (which is $|E_\xi\chi|^2\times\mu$) was assumed to be non-zero (4.2) shows that the measure of $\widetilde{E}_\chi\tilde{\chi}u\restriction_\omega$ is non zero and consequently there is a point in the intersection of the supports of μ and $\widetilde{e}_\xi\tilde{\chi}$. But μ is invariant by the flow (as long as it does not intersect the obstacle) and hence, once we choose all the cut-offs above very close to the boundary of R, its support can be made intersect any neighbourhood of $\partial\mathcal{O}$. $\square$

Remark 1. In the proof above the smoothness, the convexity, and even the connectivity of the obstacle played no role (and we could take $\Theta = \emptyset$ provided that $V \neq \emptyset$). Consequently, the result holds for any obstacle (sufficiently smooth in the case of Neumann boundary conditions) and consequently to the special case of pseudointegrable billiards (see for instance [3] for motivation and description). By an elementary reflection principle, the result also holds for an obstacle inside a square with Dirichlet or Neumann conditions on the boundary of the square.

Remark 2. The proof above gives in fact the following estimate for any open neighbourhood of the obstacle:

$$\begin{gathered}\exists C; \forall u,\, f \in L^2(S) \text{ solutions of } (-\Delta + \lambda)u = f, \qquad u\restriction_{\partial S} = 0 \\ \|u\|_{L^2(S)} \leq C \left(\|f\|_{L^2(S)} + \|u \mathbb{1}_V\|_{L^2(V)} \right)\end{gathered} \tag{4.3}$$

and according to [7, Theorem 4], this implies that the Schrödinger equation in S is exactly controlable by V in finite time. In fact, by working on the time evolution equation, we could strenghten this result allowing an arbitrarily small time. This latter result was previously known [13] for the particular case $\Theta = \emptyset$ ($S = \mathbb{T}^2$) but the proof was based on subtle results about Fourier series [14].

Remark 3. As shown in [7, Theorem 2′], the results of Ikawa and Gérard on scattering by two convex obstacles (see [7] and references given there) give an estimate on the maximal concentration of an eigenfunction (or a quasimode) on a closed orbit in a Sinai billiard. Let $\chi \in \mathcal{C}^\infty(S; [0,1])$ be supported in a small neighbourhood of a closed transversally reflecting orbit shown in Fig. 4. Then for any family $(-\Delta - \lambda)u_\lambda = \mathcal{O}(\lambda^{-\infty})$, $\|u_\lambda\| = 1$,

$$C \int_S |u(x)|^2 (1 - \chi(x)) dx \geq \frac{1}{\log \lambda}, \tag{4.4}$$

that is a concentration on a closed trajectory, if at all possible, has to be very weak. For more results on the weak concentration on hyperbolic orbits, and for pointers to the literature, we refer to [7].

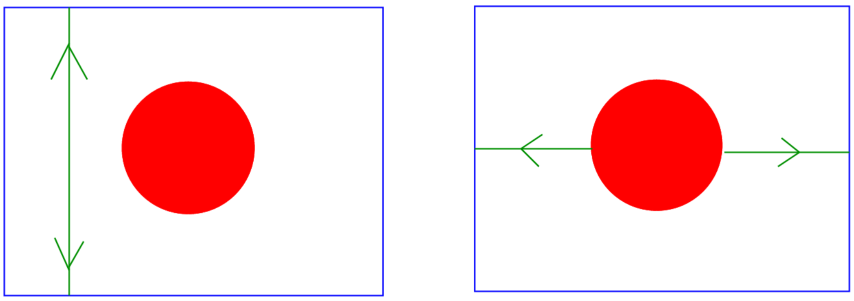

FIGURE 4. A bouncing ball trajectory (left) and a hyperbolic trajectory (right) in the Sinai billiard. By Theorem 1, no concentration is possible on the bouncing ball orbit. Estimate (4.4) allows only very weak concentration on the hyperbolic orbit, and in fact none at all is expected.

5. An open problem

The basic mathematical result in the theory of quantum chaos is the following theorem announced by Shnirelman in 1974 and first proved in the case of hyperbolic surfaces by Zelditch in 1985:

Theorem 3. *Suppose that the billiard flow on a bounded domain with boundary, Ω, is ergodic. Let u_j be the sequence of normalized eigenfunctions of the Dirichlet (or Neumann) Laplacian,*

$$-\Delta u_j = \lambda_j^2 u_j\,, \quad u_j\restriction_{\partial\Omega}=0\,, \quad \int_\Omega |u_j(x)|^2 dx = 1\,. \tag{5.1}$$

Then there exists a sequence $\{j_k\}_{k=1}^\infty \subset \mathbb{N}$ of density one, that is, $\lim_{N\to\infty}(\max_{j_k\leq N} k)/N = 1$, such that for any nice[2] *open subset V, of Ω,*

$$\lim_{k\to\infty}\int_V |u_{j_k}(x)|^2 dx = \frac{\mathrm{Area}(V)}{\mathrm{Area}(\Omega)}\,. \tag{5.2}$$

This means that for almost all eigenfunctions there cannot be any concentration: they have to be uniformly spread out in the billiard table. The integral of the square of the eigenfuction over V is interpreted as the probability of finding the quantum state in V. A stronger version of the theorem gives a phase space version of this statement.

Theorem 3 was first proved for convex billiards (in particular the Bunimovich billiard) by Gérard-Leichtnam [11], and for arbitrary manifolds with piecewise smooth boundaries by Zelditch-Zworski [18]. We refer to these papers and to [1],[10],[17] for history and pointers to the literature.

One question which is still mysterious to mathematicians and physicists alike is if the quantum states of a classically ergodic system (in our case, solutions of the Helmholtz equation for an ergodic billiard) can concentrate on the highly unstable closed orbits of the classical flow, or on some invariant tori formed by such orbits. Theorem 3 allows the possibility of such concentration on sequences of density zero.

A system is called *quantum unique ergodic* if there is no such concentration – see [17] and references given there. In particular, quantum unique ergodicity means that (5.2) holds for the *full* sequence of eigenfuctions, that is

$$\lim_{j\to\infty}\int_V |u_j(x)|^2 dx = \frac{\mathrm{Area}(V)}{\mathrm{Area}(\Omega)}\,. \tag{5.3}$$

Neither Bunimovich nor Sinai billiards are expected to be quantum unique ergodic: the full set of bouncing balls filling the maximal rectangles of the billiards could be a region of concentration. Theorem 1 above shows that eigenfunctions cannot concentrate on any smaller set of bouncing ball orbits.

Motivated by this expectation we formulate three natural problems of increasing difficulty. Let Ω be the Bunimovich billiard and R its rectangular part. Can we prove the following concentration results:

$$(-\Delta - \mu_k)v_k = o(1)\,, \quad v_k|_{\partial\Omega} = 0\,, \ \ \mu_k \to \infty\,, \quad \int_\Omega |v_k|^2 = 1\,, \quad \int_R |v_k|^2 \to 1\,. \tag{5.4}$$

$$(-\Delta - \mu_k)v_k = \mathcal{O}(\mu_k^{-\infty})\,, \quad v_k|_{\partial\Omega} = 0\,, \ \ \mu_k \to \infty\,, \quad \int_\Omega |v_k|^2 = 1\,, \quad \int_R |v_k|^2 \to 1\,. \tag{5.5}$$

[2] By nice we mean that the boundary of the open set has measure zero. We are grateful to Patrick Gérard for pointing out this condition.

$$(5.6) \qquad (-\Delta - \mu_k) v_k = 0 \,, \quad v_k|_{\partial\Omega} = 0 \,, \quad \mu_k \to \infty \,, \quad \int_\Omega |v_k|^2 = 1 \,, \quad \int_R |v_k|^2 \to 1 \,.$$

The proof of Theorem 1 shows that the trivial quasi-mode concentrating inside of R, $(-\Delta - \mu_k)u_k = \mathcal{O}(1)$, cannot be improved without going all the way to the boundary of R. Hence (5.4) is the first non-trivial statement one can make. The last statement (5.6) is very difficult as it is hard to distinguish eigenfuctions from quasimodes given in (5.5)[3]

References

[1] A. Bäcker, R. Schubert, and P. Stifter. On the number of bouncing ball modes in billiards. *J. Phys. A: Math. Gen.* 30:6783-6795, 1997.

[2] C. Bardos, G. Lebeau and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.* 30:1024–1065, 1992.

[3] E. Bogomolny, U. Gerland, and C. Schmit, Models of intermediate spectral statistics, *Phys. Rev. E* 59:1315-1318, 1999.

[4] N. Burq Control for Schrodinger equations on product manifolds *Unpublished*, 1992

[5] N. Burq. Semi-classical estimates for the resolvent in non trapping geometries. *Int. Math. Res. Notices*, 5:221–241, 2002.

[6] N. Burq and P. Gérard, Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes, *Comptes Rendus de L'Académie des Sciences*, 749–752,t.325, Série I, 1996

[7] N. Burq and M. Zworski. Geometric control in the presence of a black box. *Jour. Amer. Math. Soc.* , 17:443-471, 2004.

[8] N. Burq and M. Zworski. Bouncing ball modes and quantum chaos, to appear in *SIAM Review*, 2004.

[9] N. Burq and G.Lebeau, Mesures de défaut de compacité, application au système de Lamé, *Ann. Sci. École Norm. Sup. (4)*, No 34, 817-870, 2001.

[10] H. Donnelly. Quantum unique ergodicity *Proc. Amer. Math. Soc.* 131:2945-2951, 2003.

[11] P. Gérard and E. Leichtnam, Ergodic Properties of Eigenfunctions for the Dirichlet Problem, *Duke Mathematical Journal,* No 71, 559–607, 1993

[12] A. Haraux. Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire, *J. Math. Pures Appl.* 68-4:457–465, 1989.

[13] S. Jaffard Contrôle interne exact des vibrations d'une plaque rectangulaire. *Portugal. Math.* 47 (1990), no. 4, 423-429.

[14] J.P. Kahane Pseudo-périodicité et séries de Fourier lacunaires *Annales Sc. de l'Ecole Normale Supérieure* 79, 1962.

[15] J. Marzuola, Eigenfunctions for partially rectangular billiards, in preparation.

[16] R.B. Melrose and J. Sjöstrand, Singularities of Boundary Value Problems I & II, *Communications in Pure Applied Mathematics*, 31 & 35, 593- 617 & 129-168, 1978 & 1982.

[17] S. Zelditch. Quantum unique ergodicity. *Proc. Amer. Math. Soc.* 132:1869-1872, 2004.

[18] S. Zelditch and M. Zworski. Ergodicity of eigenfunctions for ergodic billiards. *Comm. Math. Phys.*, 175:673–682, 1996.

Université Paris Sud, Mathématiques, Bât 425, 91405 Orsay Cedex
E-mail address: Nicolas.burq@math.u-psud.fr

Mathematics Department, University of California, Evans Hall, Berkeley, CA 94720, USA
E-mail address: zworski@math.berkeley.edu

[3]Prizes were offered by the second author to the participants of this conference for solutions of these problems.