Linear vs Chaotic

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Plan of the talk

- Dynamics and statistics
- Zeta functions
- Integrability, chaos and beyond

Dynamical systems

Dynamical systems: a statistical approach

Dynamical systems: a statistical approach

Linear

Non-linear

Dynamical systems: a statistical approach

Completely integrable

Chaotic

For general aspects see

Brin–Stuck *Introduction to Dynamical Systems*

The billiard on the right is a form of Sinai billiard.

For the discussion of chaotic flows on billiard tables see Wojtkowski

`http://wmii.uwm.edu.pl/~wojtkowski/hb11.pdf`

and references given there.

We should stress that the results discussed below are really for smooth dynamical systems. There are technical difficulties in extending them (e.g. meromorphy of the Ruelle zeta functions) to billiards because of singularities and the presence of glancing trajectories.

In the chaotic case positions and directions get uniformly distributed:

Question: How long do we have to wait to have uniform distribution?

The length of time needed to have uniform distribution can be estimated when we know *exponential decay of correlations*

$$\int f(\varphi_t(x))g(x)dm(x) = \int f(x)dx \int g(x)dx + \mathcal{O}(e^{-\gamma t})$$

This has a long tradition but general results are recent:

Dolgopyat *On decay of correlations in Anosov flows,* Ann. of Math. **147**(1998)

Liverani *On contact Anosov flows,* Ann. of Math. **159**(2004),

Tsujii *Contact Anosov flows and the FBI transform,* Erg. Th. Dyn. Syst., **32**(2012)

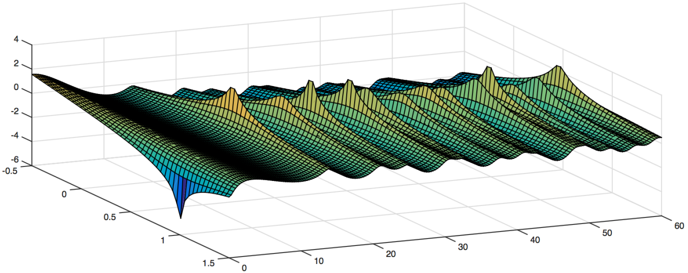
Nonnenmacher–Zworski *Decay of correlations for normally hyperbolic trapping,* Inv. Math., **200**(2015)

The most famous function of mathematics: Riemann zeta function

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \quad p = \text{a prime number}$$

$\zeta(s)$ has a pole (singularity) at $s = 1$.

It has a lot of important zeros



$-\log|\zeta(s)|$

A dynamical analogue: Ruelle zeta function

Replace primes with prime closed orbits

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Replace p by $\log T_\gamma$ where T_γ is the length of a prime closed orbit.

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$$\zeta_{\mathrm{D}}(s) = \prod_{\gamma} (1 - e^{-sT_\gamma})^{-1}$$

Replace p by e^{T_γ} where T_γ is the length of a prime closed orbit.

It turns out that the zeros and poles of ζ_{D} contain information about statistical properties of the chaotic dynamical system.

That includes the time at which we achieve uniform distribution.

For an introduction to dynamical zeta functions and to the literature see

http://homepages.warwick.ac.uk/~masdbl/grenoble-16july.pdf

Recent papers about meromorphic continuation of dynamical zeta functions:

Giulietti–Liverani–Policott *Anosov flows and dynamical zeta functions*, Ann. of Math. **178**(2013)

Dyatlov–Zworski *Dynamical zeta functions for Anosov flows via microlocal analysis* http://arxiv.org/abs/1306.4203

Dyatlov–Guillarmou *Pollicott-Ruelle resonances for open systems*, http://arxiv.org/abs/1410.5516

Microlocal approach to Anosov systems started with Faure–Sjöstrand *Upper bound on the density of Ruelle resonances for Anosov flows*, Comm. Math. Phys. **308**(2011)

Computational methods for zeta functions were developed by Cvitanovic, Eckhardt, Gaspard...

A recent mathematical account and references:

Borthwick–Weich *Symmetry reduction of holomorphic iterated function schemes and factorization of Selberg zeta functions*, `http://arxiv.org/abs/1407.6134`

A slightly different zeta function is needed to obtain the rate of decay to equilibrium: in addition to closed orbits it also includes the instability factors:

$$\zeta_1(s) := \exp\left(- \sum_{\gamma} \frac{T_\gamma^{\#} e^{-sT_\gamma}}{T_\gamma |\det(I - \mathcal{P}_\gamma)|} \right).$$

Trouble with all this: Few real systems are purely completely integrable or purely chaotic.

The simplest (?) flow exhibiting chaotic behaviour:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_2 x_3, \quad \dot{x}_3 = 1 - x_2^2.$$

Technical asides:

simple: it is the contact flow for $e^{-|x|^2/2}(x_2 dx_1 + dx_3)$

chaotic behaviour: positive Lyapounov exponents

These ordinary differential equations are called the Nosé–Hoover system and have origins in molecular dynamics.

They were rediscovered by Sprott in a computer search for simple systems with positive Lyapunov exponents. The system is simpler than the famed Lorenz equations and easier remember for mathematicians because of the simple contact form.

For a recent account and references see

Jafari–Sprott–Golpayegani *Elementary quadratic chaotic flows with no equilibria*, Phys. Lett. A **377**(2013).

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Completely integrable Chaotic

A useful visualization: Poincaré section

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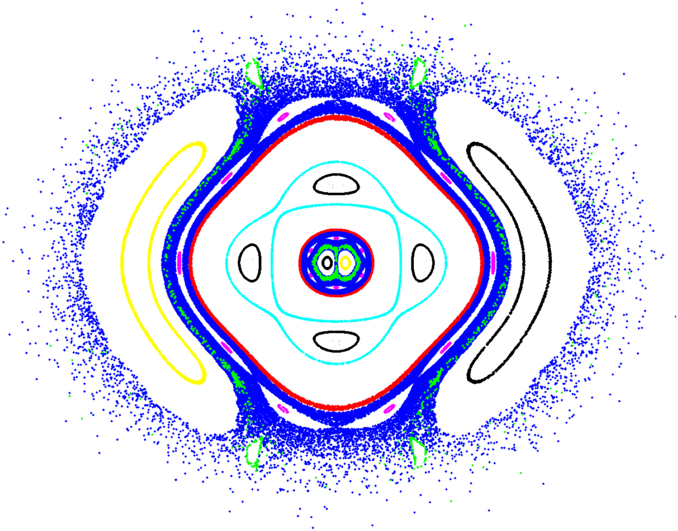


Question 1: Do we have equidistribution in the chaotic sea?

chaotic sea: the blue region

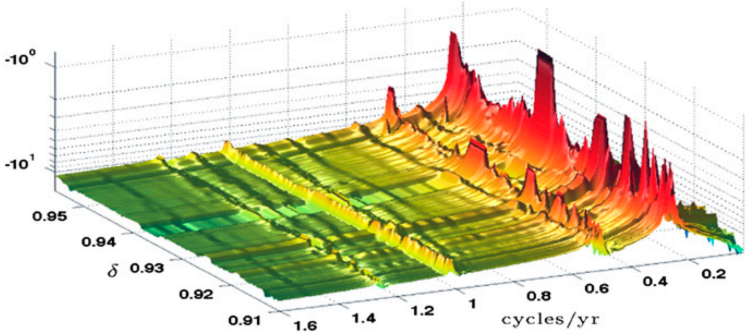
Although the movies and the picture suggest existence of invariant sets of positive measure, failure of ergodicity for the Nosé–Hoover was established only recently in

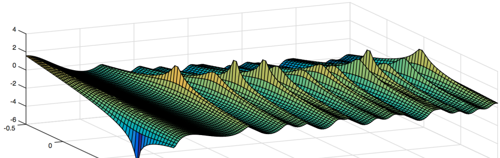
Legoll–Luskin–Moeckel *Non-Ergodicity of the Nosé–Hoover thermostatted harmonic oscillator*, Arch. Ration. Mech. Anal. **184**(2007).

Question 1: Do we have equidistribution in the chaotic sea?

Question 2: What happens in many degrees of freedom/infinite dimensions?

A relevant example: Power spectrum of El Niño





The El Niño example comes from

Chekroun–Neelin–Kondrashov–McWilliams–Ghil, *Rough parameter dependence in climate models and the role of Pollicott–Ruelle resonances,* Proc. Nat. Acad. Sci. **111**(2014)

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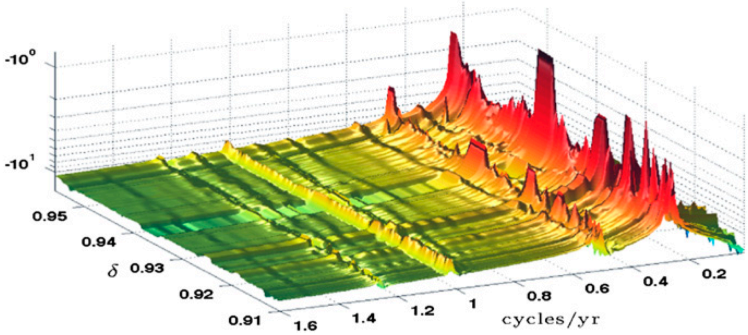

Figure credit: Proc. Nat. Acad. Sci. 2014

To conclude:

- Can we bring methods which have been successful in the study of chaotic systems to the study of mixed systems?
- What happens on the quantum level? (Mathematically, what are possible results about eigenvalues and eigenfunctions.)
- Can we visualize/understand complicated multidimensional systems using ideas from chaotic dynamics?
- Can we find complete integrability behind some interesting mixed systems? (Just as random matrix theory models quantum systems with underlying chaotic dynamics.)

For a recent result for mixed systems close to completely integrable systems see

Guardia–Kaloshin–Zhang *A second order expansion of the separatrix map for trigonometric perturbations of a priori unstable systems*, `http://arxiv.org/abs/1503.08301`

For a survey of results on partially hyperbolic systems see Hasselblatt–Pesin

`https://www.math.psu.edu/pesin/papers_www/HP-survey.pdf`

For surveys of problems in *quantum chaos* see Nonnenmacher

`http://arxiv.org/abs/1005.5598`
`http://arxiv.org/abs/1105.2457`

For results on eigenfunction statistics for (special) mixed systems:

Galkowski `http://arxiv.org/abs/1209.2968`
Riviere `http://arxiv.org/abs/1209.3576`
Gomes `http://arxiv.org/abs/1504.07332`.