Carleman estimates, unique continuation and applications

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Introduction

The unique continuation problem. Consider a partial differential operator $P(x, D)$ and an oriented hypersurface Σ. Denote the two sides of Σ by Σ^+ and Σ^-. Then a typical formulation for an unique continuation result is

Let u be a solution for $P(x, D)u = 0$. Assume that $u = 0$ in Σ^+. Then $u = 0$ near Σ in Σ^-. (1)

A principal aim of the work in this area is that, given an operator P, to determine the class of hypersurfaces S for which the above unique continuation property (UCP) holds. Most results of this type are in effect local. Hence, a more appropriate formulation of the above property is

Let u be a solution for $P(x, D)u = 0$. Let $x_0 \in \Sigma$ and V be a neighbourhood of x_0. Assume that $u = 0$ in $\Sigma^+ \cap V$. Then $u = 0$ near x_0 (in Σ^-).

The Cauchy problem The unique continuation problem is related to the Cauchy problem. If P is a partial differential operator of order m and the surface Σ is noncharacteristic then, given a smooth function u, we can define the Cauchy data of u on Σ as

$$(u_{|\Sigma}, \frac{\partial u}{\partial \nu}_{|\Sigma}, \cdots, \frac{\partial^{m-1} u}{\partial \nu^{m-1}}_{|\Sigma}) \tag{2}$$

The Cauchy problem for the operator P with initial data on Σ can be phrased as

Given functions $(u_0, u_1, \cdots, u_{m-1})$ on Σ, find u in Σ^- so that:

$$\begin{cases} P(x, D)u = 0 & \text{in } \Sigma^+ \\ (u_{|\Sigma} = u_0, \frac{\partial u}{\partial \nu}_{|\Sigma} = u_1, \cdots, \frac{\partial^{m-1} u}{\partial \nu^{m-1}}_{|\Sigma} = u_{m-1} & \text{on } \Sigma \end{cases} \tag{3}$$

We say that the above Cauchy problem is well-posed if for each Cauchy data

$$(u_0, u_1, \cdots, u_{m-1})$$

the equation (3) has an unique solution u.

Now one can see that the unique continuation property is equivalent to the uniqueness in the Cauchy problem. In particular if the Cauchy problem is well-posed then the unique continuation property also holds. Hence, it makes sense to study the unique continuation problem only when the Cauchy problem is ill-posed.

Stability estimates Take a typical unique continuation result of the form " Let $u \in H^1$ solve $P(x,D)u = 0$. Then $u = 0$ in ω implies $u = 0$ in $\Omega \supset \omega$. Then a simple compactness argument shows the existence of a continuity modulus m such that

$$|u|_{L^2(\Omega)} \leq m(|u|_{H^1(\omega)}) \quad \text{if} \quad |u|_{H^1(\Omega)} \leq 1 \tag{4}$$

Such an estimate is called a stability estimate. In particular problems the interest is to determine the behaviour of the best function m for which (4) holds.

Note that such an estimate says nothing of the regularity of the function u. However, sometimes one can obtain a stronger version of (4), namely

$$|u|_{H^1(\Omega)} \leq m(|u|_{H^1(\omega)}) \quad \text{if} \quad |u|_{L^2(\Omega_\epsilon)} \leq 1 \tag{5}$$

The latter estimate also says that if $u \in H^1(\omega) \cap L^2(\Omega_\epsilon)$ then $u \in H^1(\Omega)$.

If the function m can be taken linear then the restriction $|u|_{L^2(\Omega_\epsilon)} \leq 1$ can be dropped, and the following stronger estimate holds:

$$|u|_{H^1(\Omega)} \leq m(|u|_{H^1(\omega)}) \tag{6}$$

In accordance with the control theory terminology, we call (6) an observability estimate. The meaning of this is that one can predict u in Ω by observing it in the smaller set ω.

Carleman estimates The oriented surface Σ can be represented as $\Sigma = \{\phi = 0\}$, where ϕ is a C^2 function which vanishes simply on Σ and $phi > 0$ in Σ^+. The most common way of proving unique continuation results is by using Carleman estimates. The Carleman estimates are just regular estimates for pde's, with an additional feature, namely that they contain a one parameter family of exponential weights. The simplest Carleman estimates, say for the wave equation, have the form

$$\tau |e^{\tau\phi} u|_{H^1}^2 \leq c |e^{\tau\phi} P(x,D) u|_{L^2}^2, \quad \tau \geq \tau_0 \tag{7}$$

for u supported in a fixed compact set.

How does one get unique continuation from the Carleman estimates ? Assume first that u has compact support and that $P(x,D)u = 0$ in $\Sigma^+ = \{\phi > 0\}$. Then the RHS in (7) goes to 0 as $\tau \to \infty$. Hence the LHS also goes to 0, which implies that $u = 0$ in Σ^+. Now go one step further, and assume only that supp $u \cap \Sigma^+$ is compact. Then we can cutoff u in Σ^- to make it have compact support, and the same argument works. Hence, we have proved that (7) implies the following version of 1:

em Assume that supp $u \cap \Sigma^+$ is compact and that $P(x, D)u = 0$ in Σ^+. Then $u = 0$ in Σ^+ (8)

The relation between (1) and (8) can be seen in the following picture. Obtaining (1) near some point $x_0 \in \Sigma$ reduces to (8) for a modified surface Σ'.

Why Carleman estimates ? Well, assume that we are attempting to show (1) and that using some other devices (e.g. propagation of singularities) we have proved that u is smooth near Σ. This still leaves room for functions u which decay rapidly near Σ. But, intuitively, one cannot "see" this behaviour with usual energy estimates. This is the task of the exponential weight, which, as τ increases, "highlights" the behavior of u exactly near Σ.

The conjugated operator Is proving an estimate such as (7) more difficult than proving such an estimate without the exponential weight ? To investigate this let us try to eliminate the exponential weight from (7). Thus, set $v = e^{\tau\phi}u$. Then

$$e^{\tau\phi}P(x, D)u = e^{\tau\phi}P(x, D)e^{-\tau\phi}v = P_\tau(x, D, \tau)v$$

Hence, (7) reduces to

$$\tau|v|^2_{H^1} \leq c|e^{\tau\phi}P_\tau(x, D, \tau)u|^2_{L^2}, \quad \tau \geq \tau_0 \tag{9}$$

Here P_τ has the form

$$P_\tau(x, D, \tau) = e^{\tau\phi}P(x, D)e^{-\tau\phi} = P(x, D + i\tau\nabla\phi) \tag{10}$$

i.e. it is the conjugate of P with respect to the exponential weight $e^{\tau\phi}$.

Thus, we have elliminated the exponential weight from the estimates at the expense of replacing the operator P by the conjugated operator P_τ. Does this cause any troubles ? At first sight it would seem that P_τ equals P modulo lower order terms, therefore one is tempted to say that they have similar properties.

The critical fact is, however, that the estimate (7) is an uniform estimate in τ. Hence, we cannot afford to treat the τ's in P_τ as lower order terms. In effect, as it turns out, τ should have the same weight as a full derivative. Consequently, P_τ and P are structurally different, and separate estimates are required for P_τ.

Other types of unique continuation problem. Why assume from the begining that u vanishes in an open set, and not in a smaller set ? For operators with nonanalytic coefficients this seems to be intrinsically related to propagation of singularities. It appears that one needs to know that each possible propagating singularity would have to intersect the set where u is known to be 0. Consequently, there are two exceptions to this.

The first exception is for elliptic or subelliptic equations, for which there are no singularities which propagate. Then one can sometimes obtain a *strong unique continuation property* (SUCP) which has the form

Let u be a solution for $P(x, D)u = 0$. Assume that u vanishes of infinite order at some x_0. Then $u = 0$ near x_0

The SUCP has been extensively studied, and for a review containing the latest results one could see [**?**]. It is not the aim of this monograph to consider the SUPC for elliptic equations. However, we do consider it for parabolic equations in Chapter 3.10.

The second exception is for anisotropic equations, e.g. for the Schroedinger equation. There the speed of propagation is infinite, therefore one can hope to get some results at fixed time. To show a tipical result, fix t_0 and let $\Sigma \subset \{t = t_0\}$ be an oriented surface.

Let u be a solution for $P(x, D)u = 0$. Assume that u vanishes of "sufficiently high order" in Σ^+. Then u vanishes of "sufficiently high order" near Σ in $\{t = t_0\}$.

Results of this type are presented in Chapter 3.9.

Chapter 1

Preliminaries

1.1 Partial differential operators

For a partial differential operator $P(x, D)$ of order m in R^n of the form

$$P(x, D) = \sum_{|\alpha| \leq m} c_\alpha D^\alpha$$

we define its principal symbol as a function in T^*R^n,

$$p(x, \xi) = \sum_{|\alpha| = m} c_\alpha D^\alpha \tag{1.1}$$

The characteristic set of the operator P is defined as

$$\text{char } P = \{(x, \xi) \in R^{2n} / p(x, \xi) = 0,\ \xi \neq 0\}. \tag{1.2}$$

The Hamilton field associated to a real symbol P is

$$H_p = p_\xi(x, \xi) \frac{\partial}{\partial x} - p_x(x, \xi) \frac{\partial}{\partial \xi}$$

If the coefficients of P are at least C^2 then the trajectories of the Hamilton field are uniquely determined and are called bicharacteristic rays, or simply bicharacteristics. Observe that char P is invariant with respect to the bicharacteristic flow. The bicharacteristics contained in char P are called null bicharacteristics, and play a fundamental role in the study of partial differential operators.

Given two symbols p and q we can define their Poisson bracket as

$$\{p, q\} = H_p q = -H_q p = p_\xi q_x - q_\xi p_x \tag{1.3}$$

If p is a real symbol, then the above Poisson bracket has a simple geometrical interpretation. Namely, $\{p, q\}$ is the derivative of q along the bicharacteristics of P.

Definition 1.1 *The operator P is said to be of real principal type if its principal symbol $p(x,\xi)$ is real and $\nabla_{x,\xi} p(x,\xi) \neq 0$ on* char *P.*

If P is of real principal type then char P is a codimension 1 surface.

A second condition used throughout the book is the principal normality condition on P.

Definition 1.2 *The operator P is principally normal if*

$$|\{\bar{p}, p\}| \leq c|p||\xi|^{m-1} \tag{1.4}$$

Of course, this condition is fulfilled whenever P has a real principal symbol, and also whenever P is elliptic. This includes all the examples studied here.

1.2 Pseudoconvex functions and surfaces

If Σ is a C^2 oriented hypersurface, we can represent it as a level set of a C^2 function

$$\Sigma = \{\phi = 0\}$$

where $\nabla\phi \neq 0$ on Σ and $\phi > 0$ on Σ^+. This representation is not unique. However, it is unique modulo multiplication by C^2 positive functions.

The strong pseudoconvexity condition is

Definition 1.3 *Let S be a C^2 hypersurface. Let ϕ be a real valued C^2 function vanishing simply on S. We say that S is strongly pseudoconvex with respect to P at $x_0 \in S$ if*

$$Re\{\bar{p}, \{p, \phi\}\}(x_0, \xi) > 0 \text{ whenever} \tag{1.5}$$

$$p(x_0, \xi) = \{p, \phi\}(x_0, \xi) = 0 \ , \ \xi \neq 0 \tag{1.6}$$

$$\{\bar{p}(x_0, \xi - i\tau\nabla\phi), p(x_0, \xi + i\tau\nabla\phi)\}(x_0, \xi)/\tau i > 0 \quad \text{whenever} \tag{1.7}$$

$$\{p(x_0, \xi + i\tau\nabla\phi) = \{p(x_0, \xi + i\tau\nabla\phi), \phi\}(x_0, \xi) = 0, \tau > 0\} \tag{1.8}$$

In order for this definition to be correct, one needs to verify that it does not depends on the choice of the function ϕ, i.e. that is invariant with respect to substitutions of the form $\phi \to g\phi$, with $g > 0$.

Note that, given that P is principally normal, (1.6) is the limiting case of (1.8) as $\tau \to 0$. Indeed, the linearization of p_τ in τ at $\tau = 0$ is

$$p_l(x, \xi, \tau) = p(x, \xi) + i\tau\{p, \phi\}(x, \xi)$$

Hence, if $p(x_0, \xi) = \{p, \phi\}(x_0, \xi) = 0$ then

$$\lim_{\tau \to 0} \{\bar{p}(x_0, \xi - i\tau\nabla\phi), p(x_0, \xi + i\tau\nabla\phi)\}(x_0, \xi)/\tau i = 2\mathrm{Re}\ \{\bar{p}, \{p, \phi\}\}(x_0, \xi)$$

Following is the definition of strongly pseudoconvex functions:

Definition 1.4 *We say that the C^2 function ϕ is strongly pseudoconvex at x_0 with respect to P at x_0 if*

$$Re\{\bar{p}, \{p, \phi\}\}(x_0, \xi) > 0 \ whenever \tag{1.9}$$

$$p(x_0, \xi) = 0\ ,\ \xi \neq 0 \tag{1.10}$$

$$\{\bar{p}(x, \xi - i\tau\nabla\phi), p(x, \xi + i\tau\nabla\phi)\}/\tau i > 0 \ on\ \{p(x, \xi + i\tau\nabla\phi) = 0, \quad \tau \geq 0,\ (\xi, \tau) \neq 0\} \tag{1.11}$$

The relationship between strongly pseudoconvex functions and surfaces is described next.

Theorem 1.5 *Let P be a principally normal operator. Then*

i) Any nondegenerated level set of a function which is strongly pseudoconvex with respect to P is a strongly pseudoconvex surface with respect to P.

ii) Any surface which is strongly pseudoconvex with respect to P is a level surface for some strongly pseudoconvex function with respect to P.

iii) The strong pseudoconvexity condition for both functions and surfaces is stable with respect to small C^2 perturbations.

Proof : Parts (i),(iii) follow directly from the definitions. For (ii), let ϕ be as in Definition . This implies that if c is sufficiently large then

$$\frac{1}{\tau i}\{\overline{p_\tau}, p_\tau\}(x_0,\xi) + c(|\{p_\tau,\phi\}|^2 + \tau^{-2}|p_\tau|^2) \geq d(\xi^2+\tau^2)^{m-1} \tag{1.12}$$

Then we claim that the function $\psi = e^{\lambda\phi}$ satisfies (1.10), (1.11), provided that λ is sufficiently large. Indeed, compute

$$\begin{aligned}\frac{1}{\tau i}\{\overline{p(x,\xi+i\tau\nabla\psi)}, p(x,\xi+i\tau\nabla\phi)\} &= \frac{1}{\tau i}\{\overline{p(x,\xi+i\lambda\tau\psi\nabla\psi)}, p(x,\xi+i\tau\lambda\psi\nabla\phi)\} \\ &= \frac{1}{\tau i}\{\overline{p_{\lambda\tau}}, p_{\lambda\tau}\} + 2\lambda|p_{\lambda\tau}|^2\end{aligned}$$

If we combine this with (1.12) then for $\lambda > c$ we get

$$\frac{1}{\tau i}\{\overline{p(x,\xi+i\tau\nabla\psi)}, p(x,\xi+i\tau\nabla\phi)\}\tau^{-2}|p_\tau|^2) \geq d(\xi^2+\tau^2)^{m-1} \tag{1.13}$$

This implies (1.10), (1.11).

1.3 Pseudoconvexity revised: anisotropic operators

Consider the Schroedinger operator

$$P(x,\partial) = i\partial_t - A(t,x,\partial_x)$$

where A is a second order elliptic operator. What is the principal symbol of P ? According to 1.1, it should be $a(x,t,\xi)$. However, the word "principal" tends to indicate that this symbol essentially describes the properties of the operator. This is outright false in this setting, since the Schroedinger equation and a second order elliptic equation are completely different. Consequently, it makes more sense to define the principal symbol as

$$p(x,\xi) = s + a(t,x,\xi)$$

where s is the time Fourier variable.

This symbol, however, is no longer homgeneous of order 2. Or is it ? It actually is if we twist a bit the meaning of the word homogeneous. Namely, redefine

$$\lambda(s,\xi) = (\lambda^2 s, \lambda\xi)$$

Of course, this changes the meaning of the order of a partial differential operator; in this setting D_x has order 1, but D_t has order 2. If we take an operator of the form $D_t^{\alpha_0} D_x^{\alpha'}$ then its order has to be $2\alpha_0 + |\alpha'|$. Thus, redefine

$$|\alpha| = 2\alpha_0 + \alpha_1 + \cdots + \alpha_n$$

We call the operators associated to this setting anisotropic operators.

Of course one can try to fit all other operators into this framework. For instance, in this section the operator $-\Delta_{t,x}$ has order four and principal symbol s^2. This would definitely be misleading. The philosophy is that for each operator there exists a natural setting in which it can be studied. Elliptic and hyperbolic equations, for instance, fit just fine into the classical (isotropic) setting. On the other hand, the parabolic equations, the Schroedinger equation, and the Euler-Bernoulli plate model correspond to the framework developed in this section. The KdV equation fits into neither of these two and requires its own setting.

To avoid excessive generality we confine ourselves to the classical (isotropic) and the parabolic setting. Based on this the interested reader can easily adapt the arguments to other types of equations.

What is the Hamilton field associated to an anisotropic operator P in this context ? To understand this, look at the order of the terms which appear in the classical Hamilton field. The terms $p_x \frac{\partial}{\partial \xi}$, $p_\xi \frac{\partial}{\partial x}$ have order $m-1$. On the other hand, the terms $p_t \frac{\partial}{\partial s}$, $p_s \partial \partial t$ have order $m-2$. It is natural now to drop the lower order terms and set

$$H_p = p_\xi \frac{\partial}{\partial x} - p_x \frac{\partial}{\partial \xi} \tag{1.14}$$

The Poisson bracket is altered correspondingly,

$$\{p, q\} = H_p q = -H_q p = p_\xi q_x - q_\xi p_x \tag{1.15}$$

(no time derivatives).

Look now at the definitions of the strongly pseudoconvex surfaces. In factors of the form $D_t + i\tau\phi_t$ the second term has order 1 while the first has order two. Hence, to be consistent with the previous reasoning, we drop the lower order term. Consequently, given also the new choice for the Poisson bracket, the pseudoconvexity conditions will contain no time derivatives of ϕ, therefore they are completely uncoupled in time. This implies that one can apply these conditions to surfaces which live on time sections:

Definition 1.6 *Let $S \subset \{t = t_0\}$ be a C^2 hypersurface. Let ϕ be a real valued C^2 function vanishing simply on S. We say that S is strongly pseudoconvex with respect to P at $(t_0, x_0) \in S$ if*

$$Re\{\bar{p}, \{p, \phi\}\}(x_0, \xi) > 0 \text{ whenever} \tag{1.16}$$

$$p(x_0, \xi) = \{p, \phi\}(x_0, \xi) = 0 \ , \ \xi \neq 0 \tag{1.17}$$

$$\{\bar{p}(x_0, \xi - i\tau\nabla\phi), p(x_0, \xi + i\tau\nabla\phi)\}(x_0, \xi)/\tau i > 0 \ \ \text{whenever} \tag{1.18}$$

$$\{p(x_0, \xi + i\tau\nabla_x\phi) = \{p(x_0, \xi + i\tau\nabla_x\phi), \phi\}(x_0, \xi) = 0, \tau > 0\} \tag{1.19}$$

Definition 1.7 *We say that the function ϕ is strongly pseudoconvex with respect to P at x_0 if*

$$Re\{\bar{p}, \{p, \phi\}\}(x_0, \xi) > 0 \text{ whenever} \tag{1.20}$$

$$p(x_0, \xi) = 0 \ , \ \xi \neq 0 \tag{1.21}$$

$$\{\bar{p}(x, \xi - i\tau\nabla_x\phi), p(x, \xi + i\tau\nabla_x\phi)\}/\tau i > 0 \text{ on } \{p(x, \xi + i\tau\nabla_x\phi) = 0, \ \ \tau \geq 0, \ (\xi, \tau) \neq 0\} \tag{1.22}$$

1.4 Second order operators of real principal type

The most common class of operators arising in applications is the second order operators. This section is devoted to a detailed examination of the pseudoconvexity condition for second order operators of real principal type. The main result is

Theorem 1.8 *Let $P(x, D)$ be a second order operator of real principal type. a) Let Σ be a noncharacteristic C^2 hypersurface. Then Σ is strongly pseudoconvex with respect to P iff*

$$Re\{p, \{p, \phi\}\}(x_0, \xi) > 0 \text{ whenever} \tag{1.23}$$

$$p(x_0, \xi) = \{p, \phi\}(x_0, \xi) = 0 \ , \ \xi \neq 0 \tag{1.24}$$

Proof : We need to show that (1.8) (respectively (1.19) in the anisotropic case) is always fulfilled. If

$$p(x, \xi + i\tau\nabla\phi) = \{p(x, \xi + i\tau\nabla\phi), \phi\}$$

then $i\tau$ is a double root for the polymomial

$$q(z) = p(x, \xi + z\nabla\phi)$$

If Σ is noncharacteristic, i.e. $p(x, \nabla\phi) \neq 0$ then q is a second degree polynomial with real coefficients. Hence, its double root has to be real, which implies that $\tau = 0$.

If Σ is characteristic then q has degree at most one. Hence, in order to have a double root it needs to be identically 0. This is equivalent to

$$p(x, \xi) = \{p, \phi\}(x, \xi) = 0 \tag{1.25}$$

On the other hand

$$\begin{aligned}
\frac{1}{\tau i}\{\overline{p(x, \xi + i\tau\nabla\psi)}, p(x, \xi + i\tau\nabla\phi)\} &= \{p(x, \xi) - \tau^2 p(x, \nabla\phi), p_\xi(x, \xi)\nabla\phi\} \\
&= p_\xi(x, \xi)\phi_{xx}p_\xi(x, \xi) + \tau^2 p_\xi(x, \nabla\phi)\phi_{xx}p_\xi(x, \nabla\phi) \\
&= \{p, \{p, \phi\}\}(x_0, \xi) + \tau^2\{p, \{p, \phi\}\}(x_0, \nabla\phi)
\end{aligned}$$

This should be positive for any $\tau \geq 0$, which is equivalent to

$$\{p, \{p, \phi\}\}(x_0, \xi) > 0, \quad \{p, \{p, \phi\}\}(x_0, \nabla\phi) > 0$$

Now the first inequality above is exactly the pseudoconvexity condition at $\tau = 0$ (with ξ satisfying (1.25) while the second is the pseudoconvexity condition at $\tau = 0$, $\xi = \nabla\phi$ (which satisfies (1.25).

With small modifications the same argument aplies also to the anisotropic case, q.e.d.

1.4.1 A geometrical interpretation of pseudoconvexity

The second aim of this section is to give a simple geometrical interpretation of the pseudoconvexity condition in this case. Rewrite the pseudoconvexity condition (1.24) as

$$H_p^2\phi > 0 \quad \text{whenever} \quad \phi = H_p\phi = 0$$

Recall now that $H_p\phi$, $H_p^2\phi$ represent the first, respectively the second derivative of ϕ along the bicharacteristics of P. Then the above relation says that the null bicharacteristics of P are either transversal to Σ or, if they are tangent to Σ then they are "curved" toward $\Sigma^+ = \{\phi > 0\}$ (see the figure below).

1.5 Boundary value problems and the Lopatinskii condition

Let K be a domain in R^n with C^1 boundary ∂K. Given $x_0 \in$ choose local coordinates near it so that at x_0 we have $x_n = 0$, $dx_n \in \mathcal{N}^*\partial K$ (i.e. the plane $x_n = 0$ is tangent to ∂K at x_0)

1.5.1 The strong Lopatinskii condition

At each point $\gamma' = (x_0, \xi_0') \in T^*\partial K$ decompose $p(x_0, \xi + i\tau\nabla\phi)$ as polynomial in ξ_n:

$$p(x_0, \xi + i\tau\nabla\phi) = p^-(x_0, \xi_0', \xi_n))p^+(x_0, \xi_0', \xi_n) \prod(\xi_n - \xi_n^{(j)})^{m_j} \tag{1.26}$$

where the three factors stand for the roots with positive imaginary part, negative imaginary part, respectively for the real roots. This decomposition extends uniquely to a smooth decomposition in a conic neighbourhood of γ',

$$p(x, \xi + i\tau\nabla\phi) = p^-(x, \xi', \xi_n, \tau)p^+(x, \xi', \xi_n, \tau) \prod p_j(x, \xi', \xi_n, \tau)$$

When $\tau = 0$ define for each j:

$$n_j = min\{k \geq 0/ \ D^k_{\xi_n}\{p_j, \phi\}(\xi_n^{(j)}) \neq 0\}$$

and

$$l_j = \begin{cases} 0 & if \quad D^{n_j}_{\xi_n}\{p_j, \phi\}(\xi_n^{(j)}) < 0 \\ 1 & if \quad D^{n_j}_{\xi_n}\{p_j, \phi\}(\xi_n^{(j)}) > 0 \end{cases}$$

Define $p_0(x_0, \xi_0', \xi_n, \tau)$ by

$$p_0(x_0, \xi_0', \xi_n, \tau) = \begin{cases} p^+(x_0, \xi_0', \xi_n, \tau) \prod(\xi_n - \xi_n^{(j)})^{m_j} & if \ \tau > 0 \\ p^+(x_0, \xi_0', \xi_n, \tau) \prod(\xi_n - \xi_n^{(j)})^{m_j^+} & if \ \tau = 0 \end{cases}$$

where $m_j^+ = [\frac{m_j + n_j - l_j}{2}]$.

Definition 1.9 *We say that the boundary operators B satisfy the strong Lopatinskii condition with respect to S at $x_0 \in S \cap \partial K$ if their principal symbols $b^j(x_0, \xi + i\tau\nabla\phi)$ are complete modulo $p_0(x_0, \xi, \tau)$ as polynomials in ξ_n for each $\tau \geq 0$, $(\xi', \tau) \neq 0$.*

This definition includes the classical strong Lopatinskii condition for the case when P is strictly hyperbolic with respect to $d\phi$. That case is simpler due to the fact that $n_j = 0$ and furthermore, no real roots (for $p(x, \xi + i\tau\nabla\phi)$ as a polynomial in ξ_n) can occur when $\tau > 0$. The definition applies as well to anisotropic problems.

Remark 1.10 *a) If the strong Lopatinskii condition is fulfilled at some point x_0 then it is also fulfilled in a neighbourhood.*

b) The strong Lopatinskii condition is stable with respect to small C^1 perturbations of the coefficients and of the boundary ∂K.

It is useful to note that the above definition is coordinate independent. Indeed, one can reformulate it as it follows:

At each $\gamma \in T^*K \cap \partial K$ decompose $p(\gamma + \lambda N + i\tau d\phi)$ as polynomial in λ. Here $N \in N_{x_0}\partial K$ (the conormal bundle of the boundary ∂K) and λ plays the role of ξ_n above. With this new notations, it is clear that n_j, l_j, p_0 are invariantly defined. Then the invariant formulation of the above definition is

Definition 1.11 *We say that the boundary operators B satisfy the strong Lopatinskii condition with respect to $d\phi$ iff $b^j(x_0, \gamma + \lambda N + i\tau d\phi)$ are complete modulo $p_0(x_0, \gamma + \lambda N + i\tau d\phi)$ as polynomials in λ for each $\tau \geq 0$.*

1.5.2 The weak Lopatinskii boundary condition

With the same notations as in the previous section assume that at each $\gamma' = (x, \xi')$ the symbol $p(x_0)$ has at most one multiple root as polynomial in ξ_n, and that root is double.

1.5.3 Second order operators

Theorem 1.12 *Assume that P is a second order operator of real principal type. Let B be as above. Then B satisfies the strong Lopatinskii condition with respect to $d\phi$ iff either*

i) The pair (P, B) generates a strongly well-posed hyperbolic problem and $d\phi$ is time-like.

ii) $\dfrac{\partial \phi}{\partial \nu} < 0$ *and* B *is the Dirichlet boundary condition.*

iii) $\dfrac{\partial \phi}{\partial \nu} < 0$, B *is any Neuman boundary condition and* P *is elliptic.*

iv) $\dfrac{\partial \phi}{\partial \nu} < 0$, B *is any Neuman boundary condition and*

In this case one can choose local coordinates near x_0 so that $p(x_0, \xi)$ has the form

$$p(x, \xi) = \xi_n^2 - r(x, \xi')$$

a) The case $\tau > 0$.

In this case the two roots are

$$\xi_n = -i\tau\phi_n \pm (r(x, \xi + i\tau\nabla'\phi))$$

At least one should have positive imaginary part. This clearly happens if $\phi_n < 0$.

Suppose that $\phi_n \geq 0$. Then $r(x, \xi + i\tau\nabla'\phi) \not\in R^+$. Hence, $r(\nabla\phi) > 0$ and $r(\xi) < 0$ on $r(\xi, \nabla\phi) = 0$. This implies that R is hyperbolic and $\nabla'\phi$ is time-like with respect to R, and further that P is hyperbolic and ∂K is time-like.

In addition, $p(x, \xi + i\tau\nabla\phi)$ should have no real roots when $\tau > 0$ therefore, as argued before for r, $\nabla\phi$ is also hyperbolic with respect to P.

Consequently, either $\dfrac{\partial \phi}{\partial \nu} < 0$ or we are dealing with a strongly well-posed hyperbolic initial-boundary value problem.

Conversely, assume that $\dfrac{\partial \phi}{\partial \nu} < 0$.

Then p_0 has degree either 0 or 1. Furthermore, if it has degree 1 then it has the form

$$p_0 = \xi_n - \alpha, \quad \text{Im } \alpha > 0$$

In either case the strong Lopatinskii condition follows.

b) The case $\tau = 0$.

In the region $r < 0$ we have two conjugate imaginary roots, therefore

$$p_0 = \xi_n + i\sqrt{-r}$$

and the strong Lopatinskii condition follows.

In the region $r = 0$ we have either $n_j = 0$ or $n_j = 1$ and $l_j = 0$. Hence $m_j^+ = 1$ therefore

$$p_0 = \xi_n$$

In the region $r > 0$ p has two factors,

$$p_{1,2} = \xi_n \pm r^{1/2}$$

Then

$$\{p_1, \phi\} + \{p_2, \phi\} = 2\phi_n < 0$$

therefore at least one is negative. Hence p_0 has degree 0 or 1. In the first case, the Lopatinskii condition follows. In the second, we need

$$l(x, \xi) \neq r^{1/2} \quad if \{r, \phi\} \geq 0$$

and

$$l(x, \xi) \neq -r^{1/2} \quad if \{r, \phi\} \leq 0$$

1.6 Notes

1.6.1 Characteristic vs. pseudoconvex surfaces

The pseudoconvexity condition for a surface Σ does not require that the surface be noncharacteristic. However, if Σ is characteristic at some x_0 then (1.8) should hold at $x = x_0, \xi = 0$. Note that $p(x_0, \nabla\phi) = 0$ implies that

$$\{p, \phi\}(x_0, \nabla\phi) = 0$$

Then (1.8) at $x = x_0, \xi = 0$ can be rewritten as

$$\{p, p(x, \nabla\phi)\}(x_0, \nabla\phi(x_0)) > 0 \tag{1.27}$$

In other words, the surface Σ should change type at x_0 along the bicharacteristic conormal to Σ at x_0.

1.6.2 Critical points for pseudoconvex functions

As far as pseudoconvex functions are concerned, another degenerate situation is when $\nabla\phi = 0$ at some point x_0. Then the pseudoconvexity condition for ϕ with respect to P necessarily fails when $\xi = 0$, $\tau > 0$. However, it could still hold except for this limiting case, therefore it makes sense to use a weaker replacement for (1.22), namely

$$\{\bar{p}(x,\xi - i\tau\nabla_x\phi), p(x,\xi + i\tau\nabla_x\phi)\}/\tau i > c(|\xi| + \tau|\nabla\phi|^{2m-2})$$

$$\text{on } \{p(x,\xi + i\tau\nabla_x\phi) = 0, \quad \tau \geq 0, \ (\xi,\tau) \neq 0\} \tag{1.28}$$

When x is in a sufficiently small neighbourhood of x_0 this condition reduces to

$$p_\xi(x,\xi + i\tau\nabla_x\phi)(D^2\phi(x_0))p_\xi(x,\xi - i\tau\nabla_x\phi) > c(|\xi| + \tau|\nabla\phi|^{2m-2})$$

$$\text{on } \{p(x,\xi + i\tau\nabla_x\phi) = 0, \quad \tau \geq 0, \ (\xi,\tau) \neq 0\}$$

A sufficient condition for this is

$$p_\xi(x_0,\xi)(D^2\phi(x_0))p_\xi(x_0,\overline{\xi}) > 0 \text{ on } \{p(x,\xi) = 0, \quad \xi \in C, \quad \xi \neq 0\} \tag{1.29}$$

This condition is also necessary in the simplest case, when the Hessian $D^2\phi(x_0)$ is nondegenerate. Then $|\nabla\phi| \approx |x - x_0|$.

Chapter 2

The pdo calculus

2.1 The τ- dependent spaces and operators

Treating τ as a derivative requires changing the H^s scale of spaces into H^s_τ defined by

$$u \in H^s_\tau \iff \hat{u}(|\xi|^2 + \tau^2)^{s/2} \in L^2$$

A partial differential operator of order m is, in this setting, an operator of the form

$$P(x, D, \tau) = \sum_{|\alpha|+\beta \leq m} c_{\alpha,\beta}(x) D^\alpha \tau^\beta$$

Its principal symbol is then

$$p(x, \xi, \tau) = \sum_{|\alpha|+\beta \leq m} c_{\alpha,\beta}(x) \xi^\alpha \tau^\beta$$

Define the following classes of symbols:

$$\begin{aligned} &S^p_\tau = \{a(x, \xi, \tau)/|D^\alpha_x D^\beta_\xi a| \leq c_{\alpha,\beta}(1 + |(\xi, \tau)|)^{p-|\beta|}\} \\ &C^k S^p_\tau = \{a(x, \xi, \tau)/|D^\alpha_x D^\beta_\xi a| \leq c_{\alpha,\beta}(1 + |(\xi, \tau)|)^{p-|\beta|},\ |\alpha| \leq k\} \end{aligned}$$

We use the subscript cl to indicate the corresponding spaces of homogeheous symbols (e.g. $S^{k,p}_{cl}$).

The pseudodifferential calculus for operators with symbols in these classes mirrors the calculus for the corresponding classical framework. The main point is that the properties of such operators hold uniformly τ. Following we include for reference some results:

Theorem 2.1 *(mapping properties) Let $a \in C^k S^m_{cl}$. Then*

$$A(x, D, \tau) : H^{s+m}_\tau \to H^s_\tau \qquad |s| \leq k$$

Theorem 2.2 *(composition) a) $a \in C^k S^p_{cl}, b \in S^m$. Then*

$$A(x,D,\tau)B(x,D,\tau) - (AB)(x,D,\tau) : H_\tau^{s+m+p} \to H_\tau^s \qquad |s| \leq k$$

b) $a \in C^1 S^j_{cl}, b \in C^1 S^m$, $0 \leq j \leq 1$. Then

$$A(x,D,\tau)B(x,D,\tau) - (AB)(x,D,\tau) : H_\tau^{s+m+j+1} \to H_\tau^s \qquad 0 \leq s \leq 1-j$$

Theorem 2.3 *(commutators) Let $a \in C^1 S^j_{cl}, b \in C^1 S^k_{cl}$, $0 \leq j \leq k \leq 1$. Then*

$$[A(x,D,\tau), B(x,D,\tau)] H_\tau^{s+j-1} \to H_\tau^s \qquad 0 \leq s \leq 1-k$$

Theorem 2.4 *(adjoints) Let $a \in C^1 S^j_{cl}$, $0 \leq j \leq 1$. Then*

$$A^*(x,D,\tau) - A(x,D,\tau)^* : H_\tau^{s+j-1} \to H_\tau^s \qquad |s| \leq 1-j$$

Theorem 2.5 *(Garding's inequality) Let $a \in C^1 S^2_{cl}$ so that*

$$a(x,\xi,\tau) + a(x,\xi,\tau)^* \geq 2c(|\xi|^2 + \tau^2)$$

Then

$$\text{Re } (A(x,D)u, u) \geq c|u|^2_{1,\tau} - d|u|_0^2$$

2.2 The calculus for the L^2 estimates

This section contains the main ingredients necessary for the L^2 estimates.

2.2.1 An application of Garding's theorem

We start with the following simple result:

Lemma 2.6 *Assume that P, Q are partial differential operators with real C^1 coefficients.*

a) If P, Q have both order m then

$$|\text{Im } < Pu, Qu > | \leq c|u|_{m,\tau}|u|_{m-1,\tau}$$

b) If P has order m and Q has order $m-1$ then

$$|\text{Im } < Pu, Qu > | \leq c|u|^2_{m-1,\tau}$$

Proof : Start from

$$2\text{Im} \ < Pu, Qu >= -i(< Pu, Qu > - < Qu, Pu >)$$

and integrate by parts.

The main result in this section is

Theorem 2.7 *a) Let P, Q be partial differential operators of order $m+1$, respectively m with real C^1 coefficients. Let $R_1, ...R_j$ be partial differential operators of order m with C^0 coefficients. Assume that*

$$\{p, q\} > c|(\xi, \tau)|^{2m} \text{ on } U\text{char } R_j$$

while

$$p = q = 0 \text{ on } U\text{char } R_j$$

Then there exists $d > 0$ so that

$$|u|_{m,\tau} \leq 2Im < Pu, Qu > +d \sum |Ru|^2 \tag{2.1}$$

for sufficiently large τ.

b) Let P, Q be partial differential operators of order m, respectively $m-1$ with real C^1 coefficients. Let $R_1, ...R_j$ be partial differential operators of order m with C^1 coefficients. Assume that

$$\{p, q\} > c|(\xi, \tau)|^{2m-2} \text{ on } U\text{char } R_j$$

while

$$p = q = 0 \text{ on } U\text{char } R_j$$

Then there exists $d > 0$ so that

$$c|u|_{m-1,\tau} \leq 2Im < Pu, Qu > +d \sum |Ru|^2_{-1,\tau} \tag{2.2}$$

for sufficiently large τ.

Proof : a) Choose d sufficiently large so that

$$\{p, q\} + p_{x\xi}q - q_{x\xi}p + d \sum |r_j|^2 > c|(\xi, \tau)|^{2m} \tag{2.3}$$

Then we claim that (2.1) holds.

First we reduce the problem to the smooth coefficient case. Consider approximations of P, Q, R_j in the corresponding classes of operators, with smooth coefficients. Then (2.3) is preserved, while the change in the RHS in (2.1) is negligible.

Now we prove that (2.3) implies (2.1) in the smooth coefficient case. We have

$$2Im < Pu, Qu > +d \sum |Ru|^2 =< Au, u >$$

where A is a partial differential operator of order $2m$,

$$A(x, D) = -i(P^*Q - Q^*P) + \sum R_i^* R_i$$

with principal symbol

$$A(x, \xi) = \{p, q\} + p_{x\xi} q - q_{x\xi} p + d \sum |r_j|^2$$

By (2.3) Garding's inequality gives

$$< Au, u >\geq c|u|^2_{m,\tau} - c_1 |u|^2_{m-1,\tau}$$

which implies (2.1) if τ is sufficiently large.

b) Choose d sufficiently large so that

$$\{p, q\} + p_{x\xi} q - q_{x\xi} p + d \sum |r_j|^2 (\xi^2 + \tau^2)^{-1} > c|(\xi, \tau)|^{2m-2} \tag{2.4}$$

Then we claim that (**??**) holds.

We argue as in case (a). First, the problem reduces to the smooth coefficient case. Then we have

$$2Im < Pu, Qu > +d \sum |Ru|^2 =< Au, u >$$

where A is a pseudodifferential operator of order $2m - 2$ with principal symbol

$$A(x, \xi) = \{p, q\} + p_{x\xi} q - q_{x\xi} p + d \sum |r_j|^2 (\xi^2 + \tau^2)^{-1}$$

Then (**??**) follows from (2.4) by Garding's inequality.

2.2.2 Regularization

Many of the Carleman estimates are formally obtained only for sufficiently smooth functions. In order to lower the required a-priori regularity, we need to develop suitable regularization devices. We start with a simple result:

Theorem 2.8 *Let P be a partial differential operator of order m with C^1 coefficients. Let $u \in H^{m-1}$ such that $P(x,D)u \in L^2$. Then there exists a sequence $u_\epsilon \in H^m$ so that*

$$u_\epsilon \to u \qquad \textit{in } H^{m-1}$$

$$P(x,D)u_\epsilon \to P(x,D)u \qquad \textit{in } L^2$$

Proof : Let $\epsilon > 0$ and set

$$u_\epsilon = (1 + \epsilon \ll)^{-1} u$$

Clearly $u_\epsilon \in H^m$, $u_\epsilon \to u$ in H^{m-1}. On the other hand,

$$Pv_\epsilon = (1 + \epsilon \ll)^{-1} P_\tau v + [P_\tau, (1 + \epsilon \ll)^{-1}]v$$

The first RHS term converges to $P_\tau v$ in L^2. The second RHS term equals

$$(1 + \epsilon \ll)^{-1}[P_\tau, L]\epsilon(1 + \epsilon \ll)^{-1} v$$

which converges to 0 in L^2 since by Theorem [**?**] $[C^1, L]$ is L^2 bounded and

$$\epsilon(1 + \epsilon \ll)^{-1} v \to 0 \ \text{ in } \ H^m$$

2.3 The calculus for estimates at other energy levels

2.3.1 Function spaces for the coefficients

Define the following family of spaces:

Definition 2.9 *Let $s \geq 1$. We say that $a(x) \in \Xi_s$ iff*
(i) $a \in C^1$.
(ii) $\nabla a : H^{s-1} \to H^{s-1}$.

If $s > n/2 + 1$ then it is easy to see that $\Xi_s = H^s$. If $1 < s \leq n/2 + 1$ then we have the simple inclusion

$$C^1 \cap H^{s,p} \subset \Xi_s, \quad 1/p = \frac{s-1}{n}$$

However, the converse is false.

2.3.2 Commutator estimates

Theorem 2.10 *Let* $s \geq 1$, $f \in \Xi_s$. *If* $a(x,\xi) \in S^r$ *with* $|r| \leq s$ *then*

$$[A, f] : H^q \to H^{q-r+1}, \qquad -s + r^+ \leq q \leq s - 1 + r_- \tag{2.5}$$

Proof : *First reduction* to the case $r \geq 0$. Note that w.a.r.g. we can assume that a is elliptic. Then the reduction follows from the formula

$$[A, f] = A[A^{-1}, f]A + S^{-\infty} f + f S^{-\infty}$$

Second reduction to the case $q = s - 1$. This follows by transposition and interpolation.

Third reduction to the case $r = s$, $r = 0$. This follows by analytic interpolation.

Fourth reduction to the case $r = s$. This follows from the formula

$$S^s[f, S^0] = [f, S^s] - [f, S^s]$$

Conclusion Set $s = k + \alpha$, $k \in N^+$, $0 \leq \alpha < 1$. Then decompose $S^s = \sum D^k S^\alpha$. Consequently,

$$[f, S^s] = \sum([f, S^\alpha]D^k + S^\alpha[f, D^k])$$

Now by Theorem 2.10 $[f, S^\alpha]$ maps $H^{\alpha-1}$ into L^2. Thus, it remains to prove that

$$[f, D^k] : H^{s-1} \to H^\alpha$$

This reduces to

$$D^j \nabla f D^{k-j-1} : H^{s-1} \to H^\alpha, \quad j = 0, k-1$$

and further to

$$\nabla f : H^{\alpha+j} \to H^{\alpha+j}, \quad j = 0, k-1$$

which follows from the definition of Ξ_s.

The next theorem shows what happens when we try to obtain a second order calculus.

Theorem 2.11 *Let* $f \in C^2$, *and* $a \in S^{j+1}$, $|j| \leq 1$. *Then the operator*

$$R = [A(x, D), f] - i f_x A_\xi(x, D)$$

has the following mapping properties:

$$R : H^s \to H^{s+j-1}, \quad j_+ - 1 \leq s \leq j_-$$

Theorem 2.12 *bmo1/2*

2.3.3 Regularization

Many of the Carleman estimates are formally obtained only for sufficiently smooth functions. In order to lower the required a-priori regularity, we need to develop suitable regularization devices. We start with a simple result:

Theorem 2.13 *Let P be a partial differential operator with Ξ_r coefficients, and $1-s \leq r \leq s$. Let $u \in H^{s+m-1}$ such that $P(x,D)u \in H^s$. Then there exists a sequence $u_\epsilon \in H^{s+m}$ so that*

$$u_\epsilon \to u \qquad in\ H^{s+m-1}$$

$$P(x,D)u_\epsilon \to P(x,D)u \qquad in\ H^{s+m-1}$$

Proof : Let $\epsilon > 0$ and set

$$u_\epsilon = (1+\epsilon \ll)^{-1}u$$

Clearly $u_\epsilon \in H^{s+m}$, $u_\epsilon \to u$ in H^{m+s-1}. On the other hand,

$$Pv_\epsilon = (1+\epsilon \ll)^{-1}P_\tau v + [P_\tau, (1+\epsilon \ll)^{-1}]v$$

The first RHS term converges to $P_\tau v$ in H^s. The second RHS term equals

$$(1+\epsilon \ll)^{-1}[P_\tau, L]\epsilon(1+\epsilon \ll)^{-1}v$$

which converges to 0 in H^s since by Theorem [] $[P_\tau, L]$ is bounded from H^m_τ into L^2 and

$$\epsilon(1+\epsilon \ll)^{-1}v \to 0 \ \text{ in } \ H^m$$

The following theorem goes one step further.

Theorem 2.14 *Let $P(x,D,\tau)$ be a partial differential operator with Ξ_r coefficients, and $1-r \leq s \leq r$, $s_0 = \max\{1-r, s-1\}$. Let $v \in H^{j+m-1}$ such that $P_\tau v \in H^s$. Then the sequence $v_\epsilon \in H^{s+m-1}$ defined by*

$$u_\epsilon = (1+\epsilon \ll)^{-1}u$$

satisfies

$$u_\epsilon \to u \qquad in\ H^{s+m-2}$$

$$P_\tau u_\epsilon \to P_\tau u \qquad \text{in } H^{s-1}$$

and for any smooth function z with compact support we have

$$|z^2 P_\tau v_\epsilon|_s \le c(|z^2 v_\epsilon|_{s+m-1} + |v_\epsilon|_{s_0+m-1}) \tag{2.6}$$

Proof : As in the previous theorem we get

$$P(x,D)u_\epsilon \to P(x,D)u \qquad \text{in } H^{s-1}$$

Compute now

$$\begin{aligned} z^2 P u_\epsilon &= z^2(1+\epsilon \ll)^{-1} P u + \epsilon z^2 (1+\epsilon \ll)^{-1} [P, \ll] u_\epsilon \\ &= (S^0 z^2 + S^{-1} z + S^{-2}) P u + \epsilon (1+\epsilon \ll)^{-1} [P,\ll] z^2 u_\epsilon + \epsilon S^{-1} [P,\ll] (1+\epsilon \ll)^{-1} u \\ &+ \epsilon S^0 [z^2, [P,\ll]] (1+\epsilon \ll)^{-1} v \end{aligned}$$

Since the coefficients a_i of P are Ξ_r it follows that $[P,\ll]$ is bounded from H^{r+m-1} into H^{r-1}. On the othe hand, the double commutator $[z^2,[P,\ll]]$ can be rewritten as

$$[z^2,[P,\ll]] = [z^2,[\Xi_r,\ll]D^m] = [\Xi_r,\ll]S^{m-1} + [\Xi_r,S^0]S^m$$

Thus, by Theorem 2.10 it is mapping H^{r+m-1} into H^r. Consequently, we get

$$|z^2 P u_\epsilon|_s \le c|z^2 u_\epsilon|_{m+s-1} + c_z(|zPu|_s + |u|_{m+s-2})$$

q.e.d.

The following addition will also be useful.

Theorem 2.15 *Assume that* supp $v \subset \Sigma^+$. *Suppose $A \in OPS^m_\tau$. Then*

$$|\phi^j A v| s + j - m, \tau, \Sigma^- \le |v|_{s,\tau}$$

Proof : By interpolation and duality it suffices to consider the case when s, m are integers satisfying $s, s+m-j \ge 0$. Furthermore, by representing A as $A = BD^s$ with $B \in OPS^{m-s}_\tau$ the problem reduces further to the case $s = 0$. Hence, we want to prove that

$$|\phi^j A v| j - m, \tau, \Sigma^- \le |v|$$

for $j > m$ and v supported in Σ^+. This is equivalent to

$$|D^{j-m}\phi^j OPS_\tau^m v|_{\Sigma^-} \leq |v|$$

and after commuting D^{m-j} with ϕ^j the estimate reduces to

$$|\phi^j OPS_\tau^j v|_{\Sigma^-} \leq |v|$$

for j positive integer.

Choose new local coordinates so that $\phi(x) = x_n$. Without any restriction in generality we can assume that the symbol of $A \in S_\tau^j$ has the form

$$a(x, \xi, \tau) = b(x)c(\xi, \tau)$$

where b is smooth and $c \in S_\tau^j$. Then we can factor out c in the estimate, and what is left is to prove that

$$|x_n^j C(D, \tau)v|_{\Sigma^-} \leq |v|$$

This can be reduced to the one dimensional case by taking the tangential Fourier transform. In the one-dimensional case, the kernel $K(x, y)$ of $x_n^j C(x, D)$ satisfies

$$K(x, y) \leq |x|^j |x - y|^{-j-1}$$

We are only interested in $y > 0$, $x < 0$. In this case we get

$$K(x, y) \leq |x - y|^{-1}$$

Then the estimate

$$|x_n^j C(D, \tau)v|_{R^-} \leq |v|_{R^+}$$

follows from the L^2 boundedness of the Riesz transform.

Theorem 2.16 *Let $s > 0$. Then*

$$H_\tau^s \subset H_{0,\tau}^s + H_\tau^s(\Sigma) \tag{2.7}$$

Proof : Consider local coordinates in which $\phi(x) = x_n$. Let a be a positive, smooth, compactly supported function which equals 1 of infinite order at the origin. Then consider the operator A with symbol $a(x_n|(\xi,\tau)|)$. We claim that for $u \in H^s_\tau$ the desired decomposition is

$$u = (1-A)u + Au$$

First, prove that $Au \in H^s_\tau(\Sigma)$. For this we need to show that

$$(D,\tau)^j x_n^j A : H^s_\tau \to H^s_\tau$$

or equivalent, that

$$D^j x_n^j A : H^s \to H^s$$

Taking the tangential Fourier transform reduces the problem to a one dimensional problem of the form

$$B : H^s \to H^s$$

where B has a symbol of the form $b(x_n, x_n\xi_n)$ with b smooth, compactly supported. Then the above boundedness result is a consequence of Melrose's calculus of totally characteristic operators.

Next we prove that $(1-A)u \in H^0_s$. This follows from the stronger statement:

$$x_n^{-j}(1-A) : H^s_\tau \to H^{s-j}_\tau$$

This is equivalent to

$$x_n^{-j}(1-A)(D,\tau)^{-j} : H^s_\tau \to H^s_\tau$$

The symbol of $x_n^{-j}(1-A)(D,\tau)^{-j}$ has again the form $b(x_n|(\xi,\tau)|)$ where b is smooth and compactly supported. Then its mapping properties can be easily obtained as above.

2.4 The boundary calculus

2.4.1 Interior and boundary bilinear forms

In the sequel we use the following classes of interior bilinear forms:

Definition 2.17 *Let $\Sigma \in R$, $\mu \geq 0$. We say that an interior bilinear form B given by*

$$B(u,v) = \sum_k < A^k(x,D,\tau)u, C^k(x,D,\tau)v > \qquad (2.8)$$

is of type $(m-1,\Sigma,\mu)$*, if for each* k *we have* $a_k(x,\xi,\tau)\in C^1S_{cl}^{m,\alpha-1}$ *and* $b_k\in C^1S_{cl}^{m-1,\beta}$ *for some* α,β *such that* $\alpha+\beta=2\Sigma$ *and* $|\alpha-\Sigma|\le\mu+1/2$*.*

For an interior bilinear form as above we define the principal symbol as

$$b(x,\xi,\tau)=\sum_k a^k(x,\xi,\tau)\overline{c^k(x,\xi,\tau)}\in C^1S^{2(m-1+\Sigma)}$$

If u is a function in K for which the first $m-1$ boundary traces are well defined and $a(x,\xi,\tau)\in C^1S^{m-1,p}$ then we use the notation $A(x,D,\tau)u_{|\partial K}$ (or simply Au when no confusion is possible) for the boundary operator A applied to u, i.e.

$$A(x,D,\tau)u_{|\partial K}=\sum_{i=0}^n A_i(x,D',\tau)D_n^iu$$

Correspondingly, we define the following classes of boundary bilinear forms

Definition 2.18 *Let* $\Sigma\in R$*,* $\mu\ge 0$*. We say that a boundary bilinear form* B*, acting on the boundary traces of smooth functions* u,v *in* K*, defined by*

$$B(u,v)=\sum_{k\in K,\ finite} < A^k(x,D',\tau)u,B^k(x,D',\tau)v>_\partial \tag{2.9}$$

is of type $(m-1,\Sigma,\mu)$ *if* $a^k\in C^1S_{cl}^{m-1,\alpha}$ *and* $b_k\in C^1S_{cl}^{m-1,\beta}$ *with* $\alpha+\beta=2\Sigma$ *and* $|\alpha-\Sigma|\le\mu+1/2$*.*

According to the operator estimates in Taylor [54] 2.1, it is easy to see that if B is a boundary bilinear form of type $(m-1,\Sigma,\mu)$ then is bounded by

$$|B(u,v)\le c(|u|_{m-1,\Sigma+\mu,\partial}|v|_{m-1,\Sigma-\mu,\partial}+|v|_{m-1,\Sigma+\mu,\partial}|u|_{m-1,\Sigma-\mu,\partial}) \tag{2.10}$$

Define the principal symbol of a boundary bilinear form B by

$$B(x,\xi',\tau,\xi_n,\tilde{\xi}_n)=\sum a^k(x,\xi',\xi_n)b^k(x,\xi',\tilde{\xi}_n)$$

For $z=(z_0,...,z_{m-1})\in C^{m-1}$, $a(x,\xi,\tau)\in C^jS^{k,p}$ denote

$$a(z)=\sum_{i=0}^{m-1}a_i(x,\xi',\tau)z_i$$

To the boundary bilinear form above associate also

$$b(x,\xi',\tau)(z,z)=\sum_k < a^k(z),\overline{b^k(z)}>$$

Note that the symbol $b(x,\xi',\xi_n,\tilde{\xi}_n)$ uniquely determines the bilinear form $b(x,\xi',\tau)(z,z)$.

Definition 2.19 *We say that a boundary bilinear form B of type $(m-1, \Sigma, 0)$ is positive definite if*

$$B(z,z) \geq c \sum_{j=0}^{m-1} |(\xi', \tau)|^{2(m-j-1-\Sigma)} |z_j|^2 \tag{2.11}$$

To an interior bilinear form B as in (2.8) attach the boundary bilinear form symbol symbol

$$q_B(x, \xi', \tau, \xi_n, \tilde{\xi}_n) = \sum_k Q_{A_k,B_k}(x, \xi', \tau, \xi_n, \tilde{\xi}_n)$$

where

$$q_{P,Q}(x, \xi', \tau, \xi_n, \tilde{\xi}_n) = \frac{i}{2} \, \frac{p(x, \tau, \xi', \xi_n)\overline{q(x, \tau, \xi', \tilde{\xi}_n)} + \overline{p(x, \tau, \xi', \tilde{\xi}_n)} q(x, \tau, \xi', \xi_n)}{\xi_n - \tilde{\xi}_n} \tag{2.12}$$

Note that q_B is well-defined (i.e. a polynomial in $\xi_n, \tilde{\xi}_n$) iff Re $b(x, \xi, \tau) = 0$. If this happens then q_B is symmetric, i.e.

$$q_B(x, \xi', \tau, \tilde{\xi}_n, \xi_n) = \overline{q_B(x, \xi', \tau, \xi_n, \tilde{\xi}_n)}$$

For a bilinear form B as in (2.8) define also its subprincipal symbol $b^1(x, \xi, \tau)$ as the (formal) subprincipal symbol of the operator

$$\sum_k C_k^* A_k$$

More precisely, let

$$b^1(x, \xi, \tau) = \sum_k b^1_{A_k,C_k}(x, \xi, \tau)$$

where

$$b^1_{A,C} = \frac{i}{2}(a_\xi \bar{c}_x - a_x \bar{c}_\xi + a_{x,\xi}\bar{c} - a\bar{c}_{x,\xi}) \tag{2.13}$$

2.4.2 Green's formula and Garding's type inequalities

Various formulas are called in the literature "Green's formula". The common point is to express an interior bilinear form with imaginary principal symbol of a certain order, say k, in terms of an interior bilinear form of order $k-1$ and a boundary bilinear form of order $k-1$. Next we state and prove several results of this type which are necessary in the sequel.

We start with

Lemma 2.20 *Let B be an interior bilinear form of type $(m-1, 1/2, 0)$ with purely imaginary symbol, of the form*

$$B(u,v) = \sum < A_k(x,D,\tau), B_k(x,D,\tau) >$$

where for each k the symbols $a_k(x,\xi,\tau), c_k(x,\xi,\tau)$ are in $C^1 S_{cl}^{m,-j}$ respectively $C^1 S_{cl}^{m-1,j}$ for some $j \in \{0,1\}$.

a) Then for each large enough τ we have

$$|\text{Re } (B(u,u) - Q_B(u,u))| \leq c|u|^2_{m,-1,\tau} \tag{2.14}$$

where Q_B is a boundary bilinear form of type $(m-1,0,0)$ with principal symbol q_B.

b) Assume in addition that the symbols a_k, c_k are smooth. Then

$$|\text{Re } (B(u,u) - B^1(u,u) - Q_B(u,u))| \leq d(|u|^2_{m,-3/2,\tau} + |u|^2_{m-1,-1/2,\partial}) \tag{2.15}$$

where $Q_{B,Q}$ is as in case (a) and B^1 is an interior bilinear form of type $(m-1,0,0)$ whose symbol has real part equal to Re $b^1(x,\xi,\tau)$, *i.e. the subprincipal symbol of B.*

Proof : We prove both parts simultaneously. Denote

$$L = Op((1+|\xi'|^2+\tau^2)^{1/2})$$

Since the real part of the symbol of B is 0 it follows that B can be decomposed into a sum of finitely many terms of the form

$$\begin{aligned} Z(u,u) &= < A(x,D',\tau)L^{m-j-1}D_n^j u, C(x,D',\tau)L^{m-k-1}D_n^k u > \\ &- < (\overline{A}C)(x,D',\tau)L^{m-k-1}D_n^k u, L^{m-j-1}D_n^j u > \end{aligned}$$

with $0 \leq j,k \leq m$, $j+k \leq 2m-1$ and $a \in C^1 S_{cl}^{0,1}, c \in C^1 S_{cl}^{0,0}$, respectively

$$W(u,u) = < A(x,D',\tau)L^{m-j-1}D_n^j u, L^{m-k-1}D_n^k u > - < A(x,D',\tau)L^{m-j-2}D_n^{j+1}u, L^{m-k}D_n^{k-1} >$$

with $0 \leq j \leq m-1$, $1 \leq k \leq m-1$ and $a \in C^1 S_{cl}^{0,1}$ (respectively the corresponding classes of smooth symbols in case (b)). Then it suffices to prove the result for these two types of quadratic forms.

For Z a simple computation gives

$$\text{Re } Z(u,u) = \text{Re } < FL^{m-j-1}D_n^j u, L^{m-k-1}D_n^k u > \tag{2.16}$$

where

$$F = C(x,D,\tau)^* A(x,D',\tau) - (\overline{A}C)(x,D',\tau)^*$$

It is easy to see that $q_Z = 0$. Then to conclude the proof in case (a) it suffices to note that the operator F is L^2 bounded (see Propositions 4.2.A,4.2.C in Taylor [53]). Then the RHS term in (2.16) is controlled by the RHS term in (2.14).

In case (b) F is a pdo in $OPS^{0,0}$ with principal symbol $f(x,\xi,\tau)$ given by

$$f = -i(\overline{c}_{x,\xi}a + \overline{c}_\xi a_x - (a\overline{c})_{x,\xi}).$$

Hence, if we set $d^j(\xi,\tau) = |(\xi',\tau)|^{m-j-1}\xi_n^j$ then the real part of the principal symbol of the RHS in (2.16) is

$$d^j d^k \text{Re } f = -d^j d^k (\text{Im } a_\xi \overline{c})_x$$

On the other hand the symbol z^1 is given by

$$\begin{aligned} z^1(x,\xi,\tau) &= \frac{i}{2}((ad^j)_\xi \overline{c}d^k - ad^j(\overline{c}d^k)_\xi - (\overline{a}cd^k)_\xi d^j + \overline{a}cd^k d^j_\xi)_x \\ &= i(d^j_\xi d^k - d^k_\xi d^j)(\text{Re } a\overline{c})_x + d^j d^k(-\text{Im } (a_\xi \overline{c}) + i\text{Re } (a\overline{c}_\xi))_x \end{aligned}$$

Hence the real part of the symbol of the RHS quadratic form in (2.16) equals the real part of z^1, which implies (b) for Z.

For W integration by parts gives

$$\begin{aligned} W(u,u) &= < [A,L]L^{m-j-1}D_n^j u, L^{m-k-1}D_n^k u > + < iA_{x_n}L^{m-j}D_n^j u, L^{m-k-1}D_n^{k-1}u > \\ &+ i < AL^{m-j-1}D_n^j u, L^{m-k}D_n^{k-1}u >_\partial \end{aligned} \tag{2.17}$$

The third RHS term is a boundary quadratic form with symbol $ia(x,\xi',\tau)l(\xi',\tau)^{2m-j-k-1}\xi_n^j\tilde{\xi}_n^{k-1}$ which, after symmetrization, equals q_W.

The proof for part (a) is concluded since the first two RHS terms are controlled by $|u|^2_{m,-1,\tau}$ (i.e. the RHS in (2.14)); indeed, according to Taylor [53] Prop. 1.1.C, respectively Prop. 4.2.A, the operators A_{x_n}, respectively $[A,L]$ are L^2 bounded.

For part (b) a computation similar to the one for Z shows that the principal symbol of the quadratic form given by the first two RHS terms in (2.17) and the subprincipal symbol w^1 of W have the same real part. This concludes the proof of the Lemma.

Lemma 2.21 *Let B be an interior bilinear form of type $(m-1, \Sigma, \mu)$ with purely imaginary principal symbol. Let $\alpha = \max\{0, \mu - 1/2\}$. Then for each large enough τ we have*

$$|\text{Re }(B(u,u) - Q_B(u,u))| \leq c|u|_{m,\Sigma+\mu-3/2,\tau}|u|_{m,\Sigma-\mu-3/2,\tau} \tag{2.18}$$

where $Q_{B,Q}$ is a boundary bilinear form of type $(m-1, \Sigma - 1/2, \alpha)$ with principal symbol q_B.

In particular we have

$$|\text{Re } B(u,u)| \leq c(|u|_{m,\Sigma+\mu-3/2,\tau}|u|_{m,\Sigma-\mu-3/2,\tau} + |u|_{m-1,\Sigma+\alpha-1/2,\partial}|u|_{(\Sigma-\alpha-1/2,\partial)}) \tag{2.19}$$

Remark 2.22 *If we represent the bilinear form B as in (2.8) then, by the closed graph theorem, the constant c in (2.21) is bounded by a sum of products of seminorms of the symbols A_k, B_k in the corresponding spaces of homogeneous symbols as in Definition 2.17.*

Proof : The problem reduces to the case $\Sigma = 1/2$ by the substitution $v = L^{\Sigma-1/2}u$. Next we reduce the problem to the case $\mu = 0$.

Assuming $\Sigma = 1/2$ consider a term in B of the form

$$< A(x, D', \tau)L^{m-i-1+\alpha}D_n^i u, C(x, D', \tau)L^{m-j-\alpha}D_n^j u >, \quad 1/2 \leq \alpha \leq \mu + 1$$

where $a(x, \xi', \tau), c(x, \xi', \tau) \in C^1 S_{cl}^{0,0}$, and $i, j \leq m, i + j \leq 2m - 1$.

If $\alpha \geq 1$ then we can substitute it by $\alpha - 1$ above without modifying the symbol of B and q_B, with an error of

$$< [A, L]L^{m-i-2+\alpha}D_n^i u, CL^{m-j-\alpha}D_n^j u > + < AL^{m-i-2+\alpha}D_n^i u, [C, L]L^{m-j-\alpha}D_n^j u >$$

According to the commutator estimates in Taylor [53] 4.3 the two commutators above are L^2 bounded therefore the error is bounded by $c|u|_{m,\alpha-2,\tau}|u|_{m,-\alpha,\tau}$ which, by interpolation, is further controlled by the RHS in (2.18).

Iterating this argument the problem reduces to the case $0 \leq \alpha \leq 1$. Hence, it suffices to prove the result in the case when $\mu = 0$. If $0 \leq \alpha \leq 1$ then a similar argument allows us to replace α with 1. Thus, we have reduced the proof of this Lemma to the previous Lemma.

Lemma 2.23 *Let B be an interior bilinear form of type $(m-1, 0, 1/2)$ such that its symbol satisfies*

$$\text{Re } b(x, \xi, \tau) \geq c|(\xi, \tau)|^{2(m-1)}$$

Then for large enough τ we have

$$\text{Re } B(u,u) \geq c|u|^2_{m-1,\tau} - d(|u|^2_{m-1,-1/2,\partial} + |u|^2_{m,-3/2,\tau}) \tag{2.20}$$

Proof : Due to the previous lemma we can assume w.a.r.g. that B has real symbol. Using Lemma **??** decompose the symbol $b(x,\xi',\xi_n)$ as

$$b(x,\xi',\xi_n) = c(\tau^2+\xi^2)^{2m-2} + \sum_{j=1,2} b_j^2(x,\xi',\xi_n)$$

where $b_j \in C^1 S^{m-1,0}$, $j = 1, 2$, are real symbols.

Then Lemma 2.21 gives

$$|\text{Re } B(u,u) - c|u|^2_{m-1,\tau} - \sum_j |B_j(x,D,\tau)u|^2_0| \leq d(|u|^2_{m,-3/2,\tau} + |u|^2_{m-1,-1/2,\partial}).$$

This implies (2.20), q.e.d.

Proposition 2.24 *Let B be an interior bilinear form of type $(m-1,0,1/2)$ such that*

$$\text{Re } b(x,\xi,\tau) > 0 \quad \textit{on } \text{char } P_\tau$$

Then for any large enough τ we have

$$\text{Re } B(u,u) \geq c|u|^2_{m-1,\tau} - d(|u|^2_{m-1,-1/2,\partial} + |P_\tau(x,D,\tau)u|^2_{0,-1,\tau}) \tag{2.21}$$

Proof : According to Lemma **??** there exists a symbol $q \in S^{m-1,-1}_{cl}$ such that

$$\text{Re } (b(x,\xi,\tau) + p_\tau(x,\xi,\tau)q(x,\xi,\tau)) > c|(\xi,\tau)|^{2(m-1)} \tag{2.22}$$

Then we apply Lemma 2.23 to the bilinear form

$$B(u,u) + < Q(x,D,\tau)u, P_\tau(x,D,\tau)u >$$

of type $(m-1,0,1/2)$. Since

$$|u|_{m,-3/2,\tau} \leq \tau^{-1/2}|u|_{m,-1,\tau}$$

this implies

$$\text{Re }(B(u,u)+ < Q(x,D,\tau)u, P_\tau(x,D,\tau)u >) \geq c|u|^2_{m-1,\tau} - d(|u|^2_{m-1,-1/2,\partial} + \tau^{-1}|u|^2_{m,-1,\tau}) \tag{2.23}$$

Since the boundary is noncharacteristic we can estimate the $m-th$ order derivative of u in the normal direction in terms of its first $m-1$ normal derivatives and $P_\tau(x,D,\tau)u$, i.e.

$$|u|^2_{m,-1,-\tau} \leq c(|u|^2_{m-1,\tau} + |P_\tau(x,D,\tau)u|^2_{0,-1,\tau})$$

Combining the last inequality with (2.23) gives

$$B(u,u) \geq (c - \frac{d}{\tau})|u|^2_{m-1,\tau} - d(|u|^2_{m-1,-1/2,\partial} + |P_\tau(x,D,\tau)u|^2_{0,-1,\tau})$$

This implies (2.21) for large enough τ, q.e.d.

The next Proposition is in some sense a refinement of Lemma 2.21.

Proposition 2.25 *Let B, W be interior bilinear forms of type $(m-1, 1/2, 0)$, respectively $(m-1, 0, 0)$, such that*

$$\text{Re } b(x,\xi,\tau) = 0 \tag{2.24}$$

and

$$\text{Re }(b^1(x,\xi,\tau) + w(x,\xi,\tau)) > c|(\xi,\tau)|^{2(m-1)} \textit{ on } \text{char } P_\tau \tag{2.25}$$

Then for any large enough τ and for each $u \in H^m(K)$ we have

$$\text{Re }(B(u,u) + W(u,u) - Q_B(u,u)) \geq c|u|^2_{m-1,\tau} - d(|P_\tau u|^2_{0,-1,\tau} + |u|^2_{m-1,-1/2,\partial}) \tag{2.26}$$

Proof : To simplify the exposition assume that B has the form

$$B(u,v) = \sum_k < A^k(x,D,\tau)u, C^k(x,D,\tau)v > \tag{2.27}$$

where $a_k(x,\xi,\tau) \in C^1S^{m,-j}_{cl}$ and $c_k \in C^1S^{m-1,j}_{cl}$, $j = 0,1$. This is the case for all the applications in the sequel.

First note that the problem reduces to the case when the symbols a_k, c_k are smooth. Indeed, substitute the symbols a_k, c_k in (2.27) with smooth ϵ-approximations with respect to some suitably chosen seminorms in the classes of homogeneous symbols where they belong, so that (2.24) remains satisfied. Then the estimate in Lemma 2.21 and the remark following it show that the term

$$\text{Re }(B(u,u) - Q_B(u,u))$$

changes at most by

$$O(\epsilon)(|u|^2_{m,-1,\tau} + |u|^2_{m-1,-1/2,\partial})$$

This is further bounded by

$$O(\epsilon)(|u|^2_{m-1,\tau} + |u|^2_{m-1,-1/2,\partial} + |P_\tau u|^2_{0,-1,\tau})$$

which can be easily incorporated in the RHS in (2.26) if ϵ is small enough. Note also that (2.25) is not altered after a small enough perturbation of B as above.

Hence, assume that the symbols a_k, b_k in the definition of B are smooth. Then the subprincipal symbol $b_1(x,\xi,\tau)$ of B is also smooth, i.e. in $S^{2(m-1),0}_{cl}$. Consider an interior bilinear form B^1 with symbol $b^1(x,\xi,\tau)$.

Then the assumption (2.25) combined with Proposition 2.24 gives

$$B^1(u,u) + W(u,u) \geq c|u|^2_{m-1,\tau} - d(|u|^2_{m-1,-1/2,\partial} + |P_\tau(x,D,\tau)u|^2_{0,-1,\tau})$$

Hence in order to conclude the proof it suffices to prove that

$$|\text{Re }(B(u,u) - B^1(u,u) - Q_B(u,u))| \leq d(\tau^{-1}|u|^2_{m-1,\tau} + (|P_\tau u|^2_{0,-1,\tau} + |u|^2_{m-1,-1/2,\partial}) \quad (2.28)$$

But if we take into account the inequalities

$$|u|_{m,-3/2,\tau} \leq \tau^{-1/2}|u|_{m,-1,\tau} \leq c\tau^{-1/2}(|u|_{m-1,\tau} + |P_\tau u|_{0,-1,\tau})$$

then (2.28) is a consequence of Lemma 2.20.

The next Lemma gives a Garding's type inequality for boundary bilinear forms with positive definite symbol.

Lemma 2.26 *Assume that the boundary bilinear form B of type $(m-1,0,0)$ is positive. Then the following inequality holds*

$$\text{Re } B(u,u) \geq c|u|^2_{m-1,0,\partial} \quad (2.29)$$

for each $u \in H^0_\partial$ and each large enough τ.

Proof : Define the m-tuple of functions $v = (v_0, ...v_{m-1})$ by

$$v_i = L^{m-i-2} u_i$$

Then

$$|u|^2_{m-1,0,\partial} = \sum_{i=0}^{m-1} |v_i|^2_{1,\tau} = |v|^2_{1,\tau} \tag{2.30}$$

The quadratic form B can be written as a finite sum

$$B(u, v) = \sum_{i,j=0}^{m-1} \sum_k < A^k_{ij}(x, D', \tau) L^{n-i} D^i_n u, C^k_{ij}(x, D', \tau) L^{n-j} D^j_n u >$$

where $a^k_{ij} \in C^1 S^{0,\alpha}$, $c^k_{ij} \in C^1 S^{0,-\alpha}$ for each i, j, k, for some $|\alpha| \leq 1/2$.

Then the ellipticity condition on the symbol of B gives

$$\text{Re} \; (\sum_{i,j=0}^{m-1} a^k_{ij}(x, \xi', \tau) \overline{c^k_{ij}(x, \xi', \tau)} z_i \overline{z_j}) \geq c|z|^2 \quad z \in C^n \tag{2.31}$$

Substituting u in terms of v we get

$$B(u, u) = \sum_{i,j} < \sum_k C^k_{ij}(x, D', \tau)^* A^k_{ij}(x, D', \tau) L v_i, L v_j >$$

Define the symbols $d_{ij}(x, \xi', \tau) \in C^1 S^0$ by

$$d_{ij}(x, \xi', \tau) = \sum_k \overline{c^k_{ij}}(x, \xi', \tau) a^k_{ij}(x, \xi', \tau)$$

Then, according to the calculus for $OPC^1 S^0_{cl}$ operators (see Taylor [53], 4.2.A-B) it follows that for the operators we have

$$D_{ij}(x, D', \tau) = \sum_k C^k_{ij}(x, D', \tau)^* A^k_i(x, D', \tau) + R_{ij}$$

where the remainders R_{ij} are bounded from $H^{-1/2}_\tau$ into $H^{1/2}$. Hence we have

$$|B(u, u) - < D(x, D', \tau) L v, L v > | \leq c|v|^2_{1/2,\tau} \tag{2.32}$$

The ellipticity condition (2.31) on the symbol of B gives

$$d(x, \xi', \tau) + d(x, \xi', \tau)^* \geq c I_m > 0$$

Then Garding's inequality for symbols with limited smoothness gives (see Taylor [53], 4.3.C.)

$$\text{Re } < D(x, D', \tau)Lv, Lv > \geq c|v|^2_{1,\tau} - d|v|^2_{1/2,\tau}$$

Combining this with (2.32) we obtain (with different constants c, d)

$$\text{Re } B(u, u) \geq c|v|^2_{1,\tau} - d|v|^2_{1/2,\tau} \geq (c - d\tau^{-1})c|v|^2_{1,\tau}$$

According to (2.30) the conclusion of the Lemma follows if τ is large enough.

The next result shows how we can strengthen the previous Lemma if we have some additional information on $P_\tau(x, D, \tau)u$.

Denote by $e_j(x, \xi, \tau)$ the symbols

$$e_j(x, \xi, \tau) = \frac{p_\tau}{p_\tau^-}(x, \xi, \tau)|(\xi', \tau)|^{k-j-1}\xi_n^{j+1-k} \ j = 0, k-1.$$

where $k = ord\ p^-$. Note that these symbols might not depend smoothly on (x', ξ', τ).

Lemma 2.27 *Let $B(u, v)$ be a boundary bilinear form of type $(m-1, 0, 0)$ such that*

$$b(x, \xi, \tau)(z, z) > 0 \quad on \quad e(x, \xi, \tau)(z) = 0$$

Then the following inequality holds

$$B(u, u) \geq c|u|^2_{m-1,0,\partial} - d(\tau^{-1}|u|^2_{m-1,\tau} + |P_\tau u|^2_{-1/2,\tau}) \tag{2.33}$$

for each large enough τ and each $u \in H^m$.

Proof : We claim that there exists a boundary bilinear form $C(u, u)$ of type $(m-1, 0, 0)$ such that

(i) $c(x, \xi, \tau)(z, z) \geq 0$

(ii) $b(x, \xi, \tau)(z, z) + c(x, \xi, \tau)(z, z) > 0$

(iii) Re $C(u, u) \leq d(\tau^{-1}|u|^2_{m-1,\tau} + |P_\tau u|^2_{-1/2,\tau})$

According to Lemma 2.26 this implies (2.33).

To prove our claim it suffices to show that for each (x_0, ξ_0', τ_0) we can find a boundary bilinear form $C(u,v)$ satisfying (i),(iii) globally, and

$$c(x_0, \xi_0, \tau_0, \xi_n) \geq c \sum |e_j(x_0, \xi_0, \tau_0, \xi_n)|^2.$$

Then a global choice for C can be found by summing up (multiples of) finitely many of these local choices.

Extend the decomposition $p_\tau(x_0, \xi_0', \tau_0, \xi_n) = p_\tau^-(x_0, \xi_0', \tau_0, \xi_n) p_\tau^{0+}(x_0, \xi_0', \tau_0, \xi_n)$ smoothly to a conic neighbourhood V of (x_0, ξ_0', τ_0). This yields a local smooth extension of the symbols e_j in V. Let $\chi, \eta \in S^{0,0}$ be homogeneous cutoff symbols supported in a smaller conic neighbourhood of (x_0, ξ_0', τ_0) so that $\eta = 1$ in supp χ.

Define the symbols $f_j(x, \xi, \tau) = \chi(x, \xi', \tau) e_j(x, \xi, \tau)$, respectively

$$q(x, \xi, \tau) = \eta(x, \xi', \tau) p^-(x, \xi, \tau) + (1 - \eta)(x, \xi', \tau) p^-(x_0, \frac{|(\xi', \tau)|}{|(\xi_0', \tau_0)|}(\xi_0, \tau), \xi_n)$$

The last definition insures that $q = p_\tau^-$ in supp χ, while it doesn't go too far from its behavior at (x_0, ξ_0', τ_0). Write q in the form

$$q(x, \xi, \tau) = \sum_{j=0}^{k} a_j |(\xi', \tau)|^{k-j} \xi_n^j$$

with $a_j \in C^1 S_{cl}^{0,0}$, $a_k = 1$. A simple computation shows that

$$D_n F(x, D, \tau) u = T(x, D, \tau) F(x, D, \tau) u + G(x, D, \tau) P_\tau(x, D, \tau) u + Ru \tag{2.34}$$

where

$$t(x, \xi, \tau) = |(\xi', \tau)| \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & & \vdots \\ -a_0 & -a_1 & ... & -a_{k-1} \end{pmatrix} \quad g(x, \xi', \tau) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \chi \end{pmatrix}$$

and the remainder R is bounded from H^{m-1} into L^2.

Note that the eigenvalues of $t(x, \xi, \tau)$ (which are the roots of p^-) have negative imaginary part, therefore the system (2.34) is parabolic from the boundary into the interior. Hence, choosing a suitable scalar product in R^k, $-iT$ has a positive symbol,

$$\text{Re } < it(x, \tau, \xi') z, z > \geq c |(\xi', \tau)| |z|^2$$

Then the following version of Garding's inequality holds

$$< iT(x, D', \tau)u, u > \geq c|u|^2_{1/2,\tau} - d|u|^2_{-1/2,\tau} \tag{2.35}$$

uniformly in x_n. This can be easily proved as in [54],4.3. (even simpler, since the order of A is 1 instead of 2 there). Thus,

$$\frac{d}{dx_n}|Fu(x_n)|_{L^2} \geq c|Fu|^2_{1/2,\tau} - d|Fu|_{1/2,\tau}|GP_\tau u + Ru|_{-1/2,\tau} \geq -c(|P_\tau u|^2_{-1/2,\tau} + \tau^{-1}|Ru|^2_0)$$

which after integration in the normal direction gives

$$|Fu|^2_{L^2(\partial K)} \leq c(|P_\tau u|^2_{-1/2,\tau} + |u|^2_{m-1,-1/2,\tau})$$

But the symbol of the boundary quadratic form $|Fu|^2_{L^2(\partial K)}$ is nonnegative and equals

$$\sum_{j=0}^{k-1} |e_j(z)|^2$$

at (x_0, ξ_0', τ_0), q.e.d.

2.4.3 The strong Lopatinskii boundary condition

Let $\gamma_0 = (x_0, \xi_0') \in T^*\partial K$. Corresponding to the factorization (1.26) of p as a polynomial in ξ_n at γ_0 denote

$$p_j(\xi_n) = \frac{p(\xi_n)}{(\xi_n - \xi_n^j)^{m_j}}, \; p^{\pm} = p^+ p^-$$

Define the polynomials

$$h_{j,k}(\xi_n) = p_j(\xi_n)(\xi_n - \xi_n^j)^k \quad k = \overline{m_j^+, m_j - 1}$$

Then the relaxed strong Lopatinskii condition at γ_0 implies that the set of polynomials $H_{j,k}(\xi_n),\; E_j(\gamma_0, \xi_n),\; B^j(\gamma_0, \xi_n)$ is complete in the class of $m-1$-degree polynomials in ξ_n. Hence,

$$\sum_{j,k} |h_{j,k}(z)|^2 + |\sum b^j(z)|^2 \geq c|z|^2 \text{ on } \; Ez = 0 \tag{2.36}$$

The following Proposition is the key to using the r-strong Lopatinskii condition:

Proposition 2.28 *Let $\gamma_0 = (x_0, \xi_0') \in T^*\partial K$, and assume that the set of boundary operators B^j satisfy the r-strong Lopatinskii condition at γ_0. Then there exists real polynomials $r(\xi_n), q(\xi_n)$ of degree $m-1$ such that* *(i)* $q(\xi_n) = r(\xi_n)\{p,\phi\}(\gamma_0,\xi_n)$ *(mod* $p(\gamma_0,\xi_n)$*) and* $r(\xi_n) > 0$ *whenever* $p(\gamma_0,\xi_n) = 0$, $\xi_n \in R$.

(ii) $\mathrm{Re}\ q_{p(\gamma_0,\cdot),iq}(z,z) + \sum |b^j(z)|^2 > 0$ *on* $\{e(\gamma_0)z = 0\}$.

(iii) **Proof :** Start with the following simple construction, inspired from Sakamoto [40].

Lemma 2.29 *Let* $m \in N$. *Then for each* $\epsilon > 0$ *there exists a polynomial* w *of degree* $m-1$ *so that*

i) $w(0) > 0$ *and*

$$Q_{\xi^m,iw}(z,z) \geq \sum_{j=[m/2]}^{m-1} |z_j|^2 - \epsilon \sum_{j=0}^{[(m-1)/2]-1} |z_j|^2$$

where $z = (z_0,\cdots,z_{m-1}) \in C^m$.

ii) $w(0) < 0$ *and*

$$Q_{\xi^m,iw}(z,z) \geq \sum_{j=[(m+1)/2]}^{m-1} |z_j|^2 - \epsilon \sum_{j=0}^{[m/2]-1} |z_j|^2$$

Proof : For (i) let

$$w(\xi_n) = \sum_{j=0}^{m-1} \lambda^{a(j)} \xi_n^j$$

where $a(j) = (m+j)^2 - m^2$. Then

$$Q_{\xi^m,iw}(\xi_n,\tilde{\xi}_n) = \sum_{\substack{0\leq h,k\leq m-1 \\ m-1\leq h+k\leq 2m-2}} \lambda^{a(h+k-m+1)} \xi_n^h \tilde{\xi}_n^k$$

Hence

$$Q_{\xi^k,iw}(z,z) \geq \sum_{k=[m/2]}^{m-1} \lambda^{a(2k-m+1)}|z_k|^2 - 2 \sum_{\substack{0\leq h<k\leq m-1 \\ m-1\leq h+k\leq 2m-2}} \lambda^{a(h+k-m+1)}|z_h||z_k|$$

If $h < k$ then

$$a(h+k-m+1) \leq a(2h-m+1) + a(2k-m+1) - 2$$

therefore, if $\lambda \geq 1$ we obtain

$$Q_{\xi^k,iw}(z,z) \geq \sum_{k=[m/2]}^{m-1} \lambda^{a(2k-m+1)}|z_k|^2 - c\lambda^{-1} \sum_{j=0}^{m-1} a(2j-m+1)|z_j|^2$$

where the constant c does not depend on $\lambda \geq 1$. If we choose λ large enough this gives (i).

For (ii) replace the free term in w above (which is 1) by $-\lambda^{-1}$. Then a similar argument applies.

Proof of Proposition 2.28, continued :

Decompose $\{p, \phi\}$ at γ_0

$$\{p, \phi\} = \sum s_j(\xi_n)(\xi_n - \xi_n^{(j)})^{n_j} p_j(\xi_n) + s_{\pm} p^{\pm}$$

where $s_j \in P(m_j - n_j - 1), s_j(\xi_n^j) \neq 0$ and $s_{\pm} \in P(m - n_0 - 1, 0)$, $m_0 = \sum deg\ p_j = \sum m_j$.

Look for q of the form

$$q = \sum q_j(\xi_n)(\xi_n - \xi_n^{(j)})^{n_j} p_j(\xi_n)$$

Then

$$Q_{P,iQ}(\xi_n, \tilde{\xi}_n) = \sum_j (\xi_n - \xi_n^{(j)})^{n_j} p_j(\xi_n)(\tilde{\xi}_n - \xi_n^{(j)})^{n_j} p_j(\tilde{\xi}_n) Q_{(\xi_n - \xi_n^j)^{m_j - n_j}, iq_j}(\xi_n, \tilde{\xi}_n)$$

Recall that $m_j^+ = [(m_j + n_j - l_j)/2]$ where $l_j = 1$ if $(s_j p_j)(\xi_n^j) > 0$ and $l_j = 0$ otherwise. Then, after a change of variable $\xi_n \to \xi_n - \xi_n^j$ Lemma 2.29 implies that for each $\epsilon > 0$ we can find polynomials q_j such that

$$\text{sgn } q_j(\xi_n^{(j)}) = \text{sgn } s_j(\xi_n^{(j)}) \tag{2.37}$$

and

$$Q_{P,iQ}(z, z) \geq \sum |H_{j,k}(\gamma_0) z|^2 - \epsilon |z|^2$$

Combined with (2.36) this gives

$$q_{p(\gamma_0, \cdot), iq}(z, z) + \sum |b^j(z)|^2 > 0 \text{ on} \{Ez = 0\}$$

This concludes the proof of part (ii) of the Proposition.

For part (i) we still have to construct r. Look for r of the form

$$r(\xi_n) = \sum r_j(\xi_n) p_j(\xi_n)$$

where r_j has degree $m_j - n_j - 1$. Then it has to satisfy the condition

$$q_j = r_j s_j p_j \ (mod(\xi_n - \xi_n^j)^{m_j - n_j}) \tag{2.38}$$

Denote by M the abelian ring of polynomials in ξ_n modulo $(\xi_n - \xi_n^j)^{m_j - n_j}$. The polynomial $s_j p_j$ is invertible in M since $(s_j p_j)(\xi_n^j) \neq 0$. Then simply choose

$$r_j = q_j (s_j p_j)^{-1}$$

where all the operations are in M. Then (2.38) is fulfilled. In particular (2.37), (2.38) imply that $(p_j r_j)(\xi_n^j) > 0$ which shows that $r > 0$ at the zeros of p, q.e.d.

2.4.4 An "interpolation" inequality

Lemma 2.30 *Let η be a compactly supported smooth function in R^n. Then there exists λ_0 such that for each $\lambda \geq \lambda_0$ the following inequality holds*

$$\sum_{|\alpha|=1}^{m-1} \lambda^{\alpha} |\eta^{m-\alpha} u|_{m-|\alpha|,\tau} \leq c(|\eta^{(m)} u|_{m,\tau} + \lambda^m |u|_{0,\tau}) \tag{2.39}$$

whenever the RHS is finite.

Proof : First we prove (2.39) for $m = 2$. Using a partition of unit and a change of coordinates which flattens the boundary the problem reduces to proving the same inequality in a cube M of side 1 in R^n. Without any restriction in generality assume that

$$|\nabla \eta| \leq (4n)^{-1/2} \tag{2.40}$$

.

Decompose $M \cap \operatorname{supp} \phi$ into a countable disjoint union of dyadic cubes

$$M = \cup M_j^{\mu}$$

where each cube M_j^{μ} has sides 2^{-j} and $|\eta(x)| \in [2^{-j}, 2^{-j+2}]$ for some $x \in M_j$. Then the bound (2.40) on $\nabla \eta$ implies that

$$\eta(x) \in [2^{-j-1}, 2^{-j+2}] \quad for \quad x \in M_j$$

Proving (2.39) in M reduces to proving that in each cube M_j we have

$$\lambda |\eta D u|_{L^2}^2 \leq c(\lambda^2 |u|_{L^2}^2 + |\eta^2 D^2 u|_{L^2}^2) \tag{2.41}$$

for $\lambda \geq \lambda_0$, with λ_0, c independent of j. Rescale M_j into M. Then (2.41) reduces to

$$2^j \lambda |\eta_j D u|_{L^2}^2 \leq c(\lambda^2 |u|_{L^2}^2 + 2^{2j} |\eta_j^2 D^2 u|_{L^2}^2) \tag{2.42}$$

where $2^{-j-1} \le \eta_j \le 2^{j+2}$ in M. Hence (2.42) reduces to

$$\lambda |Du|^2_{L^2(M)} \le c(\lambda^2 |u|^2_{L^2(M)} + |D^2u|^2_{L^2(M)})$$

for large enough λ, which follows immediately from the interpolation inequality

$$|u|^2_{H^1(M)} \le c|u|_{L^2(M)}|u|_{H^2(M)}$$

To prove (2.39) for $m > 2$ use induction. Apply (2.39) for $m = 2$ to $D^{m-2}\eta^{m-2}u$. After some standard commutations this gives

$$\lambda |D^{m-1}\eta^{m-1}u|^2_{0,\tau} \le c(|D^m\eta^m u|^2_{0,\tau} + \lambda^2 |D^{m-2}\eta^{m-2}u|^2_{0,\tau})$$

Combining this with (2.39) for $m-1$ gives (2.39) for m.

Consider the tangential pdo. L_n, $n \in N$, with the symbol

$$l_n(x, \xi') = (1 + n^{-1}|(\xi', \tau)|)^{-s}$$

Lemma 2.31 *Let $0 < s \le m-1$. Then the following inequality holds*

$$\sum_{1 \le j \le s} |z^{(j)} L_n^{(s-j)/s} u|_{m-1,-j} \le c(|zL_n u|_{m-1} + |u|_{m-1,-s}) \qquad (2.43)$$

uniformly in $n \in N$, whenever $u \in H^{m-1,-s}$ and the RHS is finite.

Proof : To simplify the argument assume that s is integer. Then the above inequality reduces to

$$\sum_{1 \le j \le s} |z^{(j)} L_n^{(s-j)/s} u|_{0,-j} \le c(|zL_n u|_0 + |u|_{0,-s}) \qquad (2.44)$$

Furthermore, if we commute $z^{(j)}$ with $L_n^{-j/s}$ the inequality reduces to

$$\sum_{1 \le j \le s} |L_n^{-j/s} z^{(j)} L_n u|_{0,-j} \le c(|zL_n u|_0 + |u|_{0,-s})$$

Redenote $u := L_n^{-1}u$. Then the last inequality becomes

$$\sum_{1 \le j \le s} |L_n^{-j/s} z^{(j)} u|_{0,-j} \le c(|zu|_0 + |L_n^{-1}u|_{0,-s})$$

and further

$$\sum_{1 \le j \le s} |z^{(j)}u|^2_{0,-j} + n^{-j/s}|z^{(j)}u|^2_0 \le c(|zu|^2_0 + |u|^2_{0,-s} + n^{-1}|u|^2_0)$$

But this is a consequence of the following two inequalities:

$$\sum_{1\le j\le s} n^{-j/s}|z^{(j)}u|_0^2 \le c(|zu|_0^2 + n^{-1}|u|_0^2) \tag{2.45}$$

$$\sum_{1\le j\le s} |z^{(j)}u|_{0,-j}^2 \le c(|zu|_0^2 + |u|_{0,-s}^2) \tag{2.46}$$

The first is straightforward since our assumptions on z imply that

$$|z^{(j)}| \le c|z|^{j/s}, \quad 1 \le j < s \le m-1$$

For the second redenote $u := L^s u$ where $l(\xi,\tau) = |(\xi',\tau)|$. Then after some commutations (2.46) reduces to

$$\sum_{1\le j\le s} |z^{(j)}u|_{0,s-j}^2 \le c(|zL^s u|_0^2 + |u|_0^2)$$

which is a consequence of Lemma 2.30; indeed, due to our assumptions on z we have $z^{(j)} = z^{(s)} z_1^{s-j} q_j$ where z_1, q_j are smooth functions.

Chapter 3

Carleman estimates and unique continuation

3.1 The "classical" theory

3.1.1 The estimates

Let P be an operator of order m with C^1 coefficients. There are two cases we want to consider:

Theorem 3.1 *Assume that P is of real principal type. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Then*

$$\tau |e^{\tau\phi} u|^2_{m-1,\tau} \leq c |e^{\tau\phi} P(x,D) u|^2 \quad \tau > \tau_0 \tag{3.1}$$

whenever $u \in H^{m-1}$ is supported in K.

A stronger version of this result holds in the elliptic case:

Theorem 3.2 *Assume that P is elliptic. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Then*

$$\tau^{-1} |e^{\tau\phi} u|^2_{m,\tau} \leq c |e^{\tau\phi} P(x,D) u|^2 \quad \tau > \tau_0 \tag{3.2}$$

whenever $u \in H^{m-1}$ is supported in K.

Proof of Theorem 3.1 : By Theorem 2.8 with it suffices to show that (3.1) holds for $u \in H^m$.

With the substitution $v = e^{\tau\phi}u$ the first inequality reduces to

$$\tau |v|_{m-1,\tau}^2 \leq c|P_\tau(x,D)v|^2 \quad \tau > \tau_0 \tag{3.3}$$

Split P_τ into

$$P_\tau = P_\tau^r + i\tau P_\tau^i$$

where both P_τ^r and P_τ^i have real symbols. Then the pseudoconvexity condition (1.8) can be rewritten as

$$\{p_\tau^r, p_\tau^i\} > 0 \quad \text{on char } P_\tau$$

Hence, by Theorem 2.7 (a),

$$|v|_{m-1,\tau}^2 \leq c\text{Im } < P_\tau^r v, P_\tau^i v > +d|P_\tau v|_{-1,\tau}^2 \quad \tau > \tau_0 \tag{3.4}$$

which implies the desired conclusion.

Proof of Theorem 3.2 : The proof follows the same ideas as the proof of Theorem 3.1. Since Im p_τ no longer vanishes when $\tau = 0$ we set

$$P_\tau = P_\tau^r + iP_\tau^i$$

The pseudoconvexity condition gives again

$$\{p_\tau^r, p_\tau^i\} \geq 0 \quad \text{on char } P_\tau$$

Then Theorem 2.7(b) gives

$$|v|_{m,\tau}^2 \leq c\tau\text{Im } < P_\tau^r v, P_\tau^i v > +d|P_\tau v|^2 \quad \tau > \tau_0 \tag{3.5}$$

which leads again to the desired conclusion.

3.1.2 Unique continuation

The Carleman estimates in the previous section lead to the following unique continuation result:

Theorem 3.3 *(Hörmander) Assume that P is either elliptic or has real principal symbol. Let Σ be an oriented surface which is strongly pseudoconvex with respect to P. Then unique continuation across Σ holds for H^{m-1} solutions u to $P(x,D)u = 0$.*

Proof : By Theorem 1.5 we can represent the pseudoconvex surface Σ as a level set of a function ϕ which is strongly pseudoconvex with respect to P.

Given $x_0 \in \Sigma$ and a neighbourhood W of x_0, we want to prove that there exists a neighbourhood V of x_0 so that for any function $u \in H^{m-1}_{loc}$ supported in $\{\phi < \phi(x_0)\}$ such that $P(x,D)u = 0$ in W we have $u = 0$ in W.

Consider the function

$$\psi(x) = \phi(x) + \delta|x - x_0|^2 - \phi(x_0) - \epsilon, \quad \epsilon, \delta > 0$$

If δ is sufficiently small then, by Theorem 1.5, ψ is also strongly pseudoconvex with respect to P at x_0. Without any restriction in generality, assume that ψ is also strongly pseudoconvex with respect to P in W. Choose $\epsilon > 0$ small enough so that

$$\{\phi < \phi(x_0)\} \cap \{\psi > -\epsilon\} \subset W \tag{3.6}$$

Then let

$$V = \{psi > 0\}$$

Let z be a smooth cutoff function which equals 1 in $[0, \infty)$ and 0 in $(-\infty, -\epsilon]$. Then look at the function $w = z(\psi)u$. By (3.6), w is supported in W. On the other hand,

$$P(x,D)w = z(\psi)P(x,D)u + [P(x,D), z(\psi)]u = [P(x,D), z(\psi)]u \in L^2$$

is supported in $\{\psi > 0\}$. Since $w = u$ in V, we have reduced our problem to the following:

Suppose ψ is a smooth function which is strongly pseudoconvex with respect to P in a compact set W in R^n. Let $w \in H^{m-1}$, supported in W so that $P(x,D)w \in L^2$. If $P(x,D)w = 0$ in $\{\psi > 0\}$ then $w = 0$ in $\{\psi > 0\}$.

Now we are in a position to use the Caleman estimates. By (3.1) we have

$$\tau|e^{\tau\psi}w|^2_{m-1,\tau} \leq c|e^{\tau\psi}P(x,D)w|^2 \quad \tau > \tau_0 \tag{3.7}$$

Let $\tau \to \infty$. Since $P(x,D)w$ is supported in $\{\psi \leq 0\}$ it follows that the RHS in (3.7) converges to 0. Hence the LHS converges to 0 as well. This implies that $w = 0$ in $\{\psi > 0\}$, q.e.d.

3.1.3 Stability estimates

We start with a global result:

Theorem 3.4 *Suppose ϕ is a smooth function which is strongly pseudoconvex with respect to P in a compact set K in R^n. Then there exists $c > 0$ so that for any $\alpha < \beta < \gamma$ we have*

$$|u|_{m-1,\{\phi>\beta\}} \leq |P(x.D)u|_{\{\phi>\alpha\}}^{1-\mu} |u|_{m-1,\{\phi>\alpha\}}^{\mu}, \quad \mu > \frac{\gamma - \beta}{\gamma - \alpha} \tag{3.8}$$

whenever $u \in H^{m-1}$ is supported in $K \cap \{\phi < \gamma\}$.

Using the perturbation argument in the proof of Theorem 3.3 this implies the following local result:

Theorem 3.5 *Assume that P is either elliptic or has real principal symbol. Let Σ be an oriented surface which is strongly pseudoconvex with respect to P. Given $0 < \mu < 1$ $x_0 \in \Sigma$ and a neighbourhood W of x_0, there exists a neighbourhood V of x_0 so that for any function $u \in H_{loc}^{m-1}$ supported in $\{\phi < \phi(x_0)\}$ we have*

$$|u|_{m-1,V} \leq |P(x.D)u|_W^{1-\mu} |u|_{m-1,W}^{\mu}, \tag{3.9}$$

Proof of Theorem 3.4 : Let z be a smooth cutoff function which equals 1 in $[\alpha+\epsilon, \infty)$ and 0 in $(-\infty, \alpha]$. Then

$$P(x, D)z(\phi)u = z(\phi)P(x, D)u + [P(x, D), z(\phi)]u = [P(x, D), z(\phi)]u$$

Apply the Carleman estimates (3.1) to $z(\phi)u$. We obtain

$$\tau |e^{\tau\phi} z(\phi)u|_{m-1,\tau}^2 \leq c(|e^{\tau\phi} z(\phi)P(x, D)u|^2 + |e^{\tau\phi}[P(x, D), z(\phi)]u|^2) \quad \tau > \tau_0 \tag{3.10}$$

therefore

$$e^{\beta\tau} |u|_{m-1,\{\phi>\beta\}} \leq c(e^{\gamma\tau} |Pu|_{\{\phi>\alpha\}} + e^{(\alpha+\epsilon)\tau} |u|_{m-1,\{\phi>\alpha\}}^{\mu}$$

and further

$$|u|_{m-1,\{\phi>\beta\}} \leq c(e^{(\gamma-\beta)\tau} |Pu|_{\{\phi>\alpha\}} + e^{(\alpha+\epsilon-\beta)\tau} |u|_{m-1,\{\phi>\alpha\}}^{\mu}, \quad \tau > \tau_0$$

Now minimize the RHS with respect to τ to get (3.8) with $\mu = \frac{\gamma-\beta}{\gamma-\alpha-\epsilon}$. The restriction $\tau > \tau_0$ causes problems for the minimization only if

$$|Pu|_{\{\phi>\alpha\}} > c|u|_{m-1,\{\phi>\alpha\}}^{\mu}$$

in which case (3.8) is trivial.

3.2 Carleman estimates at other energy levels

The Carleman estimates in Theorems 3.1,3.2 and the unique continuation result in Theorem 3.3 apply to functions $u \in H^{m-1}$. What can we do about functions which have less regularity ? The idea is that similar estimates should hold at all energy levels, with one constraint: the higher (lower) the energy level, the more regularity one needs to impose on the coefficients.

To lift Theorem 3.1 at different energy levels we assume that the coefficients are in the Ξ_s spaces defined in Section 2.3.

Theorem 3.6 *Assume that $P(x, D)$ is of real principal type, with coefficients of class Ξ_s. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Then*

$$\tau |e^{\tau\phi} u|^2_{m-1+r,\tau} \leq c |e^{\tau\phi} P(x, D) u|^2_{r,\tau} \tag{3.11}$$

for $\tau > \tau_0$, $1 - s \leq r \leq s$, whenever $u \in H^{m-1+r}$ is supported in K.

Consequently, this gives the following unique continuation result:

Theorem 3.7 *Assume that $P(x, D)$ is of real principal type, with coefficients of class Ξ_s, $s \geq 1$. Let Σ be an oriented surface which is strongly pseudoconvex with respect to P. Then unique continuation across Σ holds for H^{m-s} solutions u to $P(x, D)u = 0$.*

Proof of Theorem 3.6 : Suppose that $u \in H^{r+m-1}$. With $v = e^{\tau\phi} u$ rewrite (3.11) as

$$\tau |v|^2_{m-1+r,\tau} \leq c |P_\tau(x, D, \tau) v|^2_{r,\tau} \tag{3.12}$$

Denote $w = \ll^r v$. Then $w \in H^{m-1}$ and

$$P_\tau w = \ll^r P_\tau v + + \ll^r [P_\tau, \ll^{-r}] \ll^{-r} P_\tau w$$

Due to the commutator estimates in Theorem 2.10,

$$[P_\tau, \ll^{-r}] : H^{m-1}_\tau \to H^r_\tau$$

Consequently,

$$|P_\tau w| \leq c |P_\tau v|_{r,\tau} + |w|_{m-1,\tau} \tag{3.13}$$

Now apply (3.3) to w and use (3.13) to get

$$\tau|w|^2_{m-1,\tau} \leq c|P_\tau v|^2_{r,\tau}$$

This implies (3.12) q.e.d.

One can see that even if the coefficients are merely C^1, the estimates are still valid in the range $0 \leq s \leq 1$. It is possible to obtain other estimates for operators with C^1 coefficients if the coefficients are placed in a different position within the operator. Namely, suppose P is an operator of the form

$$P = \sum_{|\alpha|=j}^{|\beta|=m-j} D^\alpha c_{\alpha,\beta} D^\beta$$

Theorem 3.8 *Assume that $P(x, D)$ is a partial differential operator of real principal type, as above, with coefficients of class C^1. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Then*

$$\tau|e^{\tau\phi}u|^2_{m-1+r,\tau} \leq c|e^{\tau\phi}P(x, D)u|^2_{r,\tau} \tag{3.14}$$

for $\tau > \tau_0$, $-j \leq r \leq 1-j$, whenever $u \in H^{m-1+r}$ is supported in K.

Proof : For $j = 1$ we can write

$$P = \sum_{|\alpha|=1}^{|\beta|=m-1} c_{\alpha,\beta} D^{\alpha+\beta} + R$$

where R contains $m-1$ order terms which are bounded from H^{m-1}_τ into L^2. Then the L^2 estimate holds for P.

Next set $w = \ll^r v$, and apply the L^2 estimate to w. We have

$$P_\tau w = \ll^r P_\tau v + D[\ll^r, C^1]D^{m-1}$$

The commutator is bounded from H^{1+r} into H^1, therefore the L^2 estimate for w gives

$$\tau|w|^2_{m-1,\tau} \leq c(|P_\tau v|^2_r + |v|^2_{m+r-1})$$

and further

$$\tau|v|^2_{m+r-1,\tau} \leq c(|P_\tau v|^2_r + |v|^2_{m+r-1})$$

q.e.d. The estimate for higher values of j follows iteratively by the same method.

An interesting example is contained in Section 4.1 where we consider the wave equation written in divergence form.

Note that if

$$P(x,D) = \sum_{|\alpha|\leq m} c_\alpha D^\alpha$$

then

$$P(x,D)^* = \sum_{|\alpha|\leq m} D^\alpha \overline{c_\alpha}$$

therefore

Corollary 3.2.1 *Assume that $P(x,D)$ is of real principal type, with C^1 coefficients. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Then*

$$\tau|e^{\tau\phi}u|^2 \leq c|e^{\tau\phi}P(x,D)^*u|^2_{1-m,\tau}, \qquad \tau > \tau_0 \tag{3.15}$$

whenever $u \in L^2$ is supported in K.

3.3 Carleman estimates with cutoff and continuation of regularity

One less common way of looking at the Carleman estimate (3.1) is as a regularity result:

Assume that P has Ξ_s coefficients. If a compactly supported function $u \in H^{m-s}$ satisfies $Pu \in L^2$ then $u \in H^{m-1}$.

The usefulness of this is limited by the assumption that u has compact support. This makes the above result a global one, while the interesting issue would be to have a local version of this.

Definition 3.9 *Let Σ be an oriented surface. We say that we have continuation of H^{r+m-1} regularity across Σ for functions $u \in H^s_{loc}$ with $P(x,D)u \in H^r$ if, given $x_0 \in \Sigma$, there exists a neighbourhood V of x_0 so that $u \in H^s_{loc}(V)$, $P(x,D)u \in H^r_{loc}(V)$ and $u \in H^{m+r-1}_{loc}(V \cap \Sigma^+)$ implies $u \in H^{m+r-1}_{loc}(V)$.*

Can we obtain results on continuation of regularity from (3.1) ? Not directly, since we need to cutoff u, and the commutator of P with the cutoff function is an operator of order $m-1$. The solution is to tamper a bit with the proof of the Carleman estimates in order to obtain the following version which includes the cutoff:

Theorem 3.10 *Assume that P is of real principal type, with C^1 coefficients. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Then*

$$\tau|\phi^2 e^{\tau\phi} u|^2_{m-1,\tau,\Sigma^+} \leq c(|\phi^2 e^{\tau\phi} P(x,D)u|^2_{\Sigma^+} + \tau^2 |e^{\tau\phi} u|^2_{m-2,\tau,\Sigma^+} \quad \tau > \tau_0 \tag{3.16}$$

whenever $u \in H^{m-1}$ is supported in K.

Note that this estimate is already localized in the region $\Sigma^+ = \{\phi > 0\}$. This, however, is a disadvantage when it comes to proving it. The assumption $u \in H^{m-1}$ is optimal; if u were less regular then $P(x,D)u$ would not be defined.

Proof : (i) We show first that the conclusion of the Theorem holds if we assume in addition that u is supported in Σ^+. Setting as usual $v = e^{\tau\phi}u$, (3.16) can be rewritten as

$$\tau|\phi^2 v|^2_{m-1,\tau,\Sigma^+} \leq c(|\phi^2 P_\tau v|^2_{\Sigma^+} + \tau^2|v|^2_{m-2,\tau,\Sigma^+}) \quad \tau > \tau_0 \tag{3.17}$$

To obtain this, apply (3.4) to $\phi^2 v$. This gives

$$c|\phi^2 v|^2_{m-1,\tau} \leq B(\phi^2 v) = 2Im < P^r_\tau \phi^2 v, P^i_\tau \phi^2 v > + d(|P_\tau \phi^2 v|^2_{-1,\tau} \quad \tau > \tau_0 \tag{3.18}$$

Now compute:

$$\begin{aligned} P_\tau \phi^2 &= \phi^2 P_\tau + R_{-1}, \quad R_{-1} : H^{m-2}_\tau \to H^{-1}_\tau \\ P^r_\tau \phi^2 &= \phi^2 P^r_\tau - i\phi P^i_\tau + \tau R_0, \qquad R_0 : H^{m-2}_\tau \to L^2 \\ P^i_\tau \phi^2 &= \phi^2 P^i_\tau - i\phi R_0 + R_1, \qquad R_1 : H^{m-2}_\tau \to H^1_\tau \end{aligned}$$

Using these relations we get

$$B(\phi^2 v) \leq 2\mathrm{Im} \ < \phi^2 P^r_\tau v, \phi^2 P^i_\tau v > + \tau|\phi^2 P^i_\tau v|^2 + c(|\phi^2 P^r_\tau v|^2_{-1,\tau} + \tau|v|^2_{m-2,\tau})$$

Combining this with (3.18) we get

$$c|\phi^2 v|^2_{m-1,\tau} \leq 2\mathrm{Im} \ < \phi^2 P^r_\tau v, \phi^2 P^i_\tau v > + \tau|\phi^2 P^i_\tau v|^2 + c(|\phi^2 P^r_\tau v|^2_{-1,\tau} + \tau|v|^2_{m-2,\tau}) \tag{3.19}$$

which implies (3.17).

(ii) Drop now the assumption supp $u \subset \Sigma^+$. If $v \in H^{m-2}_\tau(\Sigma^+)$ then we can find $w \in H^{m-2}(\Sigma^+)$ such that $\phi^j w \in H^{m-2+j}$ and $v - w \in H^{m-2}_0(\Sigma^+)$. Consequently, applying the first step (i) to $v - w$ yields (3.16).

Since the aim of the Carleman estimates with cutoff is to study regularity properties of solutions to pde's, one can expect that what happens in the region $\tau > 0$ is not as important. Indeed, the following modification of Theorem 3.10 is also true:

Theorem 3.11 *Assume that P is of real principal type, with C^1 coefficients. Let $K \subset\subset R^n$ and ϕ be a smooth function which is strongly pseudoconvex with respect to P on $\tau = 0$ in K. Then*

$$\tau|\phi^2 e^{\tau\phi} u|^2_{m-1,\tau,\Sigma^+} \leq c(|\phi^2 e^{\tau\phi} P(x,D)u|^2_{\Sigma^+} + \tau^3 |e^{\tau\phi} u|^2_{m-2,\tau,\Sigma^+}) \quad \tau > \tau_0 \tag{3.20}$$

whenever $u \in H^{m-1}$ is supported in K.

Again, assuming better regularity for the coefficients one can shift the energy levels in Theorem 3.10 to obtain

Theorem 3.12 *Assume that P is of real principal type, with Ξ_r coefficients. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Let $1 - s < r \leq s$, $r_0 = \max\{-s, r-1\}$. Then*

$$\tau|\phi^2 e^{\tau\phi} u|^2_{m+r-1,\tau,\Sigma^+} \leq c(|\phi^2 e^{\tau\phi} P(x,D)u|^2_{r,\tau,\Sigma^+} + \tau^2 |e^{\tau\phi} u|^2_{m+r_0-1,\tau,\Sigma^+} \quad \tau > \tau_0 \tag{3.21}$$

whenever $u \in H^{m-s}$ is supported in K and the right hand side is finite.

Proof : As usual make the substitution $v = e^{\tau\phi} u$ to reduce the estimate to

$$\tau|\phi^2 v|^2_{m+r-1,\tau,\Sigma^+} \leq c(|\phi^2 \tilde{P}_\tau(x,D)v|^2_{r,\tau,\Sigma^+} + \tau^2 |v|^2_{m+r-2,\tau,\Sigma^+} \quad \tau > \tau_0 \tag{3.22}$$

(i) Under the additional assumption that $u \in H^{m+r-1}$, supported in Σ^+, we prove that

$$\tau|\phi^2 v|^2_{m+r-1,\tau} \leq c(|\phi^2 \tilde{P}_\tau(x,D)v|^2_{r,\tau} + \tau^2 |v|^2_{m+r-2,\tau} \quad \tau > \tau_0 \tag{3.23}$$

Set $w = \ll^r v$. Then $w \in H^{m-1}$, therefore we can apply (3.19) to w to get

$$\tau|\phi^2 w|^2_{m-1,\tau,\Sigma^+} \leq c(|\phi^2 P_\tau w|^2_{\Sigma^+} + \tau^2 |w|^2_{m-2,\tau,\Sigma^+}) \quad \tau > \tau_0 \tag{3.24}$$

On the other hand Theorem 2.15 implies that

$$|\phi^j w|_{m+j-2,\tau,\Sigma^-} \leq c|v|_{m+r-2,\tau}$$

Combining this with (3.24) yields

$$\tau|\phi^2 w|^2_{m-1,\tau} \leq c(|\phi^2 P_\tau w|^2 + \tau^2 |v|^2_{m+r-2,\tau}) \quad \tau > \tau_0 \tag{3.25}$$

To switch in the above inequality from w to v compute

$$\begin{aligned}\phi^2 w &= \ll^r \phi^2 v + S^{r-1} v \\ \phi^2 P_\tau w &= [P_\tau v, \ll^r]\phi^2 v - [[P_\tau v, \ll^r], \phi^2] v + \ll^r \phi^2 P_\tau v + S^{r-1}\phi P_\tau v + S^{r-2} P_\tau v\end{aligned}$$

By theorem 2.10, $[P_\tau v, \ll^r] : H^{m+r-1} \to L^2$ and $[[P_\tau v, \ll^r], \phi^2] : H^{m+r_0-2} \to L^2$. On the other hand,

$$|\phi P_\tau v|_{r-1,\tau} \leq c(|P_\tau v|_{r-2,\tau} + |\phi^2 P_\tau v|_{r,\tau}) \leq c(|v|_{m+r_0-1,\tau} + |\phi^2 P_\tau v|_{r,\tau})$$

Applying this in (3.25) we get (3.21).

(ii) Now we want to use (3.23) to prove that (3.22) holds if $u \in H^{m+r-1}$, supported in Σ^+.

(a) If $r \geq 0$ then (3.23) and (3.22) are identical.

(b) If $2 - m \leq r < 0$ then (3.22) follows from (3.23) and the estimate

$$|\phi^2 \tilde{P}_\tau(x, D) v|^2_{r,\tau} \leq c(|\phi^2 \tilde{P}_\tau(x, D) v|_{r,\tau,\Sigma^+} + |v|_{m+r-2,\tau} \tag{3.26}$$

This estimate follows in turn by duality from the inclusion

$$H^{-r} \subset H_0^{-r} + H^{-r}(\Sigma) \tag{3.27}$$

(c) If $r < 2 - m$, then without any restriction in generality for (3.22) we assume that

$$|u|_{m+r-2,\tau} \leq c|u|_{m-r-2,\tau,\Sigma^+}$$

Afterwards, (3.22) follows from (3.23) and (3.26).

(iii) Next we want to remove the support assumption on u, i.e. to prove the result for all $v \in H^{m+r-1}$.

(a) Suppose $r \geq 2-m$. Decompose $v \in H^{m+r-2}(\Sigma^+)$ into $v_1 + v_2$ where $v_1 \in H_0^{m+r-2}(\Sigma^+)$ and $v_2 \in H^{m+r-2}(\Sigma)$. Then apply (3.22) to v_1 and transfer it to v using the better regularity of v_2.

(b) Suppose $r < 2 - m$. Decompose $v \in H^{m+r-2}$ into $v_1 + v_2$ where $v_1, v_2 \in H^{m+r-2}$ are supported in Σ^+, respectively Σ^-. Then (3.22) for v coincides with (3.22) for v_1.

(iv) Now we want to relax the regularity assumption on u, namely to prove that the result is still valid when $u \in H^{m+r_0-1}$. First observe that the arguments in steps (ii),(iii) rest unchanged, therefore it suffices to assume that u is supported in Σ^+ and prove (3.23).

Consider the approximating sequence v_ϵ for v given by Theorem 2.13. Apply (3.22) to v_ϵ. We get

$$\tau|\phi^2 v_\epsilon|^2_{m+r-1,\tau,\Sigma^+} \leq c(|e^{\tau\phi}P_\tau v_\epsilon|^2_{r,\tau} + \tau^2|v_\epsilon|^2_{m-2,\tau}) \tag{3.28}$$

The trouble is now that the above LHS norm is in Σ^+ while w_ϵ are not necessarily supported in Σ^+. This can be rectified using the support assumption on v. Namely, we have

$$|\phi^2 v_\epsilon|_{m+r-1,\tau} \leq c(|\phi^2 v_\epsilon|_{m+r-1,\tau,\Sigma^+} + |v|_{m+r-2,\tau}) \tag{3.29}$$

Theorem 2.15 gives

$$|\phi^2 v_\epsilon|_{m+r-1,\tau,\Sigma^-} \leq c|v|_{m+r-2,\tau}$$

If $m+r-1 \geq 0$ this implies (3.29). If $m+r-1 < 0$ then in addition we need to use (**??**).

Using (3.29), (3.28) gives

$$\tau|\phi^2 v_\epsilon|^2_{m+r-1,\tau} \leq c(|\phi^2 P_\tau v_\epsilon|^2_{r,\tau} + \tau^2|v|^2_{m+r-2,\tau}) \tag{3.30}$$

Now use (2.6) to get

$$\tau|\phi^2 v_\epsilon|^2_{m-1,\tau} \leq c(|\phi^2 P_\tau v|^2 + \tau^2|v|^2_{m-2,\tau}) \tag{3.31}$$

and finally, let $\epsilon \to 0$ to get (3.22), q.e.d.

Remark 3.13 *Other variants of the above theorem hold if we substitute P by P^* or otherwise if place the coefficients in a different position among the derivatives, as in Theorem 3.8.*

As a consequence of Theorem 3.12 we get the following result on continuation of regularity:

Theorem 3.14 *Assume that P is of real principal type, with Ξ_s coefficients. Let Σ be a strongly pseudoconvex surface with respect to P. Let $1-s < r \leq s$, $r_0 = \max\{-s, r-1\}$. Then we have continuation of the H^{m+r-1} regularity across Σ for functions $u \in H^{m-s}_{loc}$ satisfying $Pu \in H^r_{loc}$.*

Remark 3.15 *In effect, one can use the weaker estimates as in Theorem 3.11 to obtain the same result under the relaxed assumption that the strong pseudoconvexity condition for the surface Σ holds when $\tau = 0$.*

3.4 Boundary value problems

Let Ω be a domain in R^n with smooth boundary . The question we want to adress in this section is what type of Carleman estimates can one obtain for distributions u in Ω. Of course, such estimates should include some information about the Cauchy data of u on the boundary. The following sections study this problem in three increasingly difficult situations. First, we assume that the entire Cauchy data is known on the boundary; then we consider the case where we know only some boundary traces. In the second case, the corresponding boundary operators are first required to satisfy a strong Lopatinskii-type condition; finally, the case when only a weak Lopatinskii condition holds is considered.

3.4.1 Estimates involving all the Cauchy data on the boundary

To warm up, consider first the case when we have complete information about the boundary traces of u. Then

Theorem 3.16 *Assume that $P(x,D)$ is of real principal type, with C^1 coefficients. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Then*

$$\tau|e^{\tau\phi}u|^2_{m-1,\tau} \leq c|e^{\tau\phi}P(x,D)u|^2 + \tau|e^{\tau\phi}\,Tr\ u|_{m-1,\partial,\tau} \tag{3.32}$$

for $\tau > \tau_0$ whenever $u \in H^{m-s}$ is supported in K.

Proof of Theorem 3.16 :

We would like to use Theorem 2.7(a) for $v = e^{\tau\phi}u$. However, in order to do that we first need to extend v to a small neighbourhood of the domain Ω. Consider an extension, denoted still v, such that

$$|v|_{m-1,\tau,\Omega^c} \leq c|\mathrm{Tr}\ v|_{m-3/2,\tau,\partial} \tag{3.33}$$

If we apply Theorem 2.7(a) to the extended v and separate the Ω and the Ω^c parts then we get

$$c|v|^2_{m-1,\tau,\Omega} \leq 2\mathrm{Im}\ < P^r_\tau v, P^i_\tau v >_\Omega + d(|P_\tau v|^2_{-1,\tau,\Omega} + |v|_{m-1,\tau,\Omega^c}) + 2\mathrm{Im}\ < P^r_\tau v, P^i_\tau v >^c_\Omega$$

Use now Theorem 2.20 for the integration by parts in the last RHS term. Then we obtain

$$\begin{aligned} c|v|^2_{m-1,\tau,\Omega} \leq & 2Im < P^r_\tau v, P^i_\tau v >_\Omega + d(|P_\tau v|^2_{-1,\tau,\Omega} + |v|^2_{m-1,\tau,\Omega^c}) \\ & + Q_{P^r_\tau, P^i_\tau}(v,v) \end{aligned}$$

By (3.33) we get

$$c|v|^2_{m-1,\tau,\Omega} \leq 2Im < P^r_\tau v, P^i_\tau v >_\Omega + d(|P_\tau v|^2_{-1,\tau,\Omega} + |v|_{m-3/2,\tau,\partial}) + Q_{P^r_\tau,P^i_\tau}(v,v) \qquad (3.34)$$

Now bound the last RHS term as

$$Q_{P^r_\tau,P^i_\tau}(v,v)_{\leq} c|v|^2_{m-1,\tau,\partial}$$

Consequently, this gives

$$c|v|^2_{m-1,\tau,\Omega} \leq 2Im < P^r_\tau v, P^i_\tau v >_\Omega + d(|P_\tau v|^2_{-1,\tau,\Omega} + |v|_{m-1,\tau,\partial}) \qquad (3.35)$$

and further

$$c\tau|v|^2_{m-1,\tau,\Omega} \leq (|P_\tau v|^2_\Omega + \tau|v|^2_{m-1,\tau,\partial})$$

i.e. (3.32).

Theorem 3.16 can be shifted to other energy levels as in the proof of Theorem 3.6.

Theorem 3.17 *Assume that $P(x,D)$ is of real principal type, with coefficients of class Ξ_s. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Then*

$$\tau|e^{\tau\phi}u|^2_{m-1+r,\tau} \leq c|e^{\tau\phi}P(x,D)u|^2_{r,\tau} + \tau|e^{\tau\phi}\, Tr\ u|_{m-1+r,\partial,\tau} \qquad (3.36)$$

for $\tau > \tau_0$, $\max 1-s, -1/2 \leq r \leq s$, whenever $u \in H^{m-s}$ is supported in K.

Remark 3.18 *Why the restriction $r > -1/2$? What happens is that below that level the Cauchy data of u on is no longer well-defined. This can, however, be fixed by making an appropriate choice of the norms. The invariant way of doing it appears to be with Melrose's spaces $H^{s,k}_b$. Thus, a better reformulation of (3.32) is*

$$\tau|e^{\tau\phi}u|^2_{H^{m-1,r}_{b,\tau}} \leq c|e^{\tau\phi}P(x,D)u|^2_{H^{0,r}_{b,\tau}} + \tau|e^{\tau\phi}\, Tr\ u|_{m-1+r,\partial,\tau} \qquad (3.37)$$

The analogue of Theorem 3.12 is also true:

Theorem 3.19 *Assume that P is of real principal type, with Ξ_s coefficients. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Let $\max 1-s, -1/2 < r \leq s$ and $r_0 = \max r-1, 1-s$. Then*

$$\tau|\phi^2 e^{\tau\phi}u|^2_{m+r-1,\tau,\Sigma^+} \leq c(|\phi^2 e^{\tau\phi}P(x,D)u|^2_{r,\tau,\Sigma^+} + \tau^2|e^{\tau\phi}u|^2_{m+r_0-1,\tau,\Sigma^+} \quad \tau > \tau_0 \qquad (3.38)$$

whenever u is supported in K and the right hand side is finite.

Proof : To obtain (3.38) if u is supported in Σ^+ apply (3.35) to $\phi^2 v$ and then continue as in step (i) of the proof of Theorem 3.10. To get the same result in general, follow the same procedure as in step (ii) of Theorem 3.10.

As a consequence, we obtain the following regularity result:

Theorem 3.20 *Assume that P is of real principal type, with Ξ_s coefficients in a domain $\Omega \subset R^n$ with smooth boundary. Let Σ be a strongly pseudoconvex surface with respect to P. Let $1-s < r \leq s$.*

a) Then we have continuation of the H^{m+r-1} regularity across Σ for functions $u \in H^{m-s}_{loc}$ satisfying $Pu = 0$, $Tru \in H^{m+r-1}$.

b) If $r > -1/2$ then we have continuation of the H^{m+r-1} regularity across Σ for functions $u \in H^{m-s}_{loc}$ satisfying $Pu \in H^r$, $Tru \in H^{m+r-1}$.

The particular case when $\Sigma =$ gives:

Theorem 3.21 *Assume that P is of real principal type, with Ξ_s coefficients in a domain $\Omega \subset R^n$ with smooth boundary. Suppose that is strongly pseudoconvex with respect to P at x_0. Let $1-s < r \leq s$. If $u \in H^{m-s}$ solves $P(x,D)u = 0$ and $Tr\ u$ are H^{m+r-1} at x_0 then u is H^{m-r+1} at x_0.*

3.4.2 The strong Lopatinskii condition

Suppose now that we have a set of boundary operators B^i which satisfies the strong Lopatinskii condition with respect to $d\phi$ (described in Section 1.5.1). We denote by m_i the order of B_i. The Carleman estimate in this case has the form

Theorem 3.22 *Let P be a partial differential operator of real principal type. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Assume that the boundary operator B satisfies the strong Lopatinskii condition with respect to $d\phi$. Then*

$$\tau(|e^{\tau\phi}u|^2_{m-1,\tau} + |e^{\tau\phi}Tr\ u|^2_{m-1,\tau,\partial}) \leq c(|e^{\tau\phi}P(x,D)u|^2 + \tau|e^{\tau\phi}Bu|^2_{m-1,\tau,\partial}) \quad \tau > \tau_0 \tag{3.39}$$

whenever $u \in H^{m-1}$ is supported in K.

This estimate is also valid at other energy levels. This leads to an unique continuation result for solutions to

$$\begin{cases} P(x,D)u = 0 & in\ K \\ B(x,D)u = 0 & in\ \partial K \end{cases} \tag{3.40}$$

Theorem 3.23 *Assume that $P(x,D)$ is of real principal type, with coefficients of class Ξ_s, $s \geq 1$. Let Σ be an oriented surface which is strongly pseudoconvex with respect to P. Assume that the boundary operator B satisfies the strong Lopatinskii condition with respect to $d\phi$. Then unique continuation across Σ holds for H^{m-s} solutions u to (3.40).*

Correspondingly, we get the Carleman estimates with cutoff,

Theorem 3.24 *Let P be a partial differential operator of real principal type. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Assume that the boundary operator B satisfies the strong Lopatinskii condition with respect to $d\phi$. Then*

$$\tau(|\phi^2 e^{\tau\phi} u|^2_{m-1,\tau,\Sigma^+} + |\phi^2 e^{\tau\phi} Tr\, u|^2_{m-1,\tau,\Sigma^+,\partial} \leq c(|\phi^2 e^{\tau\phi} P(x,D)u|^2_{\Sigma^+} + |\phi^2 e^{\tau\phi} Bu|^2_{m-1,\tau,\partial,\Sigma^+} + \tau^2 |e^{\tau\phi} u|^2_{m-2,\tau,\Sigma^+}) \quad \tau > \tau_0 \tag{3.41}$$

whenever $u \in H^{m-1}$ is supported in K.

Again, the analogue estimates at other energy levels are also valid. These estimates lead to results on continuation of regularity.

Theorem 3.25 *Assume that P is of real principal type, with Ξ_s coefficients in a domain $\Omega \subset R^n$ with smooth boundary. Let Σ be a strongly pseudoconvex surface with respect to P. Assume that the boundary operator B satisfies the strong Lopatinskii condition with respect to $d\phi$. Let $1 - s < r \leq s$.*

a) Then we have continuation of the H^{m+r-1} regularity across Σ for functions $u \in H^{m-s}_{loc}$ satisfying $Pu = 0$, $Bu \in H^{m+r-1}$.

b) If $r > -1/2$ then we have continuation of the H^{m+r-1} regularity across Σ for functions $u \in H^{m-s}_{loc}$ satisfying $Pu \in H^r$, $Bu \in H^{m+r-1}$.

Proof of Theorems 3.22,3.24 :

We proceed as in the proof of Theorem 3.17 up to (3.34). Now by hypothesis we know that

$$Q_{P^r_\tau, P^i_\tau}(z,z) < 0 \quad \text{on} \quad Bz = 0$$

Consequently, if we choose c sufficiently large then we get

$$Q_{P^r_\tau, P^i_\tau}(z,z) - c|Bz|^2 < 0$$

which, by Theorem 2.29 implies that

$$-Q_{P^r_\tau, P^i_\tau}(v,v) + c|Bv|^2 > d|\text{Tr } v|^2$$

Combine this with (3.34) to get (3.39).

To obtain (3.41) start again with (3.34), but applied to $\phi^2 v$. Then use the above argument for the term $Q_{P^r_\tau, P^i_\tau}(\phi^2 v, \phi^2 v)$ and then continue as in the proof of Theorem 3.10.

3.4.3 The weak Lopatinskii condition

Many interesting problems do not satisfy the strong Lopatinskii condition, which implies that the strong estimates do not hold. However, in most cases there is some hope that some weaker estimates hold. Such estimates in the general case are highly technical and beyond the purpose of this monograph. Thus, the aim of this section is to merely highlight the main ideas, without providing complete proofs. On the other hand, some proofs are provided later for such estimates in special cases.

Based on the computations done for the case of the strong Lopatinskii condition, define

Definition 3.26 *We say that the set of boundary operators B^i satisfies only the weak Lopatinskii condition with respect to $d\phi$ if there exists a multiplier $Q \in S^{m-1,0}$ of the form*

$$q = rp^i_\tau \quad (mod \ \ p^r_\tau)$$

so that

(i) $r > 0$ on $\text{char } p_\tau$

(ii) $Q_{P,Q}(z,z) \geq 0$ on $Bz = Ez = 0$.

This may seem complicated; however, the following simple case suffices for most applications:

$$Q_{P^r_\tau, P^i_\tau}(z,z) = 0 \qquad \text{on } Bz = Ez = 0 \tag{3.42}$$

The result one can generally expect in this case is

Theorem 3.27 *Let P be a partial differential operator of real principal type. Let $K \subset\subset R^n$ and ϕ be a strongly pseudoconvex function with respect to P in K. Assume that the boundary operator B satisfies the weak Lopatinskii condition with respect to $d\phi$.*

a) Then

$$\tau(|e^{\tau\phi}u|^2_{m-1,\tau} \leq c|e^{\tau\phi}P(x,D)u|^2 \quad \tau > \tau_0 \tag{3.43}$$

whenever $u \in H^{m-1}$ *is supported in* K*.*

b) the coressponding estimate with cutoff is also valid,

$$\tau(|\phi^2 e^{\tau\phi}u|^2_{m-1,\tau,\Sigma^+} \leq c|e^{\tau\phi}P(x,D)u, \Sigma^+|^2 \quad \tau > \tau_0 \tag{3.44}$$

whenever $u \in H^{m-1}$ *is supported in* K*.*

Proof (sketch) : To keep things simple suppose that (3.42) holds.

a) Start again from (3.34). This time we know that

$$Q_{P^r_\tau, P^i_\tau}(z,z) = 0 \quad \text{on} \quad Bz = 0$$

Consequently, there exist other sets C, F of boundary operators such that

$$Q_{P^r_\tau, Q}(z,z) = < Bz, Cz > + < Ez, Fz >$$

which, by Theorem 2.21

implies that

$$| - Q_{P^r_\tau, P^i_\tau}(v,v) + < Bv, Cv > + < Ez, Fz > | \leq |\text{Tr } u|^2_{-1/2,\tau}$$

Combine this with (3.34) to get

$$\tau(|e^{\tau\phi}u|^2_{m-1,\tau} \leq c(|e^{\tau\phi}P(x,D)u|^2 + \tau|Tru|_{-1/2,\partial}) \quad \tau > \tau_0 \tag{3.45}$$

To elliminate the second RHS term one needs to have an enhanced trace regularity result:

Theorem 3.28 *Suppose that the symbols* B_j *are complete modulo* P_0*. Then there exists* $\epsilon > 0$ *such that*

$$|Tru|_{-1/2,\partial} \leq \tau^{-\epsilon}(|u|_{m-1,\tau} + |P_\tau u|)$$

For (3.44), as usual, apply the same procedure to $\phi^2 v$.

The results on unique continuation and continuation of regularity follow:

Theorem 3.29 *Assume that $P(x,D)$ is of real principal type, with coefficients of class Ξ_s, $s \geq 1$. Let Σ be an oriented surface which is strongly pseudoconvex with respect to P. Assume that the boundary operator B satisfies the weak Lopatinskii condition with respect to $d\phi$. Then unique continuation across Σ holds for H^{m-s} solutions u to (3.40).*

Theorem 3.30 *Assume that P is of real principal type, with Ξ_s coefficients in a domain $\Omega \subset R^n$ with smooth boundary. Let Σ be a strongly pseudoconvex surface with respect to P. Assume that the boundary operator B satisfies the weak Lopatinskii condition with respect to $d\phi$. Let $1 - s < r \leq s$.*

a) Then we have continuation of the H^{m+r-1} regularity across Σ for functions $u \in H^{m-s}_{loc}$ satisfying $Pu = 0$, $Bu = 0$.

b) If $r > -1/2$ then we have continuation of the H^{m+r-1} regularity across Σ for functions $u \in H^{m-s}_{loc}$ satisfying $Pu \in H^r$, $Bu = 0$.

3.5 Operators with (partially) analytic coefficients

It is by now clear that there is no significant change in the unique continuation properties if the regularity of the coefficients is upgraded from C^1 to C^∞. However, it does make a significant difference if the coefficients have instead some analyticity. A fundamental result, in this context, is

Theorem 3.31 *(Holmgren) Assume that P is a partial differential operator with analytic coefficients. Then we have unique continuation across any noncharacteristic surface for solutions u to $P(x,D)u = 0$.*

The aim of this section is to consider the more general case when the coefficients are analytic only with respect to some of the variables. As before, we use some type of Carleman estimates. The fundamental difference is that the exponential weight in the estimates is no longer scalar, but it is a pseudodifferential operator. In particular, we give a proof of Holmgreen's theorem based on Carleman estimates.

Let $\mathcal{F}$ be an analytic foliation of an open subset Ω of R^n. Let $P(x,D)$ be a partial differential operator whose coefficients are C^1 overall and analitic in the leaves of the foliation. We consider the following two cases:

(E) P is elliptic in the conormal bundle of the foliation $N^*\mathcal{F}$.

(H) P is of real principal type in $N^*\mathcal{F}$ and $N^*\mathcal{F}$ is invariant with respect to the null bicharacteristic flow.

Then

Theorem 3.32 *Let P be a partial differential operator of order m as above which satisfies either (E) or (F). Let Σ be an oriented hypersurface which is strongly pseudoconvex with respect to P in the conormal bundle of the foliation $N^*\mathcal{F}$. Then we have unique continuation across Σ for H^{m-1} solutions u to $P(x,D)u = 0$.*

One can choose local coordinates

$$x = (x_a, x_b).$$

so that the foliation $\mathcal{F}$ is generated by the functions x_a. Then the leaves are the hyperplanes $x_a = const$ and the conormal bundle of the foliation is

$$N^*\mathcal{F} = \{(x,\xi) \in T^*\Omega;\ \xi_a = 0\}$$

In these coordinates the coefficients of the partial differential operator $P(x,D)$ are analytic in x_a and C^1 in x_b. The conditions (E), (F) have the form

(E) $p(x,0,\xi_b)$ is elliptic.

(H) $p(x,0,\xi_b) = 0$ implies $p_{\xi_b}(x,0,\xi_b) \neq 0$ and $p_{x_a}(x,0,\xi_b) = 0$.

The important issue in the above theorem is the replacement of the strong pseudoconvexity condition in Hörmander's theorem. Here we use the same condition, but on a smaller subset of the cotangent bundle, namely on the set $\{\xi_a = 0\}$. The motivation for that becomes apparent if one examines the Carleman estimates below; the pseudodifferential weight there roughly cuts off the region $\{|\xi_a| > 0\}$.

3.5.1 The Carleman estimates

For $\epsilon, \tau > 0$ define the symbol

$$q^{\phi}_{\epsilon,\tau}(x,\xi) = e^{-\frac{\epsilon}{2\tau}|\xi_a|^2 + \tau\phi} \tag{3.46}$$

In the sequel we use the notation $Q^{\phi}_{\epsilon,\tau}(D,x)$ for the operator

$$Q^{\phi}_{\epsilon,\tau}(D,x)u = e^{-\frac{\epsilon}{2\tau}|D_a|^2}(e^{\tau\phi}u)$$

Let $r > 0$, $A \subset R^{n_a}$, convex, bounded , $B \subset R^{n_b}$, bounded. Let $K_r = B(A,r) \times B$ be the set in which we want to obtain the Carleman estimates. In order to have a clean calculus and a proof as simple as possible we make the following symplifying assumption on the coefficients:

" The coefficients of P can be extended as bounded analytic functions in x_a to $B(A,r) + iC^n$"

This can always be achieved locally by making suitable changes of coordinates.

Theorem 3.33 *Assume that P satisfies (E) in a domain K_r in R^n. Let ϕ be a smooth function, analytic in x_a, which is strongly pseudoconvex with respect to P in $\{\xi_a = 0\}$. Then for each small enough $\epsilon > 0$ there exist $\delta > 0$ and $c > 0$ such that for any large enough τ we have*

$$\tau^{-1}|Q^{\phi}_{\epsilon,\tau}(D,x)u|^2_{m,\tau} \leq c(|Q^{\phi}_{\epsilon,\tau}(D,x)P(x,D)u|^2_0 + |e^{\tau(\phi-\delta)}u|^2_{m-1,\tau}) \tag{3.47}$$

whenever $u \in H^{m-1}$ is supported in K_0.

Theorem 3.34 *Assume that P satisfies (H) in a domain K_r in R^n. Let ϕ be a smooth function, analytic in x_a, which is strongly pseudoconvex with respect to P in $\{\xi_a = 0\}$. Then for each small enough $\epsilon > 0$ there exist $\delta > 0$ and $c > 0$ such that for any large enough τ we have*

$$\tau|Q^{\phi}_{\epsilon,\tau}(D,x)u|^2_{m-1,\tau} \leq c(|Q^{\phi}_{\epsilon,\tau}(D,x)P(x,D)u|^2_0 + |e^{\tau(\phi-\delta)}u|^2_{m-1,\tau}) \tag{3.48}$$

whenever $u \in H^{m-1}$ is supported in K_0.

Remark 3.35 *One can shift this estimate to various energy levels if the appropriate regularity of the coefficients holds. Even if the coefficients are no better that above, one can lower the regularity of u in x_a.*

Proof of Theorems 3.33,3.34 in a special case :

We prove now the above theorems in the special case when the coefficients of P are independent of x_a and ϕ is a quadratic polinomial. On one hand the proofs are extremely simple in this case, and on the other hand this provides some insight into the problem, which is useful later on. Since the two proofs are similar, we present only the proof of Theorem 3.34.

Set

$$v = Q^{\phi}_{\epsilon,\tau}(D,x)u$$

Then after conjugation we obtain

$$Q^{\phi}_{\epsilon,\tau}(D,x)P(x,D)u = P_{\epsilon,\tau}(x,D,\tau)v$$

where

$$P_{\epsilon,\tau}(x,D,\tau) = P(x_b, D + i\tau\nabla\phi - \epsilon(\nabla^2\phi)D_a)$$

Hence $P_{\epsilon,\tau}$ is an operator of order m and moreover, if ϵ is small, it is a small perturbation of P_τ in $OPC^1S^m_\tau$. Then the strong pseudoconvexity condition on ϕ implies the following uniform inequality:

$$\{p^r_{\epsilon,\tau}, p^i_{\epsilon,\tau}\} > 0 \quad \text{on char } P_{\epsilon,\tau} \cap \{\xi_a = 0\}$$

Consequently, Theorem 2.24 implies the following uniform estimate

$$c|v|^2_{m-1,\tau} \leq 2Im < P^r_\tau v, P^i_\tau v > +d(|P_\tau v|^2_{-1,\tau} + |D_a^{m-1}v|^2) \quad \tau > \tau_0 \tag{3.49}$$

which further gives

$$\tau|v|^2_{m-1,\tau} \leq c(|P_\tau v|^2 + \tau|D_a^{m-1}v|^2) \quad \tau > \tau_0 \tag{3.50}$$

Now split the last RHS norm in the regions $|\xi_a| < \delta\tau$, respectively $|\xi_a| > \delta\tau$. We get

$$\tau|D_a^{m-1}v|^2 \leq \delta^{m-1}\tau^{2m-1}|v|^2 + \tau^{2m-1}e^{-\epsilon\delta^2\tau}|e^{\tau\phi}u|^2 \tag{3.51}$$

Combining this with (3.47) we get (3.50), q.e.d.

The similar results in the general case are considerably more technical, since this time we need to conjugate analytic functions by $Q_{\epsilon,\tau}$.

Compute first

$$Q_{\epsilon,\tau}(D,x)x_a^\alpha = (x_a + i\frac{\epsilon}{\tau}D_a)^\alpha Q_{\epsilon,\tau}(D,x)$$

Hence, formally we get

$$Q_{\epsilon,\tau}(D,x)P(x,D) = P_\tau(x_a + i\frac{\epsilon}{\tau}D_a, x_b, D)Q_{\epsilon,\tau}(D,x)$$

The operator $P_\tau(x_a + i\frac{\epsilon}{\tau}D_a, x_b, D)$ is not well-defined on any Sobolev space unless the coefficients of P are polinomials in x_a. Even then, it could have arbitrarily high order therefore any computation may seem hopeless at this point.

What we need is a slightly different viewpoint, and this is provided by the Weyl calculus. The critical observation is that

$$(x_a + i\frac{\epsilon}{\tau}D_a)^\alpha = Op^w((x_a + i\frac{\epsilon}{\tau}\xi_a)^\alpha)$$

Hence if we set

$$P_\tau(x,D) = c_\alpha(x)(D,\tau)^\alpha$$

then we have, still formally,

$$Q_{\epsilon,\tau}(D,x)P(x,D) = (Op^w(c_\alpha(x_a + i\frac{\epsilon}{\tau}\xi_a, x_b))(D,\tau)^\alpha)Q_{\epsilon,\tau}(D,x)$$

The functions $c_\alpha(z_a, x_b)$ are well-defined as bounded holomorphic functions for z_a in K_r. Extend them as smooth bounded functions supported in K_{2r}. Then the operator

$$P_{\epsilon,\tau} = p^w(c_\alpha(x_a + i\frac{\epsilon}{\tau}\xi_a, x_b))(D,\tau)^\alpha$$

is well-defined.

Our strategy is now to (i) show that $P_{\epsilon,\tau}$ is a "good enough" conjugate of P with respect to $Q_{\epsilon,\tau}$ (or of P_τ with respect to $e^{-\frac{\epsilon}{2\tau}|D_a|^2}$) and (ii) to prove that $P_{\epsilon,\tau}$ is "close" to P_τ in an apropriate sense. This is achieved in the sequel.

The following three Lemmas deal with the conjugation. Let Ω be a compact convex subset of R^n_a and $r > 0$. Denote by Z the space of functions f in C^{n_a} so that

a) f is supported in $B(\Omega, 2r) + iR^{n_a}$.

b) f is holomorphic in $B(\Omega, r) + iR^{n_a}$.

c) f is smooth and rapidly decreasing at ∞.

Let $\chi(x_a)$ be a cutoff function supported in $B(A, 2r)$, which is 1 in $B(A, r)$.

Define the remainder

$$R(f) = \chi(x_a)(Op^w(f(x + \frac{1}{\tau}\xi))e^{-\frac{\tau}{2}|D_a|^2} - e^{-\frac{\tau}{2}|D_a|^2} f(x)$$

Lemma 3.36 *Let $f \in Z$. Then*

$$|R_a w(x)| \leq c e^{-\frac{\tau}{2}r^2}|w| \tag{3.52}$$

for $\tau \geq 1$, w supported in Ω, $x \in B(\Omega, r)$.

Next let us turn our attention from scalar functions to operators. Let

$$P(x,D,\tau) = \sum_{|\alpha|\leq m} c_\alpha(x)(D,\tau)^\alpha$$

be a partial differential operator with bounded, compactly supported coefficients. Suppose that the coefficients c_α can be extended, as functions of x_a, to functions in F.

Define the candidate for the conjugated operator by

$$P_\epsilon = \sum_{|\alpha|\leq m} OP^w(c_\alpha(x_a + i\frac{\epsilon}{\tau}, x_b))(D,\tau)^\alpha$$

and the remainder

$$R(P) = e^{-\frac{\epsilon}{2\tau}|D_a|^2} P(x,D) - \chi(x_a) P_{\epsilon,\tau} e^{-\frac{\epsilon}{2\tau}|D_a|^2} \tag{3.53}$$

Lemma 3.37 *Let $P(x,D,\tau)$ be a partial differential operator as above. Then there exists $c > 0$ so that*

$$|R(P)w| \leq c e^{-\frac{\tau}{8\epsilon}r^2} |w|_{m,\tau} \tag{3.54}$$

for $\tau \geq 1$ and w supported in Ω.

For the case (H) we need a bit more

Lemma 3.38 *Let $P(x,D,\tau)$ be a partial differential operator as above. Assume in addition that P satisfies (H). Then there exists $c > 0$ so that*

$$|R(P)w| \leq c e^{-\frac{\tau}{8}r^2} (|Pw| + |w|_{m-1,\tau}) \tag{3.55}$$

for $\tau \geq 1$ and w supported in Ω.

Proof of Lemma 3.36 :

Look at the kernel $K(x,y)$ of R_a. We have

$$K(x,y) = \int (a(\frac{x+w}{2} + i\frac{1}{\tau}\xi) - a(y)) e^{i(x-w)\xi} e^{-\frac{\tau}{2}(w-y)^2} dw d\xi$$

With the change of variable

$$z = \frac{x+w}{2} + i\frac{1}{\tau}\xi$$

this becomes

$$K(x,y) = \int (a(z) - a(y)) e^{-\frac{\tau}{2}(x-y)^2} e^{\tau(z-y)(2x-z-\bar{z})} dz\, d\bar{z}$$

Write

$$a(z) - a(y) = b(z,y)(z-y)$$

with b holomorphic in the same domain as a. Then we can integrate by parts with respect to $\bar{z}$ to obtain

$$K(x,y) = \int \bar{\partial}_z b(z,y) e^{-\frac{\tau}{2}(x-y)^2} e^{\tau(z-y)(2x-z-\bar{z})} dz\, d\bar{z}$$

Now the integrand is supported in Re $z \notin B(\Omega, r)$. In this domain we estimate the exponential. It is easier to do that in the original coordinates, in which

$$K(x,y) = \int (\bar{\partial}_z b(\frac{x+w}{2} + i\frac{1}{\tau}\xi, y) e^{i(x-w)\xi} e^{-\frac{\tau}{2}(w-y)^2} dw d\xi$$

There we know that $x \in B(\Omega, r)$ and $(x+w)/2 \not\in B(\Omega, r)$. Since Ω is convex this implies $w \not\in B(\Omega, r)$. But $y \in \Omega$, hence $|w-y| \geq r$. This implies

$$K(x,y) \leq c e^{-\frac{\tau}{2} r^2}$$

q.e.d.

Lemma 3.37 is a straightforward consequence of Lemma 3.36, therefore we continue with

Proof of Lemma 3.38 : Use Cauchy's integral formula to write $p(x_a, x_b, D)$ as a superposition of

$$p(x_a, x_b, D) = \int K(x_a, z) R(z, x_b, D) dz$$

where K is analytic in x_a in $B(\Omega, r) + iR^{n_a}$ and

$$|R(z, x_b, \xi)| \leq c|P(x, \xi)| \tag{3.56}$$

By Lemma 3.36 we get the uniform estimate

$$|(e^{-\frac{\epsilon}{2\tau} D_a^2} K(z) - K_\epsilon(z) e^{-\frac{\epsilon}{2\tau} D_a^2} Rw| \leq c e^{-\frac{\tau}{8} r^2} |R(z) w| \tag{3.57}$$

Then (3.55) follows after integration from (3.57) since

$$|R(z)v| \leq c|Pv| + |v|_{m-1,\tau}$$

The last estimate is a consequence of (3.56). (keep in mind C^1 coefficients.

Now we want to prove that P_ϵ is a small perturbation of P in an appropriate sense. For this we no longer need the analyticity assumption on the coefficients. Thus assume that P and P_ϵ are as above, but with the coefficients $c_\alpha \in C^1_{x_b}(\mathcal{S}(C^{n_a})$. Fix K a compact subset of R^n.

Lemma 3.39 *Let P, P_ϵ be operators of order m as above. Then*

$$|(P - P_\epsilon)v| \leq c\frac{\epsilon}{\tau}|D_a v|_{m,\tau}$$

and

$$|(P - P_\epsilon)v|_{-1,\tau} \leq c\frac{\epsilon}{\tau}|D_a v|_{m-1,\tau}$$

Proof :

Compute

$$P_\epsilon - P = \frac{\epsilon}{\tau}(Q_{m,\epsilon}D_a + R_{m,\epsilon}) \tag{3.58}$$

Now the result follows since the m-th order operators $Q_{m,\epsilon}$, $R_{m,\epsilon}$ are bounded from H^m_τ into L^2 and from H^{m-1}_τ into H^{-1}_τ.

To see that the RHS above is small in an appropriate sense, we use the following

Lemma 3.40 *Assume that $v = e^{-\frac{\epsilon}{2\tau}D_a^2}w$. Then*

$$\frac{\epsilon}{\tau}|D_a v| \leq c|v| + e^{-\frac{\tau^2}{\epsilon}}|w|$$

The next Lemma deals with the inner products arising in the proof of the Carleman estimates:

Lemma 3.41 *Let (P, P_ϵ), (Q, Q_ϵ) be operators of order m as above with real symbols. Then*

$$|Im(< P_\epsilon u, Q_\epsilon u > - < Pu, Qu >)| \leq c\frac{\epsilon}{\tau}(|v|^2_{m,\tau} + |\frac{\epsilon}{\tau}D_a v|_{m,\tau})$$

and further

$$|Im(< P_\epsilon u, Q_\epsilon u > - < Pu, Qu >)| \leq c\frac{\epsilon}{\tau}(|v|^2_{m,\tau} + e^{-\frac{\tau}{\epsilon}}|w|_{m,\tau})$$

Proof :

Use the formula (3.58) and integrate by parts.

For the case (H) we need a stronger estimate. Fix $x_a^0 \in A$ and set

$$P_m(x_b, D) = P(x_a^0, x_b, D)$$

Now introduce the following translation invariant (in x_a) norm:

$$|v|_X^2 = |P_m v|^2 + |D_a v|_{m-1,\tau}^2$$

This norm is related to the operators P_ϵ, P by

Lemma 3.42 *Let P_ϵ be as above. Then*

$$|v|_X \leq c|Pv|^2 + |D_a v|_{m-1,\tau}^2 \tag{3.59}$$

and

$$|v|_X \leq c(|P_\epsilon v| + |(D_a, \tau)v|_{m-1,\tau}) + |\chi(\frac{\epsilon}{\tau} D_a)v|_X \tag{3.60}$$

Proof of Lemma 3.42 : a) Set

$$Q(x, D_b) = P(x, D_b, 0, 0)$$

Then it suffices to prove that

$$|P_m v| \leq c(|Qv| + |v|_{m-1,\tau}) \tag{3.61}$$

On the other hand, condition (H) gives

$$|P_m(x, \xi_b)| \leq c|Q(x, \xi_b)|$$

which implies (3.61).

b) Set

$$q_{m,\epsilon}(z_a, x_b, \xi_b) = p(z, x_b, 0, \xi_b, 0)$$

Then (3.60) would follow from

$$|P_m v| \leq c(|Q_{m,\epsilon} v| + |v|_{m-1,\tau} + |\frac{\epsilon}{\tau} D_a P_m v| \tag{3.62}$$

Due to the condition (H),

$$|q_{m,\epsilon}| \leq c|p_0|$$

Write

$$q_{m,\epsilon}(z, x_b, \xi_b) = q_{0,\epsilon}(z, x_b, \xi_b) p_m(x_b, \xi_b)$$

Then

$$q_{0,\epsilon} > 0 \quad \text{if } |\text{Im } z| \leq c$$

Let χ be a bounded smooth symbol supported away from 0 so that

$$p_{0,\epsilon} = q_{0,\epsilon}(z, x_b, xi_b) + \chi(\text{Im } z) \geq c > 0$$

Then at the operator level we get

$$|(P_{m,\epsilon} + \chi(\frac{\epsilon}{\tau} D_a) P_0) v - P_{0,\epsilon} P_0 v| \leq c|v|_{m-1,\tau}$$

This implies (3.62) provided that $P_{0,\epsilon}$ is invertible. Indeed, both $Q_{0,\epsilon}$ and $Q_{0,\epsilon}^{-1}$ are bounded operators and their product is

$$Q_{0,\epsilon}^{-1} Q_{0,\epsilon} = 1 + O(\tau^{-1})$$

Hence if τ is large enough then $Q_{0,\epsilon}$ is invertible and has a bounded inverse, q.e.d.

Lemma 3.43 *Let (P, P_ϵ), (Q, Q_ϵ) be operators of order m as above with real symbols. Assume in addition that they satisfy (H') with respect to P_0. Then*

$$|Im(< P_\epsilon u, Q_\epsilon u > - < Pu, Qu >)| \leq c\frac{\epsilon}{\tau}(|v|_X^2 + |\frac{\epsilon}{\tau} D_a v|_X^2) \tag{3.63}$$

and further

$$|Im(< P_\epsilon u, Q_\epsilon u > - < Pu, Qu >)| \leq c(\frac{\epsilon}{\tau}|P_\epsilon v| + \delta\tau|v|_{m-1,\tau} + e^{-c\delta\tau}(|Pw| + |w|_{m-1,\tau}) \tag{3.64}$$

Proof of Lemma 3.43 :

Here we want to start with a more precise description of P_ϵ, Q_ϵ which is a consequence of our assumption (H):

$$P_\epsilon = P_{0,\epsilon} P_m + P_{m-1,\epsilon} D_a + R_{m-1} \tag{3.65}$$

and

$$P - P_\epsilon = \frac{\epsilon}{\tau}[(P_{0,\epsilon} P_m + P_{m-1,\epsilon} D_a + R_{m-1}) D_a + + P_{0,\epsilon} P_m + P_{m-1,\epsilon} D_a + R_{m-1}] \tag{3.66}$$

where $R_{m-1} : H_\tau^{m-1} \to L^2$. Now use these relations to estimate

$$Im(< P_\epsilon u, Q_\epsilon u > - < Pu, Qu >)$$

All the parts can be directly bounded in terms of the RHS in (3.63) except for $\frac{\epsilon}{\tau}$ times terms of the form

$$\begin{array}{ll} (i) & Im < P_{0,\epsilon} P_m v, Q_{0,\epsilon} D_a P_m v > \\ (ii) & Im < P_{0,\epsilon} P_m v, Q_{m-1,\epsilon} D_a^2 v > \\ (iii) & Im < P_{0,\epsilon} P_m D_a v, P_{m-1,\epsilon} D_a v > \\ (iv) & Im < P_{m-1,\epsilon} D_a v, P_{m-1,\epsilon} D_a^2 v > \end{array}$$

where all the symbols involved are real.

(i) The expression above equals

$$< R P_m v, P_m v >$$

where

$$R = P_{0,\epsilon}^* Q_{0,\epsilon} D_a - D_a Q_{0,\epsilon}^* P_{0,\epsilon}$$

The pdo calculus implies that

$$R = O(1) + O(\epsilon/\tau) D_a + R_{-1} D_a$$

where $R_{-1} : L_a^2(H_b^{-1}) L^2 \to 0$ as operator in L^2, qed.

(ii) Rewrite the expression in there as

$$\text{Im } < P_{0,\epsilon} P_1 |D|^{m-1} v, Q_{0,\epsilon} D_a^2 |D|^{m-1} v >=$$

$$< R_0 P_1 |D|^{m-1} v, |D|^{m-1} v > + < R_1 |D|^{m-1} v, |D|^{m-1} v >$$

where

$$R_1 = [P_1, P_{0,\epsilon}^* Q_{0,\epsilon}] D_a^2 = D_a O(1) D_a$$

and

$$R_0 = P_{0,\epsilon}^* Q_{0,\epsilon} D_a^2 - D_a^2 Q_{0,\epsilon}^* P_{0,\epsilon} = O(1) D_a + D_a O(1/\tau) D_a$$

(iii) This can be easily reduced to (ii) by moving a D_a derivative from the left to the right.

(iv) Integrate by parts while avoiding having m D_b derivatives together. Then the expression is bounded by

$$|D_a v|_{m-1}^2 + |D_a v|_{m-1} |\frac{\epsilon}{\tau} D_a^2 v|_{m-1}$$

The last term comes from the commutator of the coefficients, which has order $\frac{\epsilon}{\tau}$.

Now (3.64) follows from (3.63) and Lemma 3.42 if we cut off at $D_a \approx \tau\sqrt{\frac{\delta}{\epsilon}}$.

Proof of Theorem 3.33 : The pseudoconvexity condition gives

$$\{\text{Re } p_\tau, \text{Im } p_\tau\} > 0 \quad \text{on } p_\tau = 0, \xi_a = 0$$

By Theorem 2.7 (b) this implies that

$$c|v|^2_{m,\tau} \leq \tau \text{Im } < P^r_\tau v, P^i_\tau v > +d(|P_\tau v|^2 + |D_a v|_{m-1,\tau}) \tag{3.67}$$

By Lemmas 3.39,3.41 this gives

$$c|v|^2_{m,\tau} \leq 2\tau \text{Im } < P^r_{\epsilon,\tau} v, P^i_{\epsilon,\tau} v > +d|P_{\epsilon,\tau} v|^2 + c_1(\epsilon|v|_{m,\tau} + |\frac{\epsilon,\tau}{D}{}_a v|_{m,\tau} + |D_a v|_{m-1,\tau})$$

and further, if ϵ is sufficiently small,

$$|v|^2_{m,\tau} \leq c(\tau|P_{\epsilon,\tau} v|^2_{m,\tau} + |\tau^{-1} D_a v|^2_{m,\tau}$$

Now suppose $v = e^{-\frac{\epsilon}{\tau}D_a^2} w$ and use Lemma 3.40 for the last RHS term to get

$$|v|^2_{m,\tau} \leq c(\tau|P_{\epsilon,\tau} v|^2_{m,\tau} + e^{-c\epsilon}|w|^2_{m,\tau}$$

Using Lemma 3.37 it follows that

$$|v|^2_{m,\tau} \leq c(\tau|e^{-\frac{\epsilon}{\tau}D_a^2} P_\tau w|^2_{m,\tau} + e^{-c\epsilon}|w|^2_{m,\tau}$$

which implies the desired conclusion.

Proof of Theorem 3.34 :

Again, by Theorem 2.7 (b) the pseudoconvexity condition gives

$$\tau|v|^2_{m-1,\tau} \leq 2\text{Im } < P^r_\tau v, \tau^{-1} P^i_\tau v > +d(\tau|P_\tau v|^{-1,\tau} + \tau|D_a w|^2_{m-2,\tau})$$

Use Lemmas 3.39, 3.43 to substitute P_τ by $P_{\epsilon,\tau}$ as before and obtain

$$\tau|v|^2_{m-1,\tau} \leq c(|P_{\epsilon,\tau} v|^2 + \tau|D_a w|^2_{m-2,\tau}) + \frac{\epsilon}{\tau}(|v|^2_X + |\frac{\epsilon}{\tau} D_a v|^2_X)$$

The next step is to substitute the X norm using the norm of $P_{\epsilon,\tau} u$. This can be done using (3.60) to get

$$\tau|v|^2_{m-1,\tau} \leq c(|P_{\epsilon,\tau} v|^2 + \tau|D_a w|^2_{m-2,\tau}) + \frac{\epsilon}{\tau}(|(D_a,\tau)v|^2_{m-1,\tau} + |\chi(\frac{\epsilon}{\tau}D_a)\frac{\epsilon}{\tau}D_a v|^2_X)$$

If $v = e^{-\frac{\epsilon}{\tau}D_a^2} w$ then Lemma 3.40 gives

$$\tau|v|^2_{m-1,\tau} \leq c(|P_{\epsilon,\tau} v|^2 + \tau e^{-c\epsilon\tau}|w|^2_{m-1,\tau}) + \tau|D_a^{-1} w|^2_X)$$

which, again by (3.60) yields the desired conclusion.

3.5.2 Unique continuation

Theorem 3.44 *Let $\mathcal{F}$ be an analytic foliation of an open subset Ω of R^n. Let $P(x, D)$ be a partial differential operator whose coefficients are C^1 overall and analitic in the leaves of the foliation, satisfying either (E) or (H). Let Σ be an oriented hypersurface which is strongly pseudoconvex with respect to P in $N^*\mathcal{F}$. Then we have unique continuation across Σ for solutions u to $P(x, D)u = 0$.*

Proof : Suppose we are in case (E). If we apply the usual argument involving a perturbation of the surface S and the cutoff of u then the problem reduces to

Suppose ϕ is a quadratic function which is strongly pseudoconvex with respect to P in the set $\{\xi_a = 0\}$. Let $u \in H^{m-1}$ be a function supported in a δ ball, with δ sufficiently small, so that $Pu \in L^2$ and $Pu = 0$ in $\{\phi > 0\}$. Then $u = 0$ in $\{\phi > 0\}$.

If we apply the Carleman estimate (3.47) to u in suitable local coordinates and let $\tau \to \infty$ then we obtain

$$|Q^{\phi}_{\epsilon,\tau} u|_{m-1,\tau} \to 0 \qquad \text{as } \tau \to \infty \tag{3.68}$$

Let v be a function whose Fourier transform has compact support. Consider the function $g : R \to R$,

$$g(t) = \int \delta(t - \phi(x))u(x)v(x)dx$$

Its Fourier transform is the entire function

$$\hat{g}(z) =< v, e^{\tau\phi} u >$$

Clearly

$$|\hat{g}(z)|_{m-1.\tau} \leq ce^{c|z|}, \quad z \in C$$

while

$$|\hat{g}(z)|_{m-1.\tau} \leq c(1 + |z|^{m-1}), \quad z \in R$$

On the other hand, (3.68) shows that $\hat{g}$ is bounded on the negative imaginary axis. Hence, we can use the Phragmen-Lindelof Theorem to conclude that

$$|\hat{g}(z)|_{m-1.\tau} \leq c(1 + |z|^{m-1}), \quad \text{Im } z < 0.$$

This implies that $g(t) = 0$ when $t > 0$. Hence, if h is a smooth function compactly supported in R^+ then

$$\int g(t)h(t) = 0$$

whic is equivalent to

$$\int u(x)v(x)h(\phi(x))dx = 0$$

This holds for any v whose Fourier transform has compact support, and, by density, for any v. Consequently we get $u = 0$ in supp $h(\phi(x))$, i.e. $u = 0$ in $\phi > 0$, q.e.d.

3.5.3 Stability estimates

As explained in the introduction, any unique continuation result is accompanied by a certain stability estimate. The challenge is then to obtain the best possible stability estimate. As opposed to the Holder stability estimates corresponding to the standard Carleman estimates, the stability estimates corresponding to the unique continuation results in this section are considerably more delicate. We start with a local result:

Theorem 3.45 *Let $\mathcal{F}$ be an analytic foliation of an open subset Ω of R^n. Let $P(x,D)$ be a partial differential operator whose coefficients are C^1 overall and analitic in the leaves of the foliation, satisfying either (E) or (H). Let Σ be an oriented hypersurface which is strongly pseudoconvex with respect to P in $N^*\mathcal{F}$.*

Then for each $x_0 \in \Sigma$ and each neighbourhood W of x_0 there exists a neighbourhoods V of x_0 so that

$$|u|_{H^{m-2}(V)} \leq c \frac{|u|_{H^{m-1}(W)}}{(\ln 1 + \frac{|u|_{H^{m-1}(W)}}{|Pu|_{L^2(W)}})} \tag{3.69}$$

whenever $u \in H^{m-1}$ is supported in Σ^+ $P(x,D)u = 0$.

The inequality (3.69) can be rewritten as

$$|u|_{H^{m-2}(V)} \leq c \frac{1}{|\ln |Pu|_{L^2(W)}|} \quad \text{whenever } |u|_{H^{m-1}(W)},\ |Pu|_{L^2(W)} \leq 1/2 \tag{3.70}$$

One can see that this inequality is weaker than the usual one (see (3.8)).

The above result is purely local. We belive, however, that the same stability result should also hold globally. Nevertheless, if one tries to iterate it directly to obtain a global result, then the function $(|\ln x| + 1)^{-1}$ is replaced by one of its iterates, of the form

$$(1 - \ln(1 - \ln(1 - \ln ...(1 - \ln x)^{-1} ...)^{-1})^{-1})^{-1}$$

The global result we can prove is only a bit weaker:

Theorem 3.46 *Let $K_0 \subset\subset K_1 \subset K_2$ be bounded subsets of R^n for which iterated application of the unique continuation result in Theorem 3.44 yields*

If $u \in H^{m-1}_{\text{loc}}$ is supported in K_2 and $Pu = 0$ in K_1 then $u = 0$ in K_1.

Then for any $\epsilon > 0$ we have

$$|u|_{H^{m-2}(K_0)} \leq c \frac{|u|_{H^{m-1}(K_1)}}{(\ln 1 + \frac{|u|_{H^{m-1}(K_1)}}{|Pu|_{L^2(K_1)}})^{1+\epsilon}} \tag{3.71}$$

whenever $u \in H^{m-1}$ is supported in K.

The idea of the proof is to obtain first an enhanced local low frequency estimate and then to iterate it, but only for the low frequencies. We continue with the crucial low frequency estimate.

Theorem 3.47 *Let $\mathcal{F}$ be an analytic foliation of an open subset K of R^n. Let $P(x, D)$ be a partial differential operator whose coefficients are C^1 overall and analytic in the leaves of the foliation, satisfying either (E) or (H). Let ϕ be an analytic function which is strongly pseudoconvex with respect to P in $N^*\mathcal{F}$.*

Let $a(\xi)$ be a smooth, compactly supported symbol, of Gevrey class α. Let $b(x)$ be a cutoff function of Gevrey class which is 1 in $\{|x| < 1\}$ and 0 in $\{|x| > 2\}$. Then given $R > 0$ there exists $r > 0, c > 0$ so that for each $x_0 \in K$ and u supported in $K \cap \phi < \phi(x_0)$ such that

$$|u|_{m-1} = 1, \quad |A(\frac{D_a}{\mu})b(R^{-1}(x - x_0))Pu| \leq e^{-\mu^\alpha} \tag{3.72}$$

we have

$$|A(\frac{D_a}{\tau})b(r^{-1}(x - x_0))u|_{m-1} \leq e^{-\tau^\alpha} \quad \tau < c\mu^\alpha \tag{3.73}$$

In other words, this theorem gives a local stability estimate in the region $\phi_0 - 3\delta \leq \phi \leq \phi_0$ only for frequencies smaller than $c|\ln|Pu||$. Surprisingly, the bound we obtain for these frequences is $e^{-c|\ln|Pu||^\alpha}$, which is incomparably better than the $|\ln|Pu||^{-1}$ that we need in order to prove Theorem 3.45, and almost as good as the $|Pu|^c$ that one obtains from the classical Carleman estimates.

Proof of Theorem 3.47 :

The idea of the proof is to cutoff u near the surface x_0 and then to use one of the estimates (3.47) or (3.48).

Next we use a perturbation argument. Given $x_0 \in K$ we consider the function

$$\psi(x) = \phi(x) + \gamma |x - x_0|^2$$

The pseudoconvexity condition is stable with respect to small C^2 perturbations. Therefore there exists γ_0 independent of x_0 so that ψ also satisfies the pseudoconvexity condition in the hypothesis of the theorem in the domain K for $\gamma < \gamma_0$. Fix $\gamma \leq \gamma_0$.

Then there exists δ independent of x_0 so that

$$\{\phi \leq 0\} \cap \{\psi > -8\delta\} \subset B(x_0, R)$$

Given δ there exists some $r > 0$, independent of x_0, so that

$$B(x_0, 2r) \subset \{\psi > -\delta\}$$

In the sequel we assume without any restriction in generality that $\psi(x_0) = 0$.

Let χ be a smooth cutoff function which is 0 on $(-\infty, -8\delta]$ and 1 in $[-7\delta, \infty)$. Then

$$P(\chi(\psi)u) = \chi(\psi)Pu + [P, \chi(\psi)]u$$

where the operator $[P, \chi]$ is supported in $\{-8\delta < \phi < -7\delta\}$. If u is supported in $\{\phi \leq 0\}$ then χu is supported in $B(x_0, R)$. Consequently, we can apply our Carleman estimates, say (3.47) to χu to obtain

$$\tau |Q^{\psi}_{\epsilon,\tau}(D,x)\chi(\psi)u|^2_{m-1,\tau} \leq c(|Q^{\psi}_{\epsilon,\tau}(D,x)\chi(\psi)P(x,D)u|^2_0 + +|[P,\chi(\psi)]u|^2 + |e^{\tau(\psi-d)}u|^2_{m-1,\tau}) \tag{3.74}$$

To continue, we need a better estimate for the norm of Pu in the RHS. This is given by the following

Lemma 3.48 *Suppose that (3.72) holds, and that ψ is quadratic in x_a. Then*

$$|Q^{\psi}_{\epsilon,\tau}(D,x)\chi(\psi)P(x,D)u| \leq e^{-\mu^\alpha}, \quad \tau < c\mu^\alpha$$

Proof : Without any restriction in generality assume that χ also cuts off the region $\psi > \delta$. By splitting it off in Fourier variable we obtain

$$\begin{aligned} |Q^{\psi}_{\epsilon,\tau}(D,x)\chi(\psi)P(x,D)u| &\leq ce^{-c\mu^2\tau^{-1}} + |A_{\mu/2}e^{\tau\phi}\chi(\psi)Pu| \\ &\leq ce^{-c\mu^2\tau^{-1}} + e^{\delta\tau-\mu^\alpha} + |A_{\mu/2}e^{\tau\phi}\chi(\psi)(1-A_\mu)Pu| \end{aligned} \tag{3.75}$$

To estimate the last term look at the Fourier transform of $e^{\tau\phi}\chi(\psi)$. Since ψ is quadratic in x we can decompose it at fixed x_0 into

$$\psi(x) = \phi(x_0) + A(x - x_0) + Q(x - x_0)$$

By looking at the power series we get

$$\partial^k e^{A\tau(x-x_0)}_{|x=x_0} \leq (c\tau)^k$$

$$\partial^k e^{\tau Q(x-x_0)}_{|x=x_0} \leq (c\tau)^{k/2}(k!)^{1/2}$$

On the other hand, for $\chi(\psi)$ we have

$$\partial^k \chi(\psi)_{|x=x_0} \leq c^k (k!)^{1/\alpha}$$

Putting the three estimates together we get

$$\partial^k e^{\tau\phi}\chi(\psi)_{|x=x_0} \leq e^{\tau\phi} c^k (\tau^k + (k!)^{1/\alpha})$$

Then for its Fourier transform we get the bound

$$\widehat{e^{\tau\phi}\chi(\psi)} \leq |\xi|^{-k} e^{\delta\tau} c^k (\tau^k + (k!)^{1/\alpha})$$

If we minimize this expression with respect to k we obtain

$$\widehat{e^{\tau\phi}\chi(\psi)} \leq e^{\delta\tau - c|\xi|^\alpha} c^k (\tau^k + (k!)^{1/\alpha}) \qquad |\xi| > C\tau$$

Hence, if $\tau < c|\mu|$ then

$$|A_{\mu/2} e^{\tau\phi}\chi(\psi)(1 - A_\mu)| < e^{\delta\tau - c\mu^\alpha}$$

Then, going back to (3.75) we obtain

$$|Q^\psi_{\epsilon,\tau}(D, x)\chi(\psi)P(x, D)u| \leq e^{-c\mu^\alpha} \quad \tau < c\mu^\alpha$$

which concludes the proof of the Lemma.

Suppose now that $d > 8\delta$ (if not, we go back to when we have chosen δ and choose a smaller one; since the function ψ is a small perturbation of ϕ, it is not difficult to see that $d > 0$, and therefore $\delta > 0$, can be chosen independently of x_0) Taking also into account Lemma 3.48, from (3.74) we obtain

$$|Q^\psi_{\epsilon,\tau}(D, x)\chi(\psi)u|_{m-1,\tau} \leq ce^\delta\tau \max\{e^{-c\mu^\alpha}, e^{-7\delta\tau}\} \tag{3.76}$$

Think now of τ as $-iz$, with z on the positive imaginary axis. We would like to extend this estimate to the upper half-space. This is done in the following

Lemma 3.49 *Let h be an analytic function in the upper halfspace, satisfying*

$$|h(z)| \leq c \max\{e^{-\mu}, e^{-7\delta \text{Im } z}\} z \in R \cup +iR_+ \tag{3.77}$$

Then there exists some $d > 0$ which does not depends on μ so that

$$|h(z)| \leq ce^{-6\delta \text{Im } z} \qquad |z| < d\mu, \text{Im } z \geq 0 \tag{3.78}$$

Proof : Let f be the bounded harmonic function in the fourth quadrant, which satisfies

$$f(z) = \max\{\mu, 7\delta \text{Im } z\} z \in R \cup -iR$$

Then

$$|h(z)| \leq ce^{f(z)}$$

therefore we need to prove that there exists some $\delta > 0$ so that

$$f(z) \leq 6\delta \text{Im } z \qquad |z| < d\mu \tag{3.79}$$

The difficulty is that we want d not to depend on μ. The solution is to scale μ away. Set $f_1(z) = (\mu)^{-1} f(z\mu)$. Then

$$f_1(z) = \max\{1, 7\delta \text{Im } z\}, \qquad z \in R \cup -iR$$

Hence, there exists $d > 0$ so that

$$f_1(z) \leq 6\delta \text{Im } z \qquad |z| < d$$

which implies (3.79).

As a consequence of the lemma, we get

$$|Q^{\psi}_{\epsilon,-iz}(D,x)\chi(\psi)u|_{m-1,\tau} \leq ce^{-5\delta \text{Im } z}, \qquad |z| \leq d\mu^{\alpha}, \quad \text{Im } z \geq 0 \tag{3.80}$$

Let η be a smooth cutoff function which equals 1 in $[-3\delta, \infty)$ and 0 in $(-\infty, 4\delta]$, of Gevrey class α. Then we would like to obtain an estimate for

$$F = A(\frac{\beta D_a}{\mu})\eta(\psi(x))u(x)$$

The idea is to foliate F with respect to the level sets of ψ:

$$F = \int \eta(t) A(\frac{\beta D_a}{\mu}) \delta(t - \phi(x)) u(z) dt$$

Using the Fourier transform in t this further gives

$$F = \int_{-\infty}^{\infty} \overline{\hat{\eta}(\overline{z}} A(\frac{\beta D_a}{\mu}) e^{-iz\phi} u dz \tag{3.81}$$

The plan is to get a good estimate of F by making a suitable modification of the integration contour in the upper half-plane. To do that, we need to get good estimates for $\hat{\eta}$ in the lower half-plane and for $A(\frac{\beta D_a}{\mu}) e^{-iz\phi} u$ in the upper half-plane.

Since η is of Gevrey class α and is supported in $[-4\delta, \infty)$, its Fourier transform satisfies

$$\hat{\eta}(z) \leq c e^{-4\delta \text{Im } z} e^{-|z|^\alpha}, \quad \text{Im } z < 0 \tag{3.82}$$

From (3.80), on the other hand, we obtain

$$|A(\frac{\beta D_a}{\mu}) \eta(\psi(x)) e^{-iz\phi} u(x)| \leq e^{(-6 + c\epsilon\beta^2) \text{Im } z} \qquad |z| = d\mu, \quad \text{Im } z \geq 0 \tag{3.83}$$

Hence, change the path in (3.81) to the contour in the following figure:

Then estimate F using the bounds in (3.83), (3.82). We obtain

$$F \leq c e^{-c\mu^\alpha} \tag{3.84}$$

Let now η_1 be a smooth cutoff function which equals 1 in $[-\delta, \infty)$ and 0 in $(-\infty, 2\delta]$. Then

$$\eta_1 A(\frac{\beta D_a}{\mu}) u = \eta_1 A(\frac{\beta D_a}{\mu}) \eta(\phi) u + \eta_1 A(\frac{\beta D_a}{\mu})(1 - \eta(\phi))$$

We use (3.84) for the fist RHS term, while for the second we estimate its kernel. Since a is of Gevrey class α we can estimate its Fourier transform by

$$\hat{a}(x) \leq d e^{-c|x|^\alpha}$$

Hence the kernel $k(x,y)$ of $\eta_1 A(\frac{\beta D_a}{\mu})(1 - \eta(\phi))$ given by

$$k(x,y) = \frac{\mu}{\beta} \hat{a}(\frac{\mu}{\beta}(x-y)) \eta_1(\phi(x))(1 - \eta(\phi(y)))$$

satisfies

$$|k(x,y)| \leq de^{-c\mu^{\alpha}}$$

(with a different c). Therefore we obtain

$$|\eta_1 A(\frac{\beta D_a}{\mu})u| \leq ce^{-c\mu^{\alpha}} \tag{3.85}$$

which, given our choices for r, δ, η_1, implies (3.71), q.e.d.

Proof of Theorem 3.45 :

Given the neighbourhood W of x_0 choose R so that $B(x_0, R) \subset W$. Then let r be as in Theorem 3.47 and $V = B(x_0, r)$. Without any restriction in generality assume that $|u|_{m-1} = 1$ and that $|Pu| = e^{-\mu}$ in $B(0, r)$. In V decompose u as

$$u = A(\frac{\beta D_a}{\mu})u + (1 - A(\frac{\beta D_a}{\mu}))u$$

Then use (3.71) for the first term and (3.72) for the second term. We obtain

$$|u|_{m-2,V} \leq d(e^{-c\mu^{\alpha}} + \mu^{-1}) \leq d\mu^{-1}$$

Due to the choice of μ in (3.72) this implies (3.69), q.e.d.

Proof of Theorem 3.46 :

The strategy of the proof is to iterate the local low-frequency result.

A. We claim that there exist $R, r > 0$ and a finite collection of points $\{x_i\}_{i=1,N}$ satisfying the following conditions:

i) $x_k \in K_0 \setminus B_k$ for all k.

ii) Lemma 3.47 holds at x_k for solutions u supported in $K_0 \setminus B_k$.

iii) $K_0 \subset B_N$

where $B_k \cup_{i<k} B(x_i, r/2)$.

Indeed, by hypothesis we can find a finite partition

$$K_0 = \cup_{A_j}$$

and associated functions ϕ_j which are strongly pseudoconvex in A_j so that the unique continuation result in K_0 is a consequence of iterating the unique continuation in A_j across level sets of ϕ_j. In other words, we can choose ϕ_j so that $\phi_j \geq 0$ in A_j and $\phi_j \leq 0$ in $K_0 \setminus \cup_{i \leq j} A_j$.

Then for $R > 0$ choose $r > 0$ small enough so that Theorem 3.47 holds for ϕ_j and $x_0 \in A_j$ for all j.

Now we can choose x_k inductively following the following algorithm:

a) x_1 is the maximum point for ϕ_1 in A_1.

b) Suppose $x_{k-1} \in A_j$.

ba) If $A_j \not\subset B_k$ then choose $x_k \in A_j$ a maximum point for ϕ_j in $A_j \setminus B_k$.

bb) Else, there exists some $h > j$ so that $\cup_{l<h} A_l \subset B_k$ but $A_h \not\subset B_k$. Then choose $x_k \in A_h$ a maximum point for ϕ_h in $A_h \setminus B_k$.

B. Now we use Lemma 3.47 for the points x_i obtained in part A. Define the functions $u_1 = u$, $u_j = (1 - b(\frac{2}{r}(x - x_{j-1})))u_{j-1}$. Let

$$\mu_1 = \ln(|Pu|_{K_1})$$

and

$$\mu_j = c\mu_j^\alpha$$

with c as in Lemma 3.47. We infer that we can apply iteratively Lemma 3.47 to conclude that

$$|A(\frac{D_a}{\mu_j})b(r^{-1}(x - x_j))u_j|_{m-1} \leq e^{-\mu_j^\alpha}$$

For $j = 1$ this follows directly from Lemma 3.47. For the induction argument, observe first that the sequence u_j stays bounded in H^{m-1} by construction. Hence, we only need to prove that

$$\begin{cases} |A(\frac{D_a}{\mu_{j-1}})gPu_{j-1}| \leq e^{-\mu_{j-1}^\alpha} \\ |A(\frac{D_a}{\mu_{j-1}})b(r^{-1}(x - x_{j-1}))u_{j-1}|_{m-1} \leq e^{-\mu_{j-1}^\alpha} \end{cases}$$

implies

$$|A(\frac{D_a}{\mu_j})gPu_j| \leq e^{-\mu_j^\alpha}$$

According to the definition of u_j we have

$$A(\frac{D_a}{\mu_j})gPu_j = A(\frac{D_a}{\mu_j})gPu_{j-1} + A(\frac{D_a}{\mu_j})[b(\frac{2}{r}(x - x_{j-1})), P]u_j$$

The first RHS term is bounded by $e^{-\mu_{j-1}^\alpha}$. For the second, denote

$$v = b(r^{-1}(x - x_{j-1}))u_{j-1}$$

Then we need to prove that

$$|A(\frac{D_a}{\mu_j})f(x)D^{m-1}v| \leq c(|A(\frac{D_a}{\mu_{j-1}})v|_{m-1} + e^{-c\mu_j}|v|_{m-1}$$

which after factoring out the derivatives reduces to

$$|A(\frac{D_a}{\mu_j})f(x)v| \leq c(|A(\frac{D_a}{\mu_{j-1}})v| + e^{-c\mu_j}|v|$$

and further to

$$|A(\frac{D_a}{\mu_j})f(x)(1 - A(\frac{D_a}{\mu_{j-1}})| \leq e^{-c\mu_j}$$

This follows since f is of Gevrey class so that its Fourier transform decays like $e^{-c|\xi|^\alpha}$ at infinity.

3.6 Elliptic equations: singular weights and strong unique continuation

This section is devoted to a class of Carleman estimates with singular weight function, for second order elliptic equations and for parabolic equations.

Note that if an operator is not elliptic and the weight function blows up at a point then the pseudoconvexity condition would be violated on all bicharacteristics passing through the singular point. Hence, weight functions which blow up at a point can be allowed only for elliptic equations.

As it turns out, even for elliptic equations the strong pseudoconvexity condition fails near the singular point, therefore we need to contend ourselves with (possibly degenerate) pseudoconvexity.

3.6.1 Second order elliptic operators

We start with a simple estimate for the Laplacian

Theorem 3.50 *Let $u \in H^2$ be supported in $\{|x| \leq \rho\}$ and away from 0. Then*

$$(\tau+1)^2 ||x|^{-\tau-1}u| \leq \rho^2 ||x|^{-\tau}\Delta u|^2 \quad \tau \in R \tag{3.86}$$

Proof : Since only a degenerate pseudoconvexity condition holds here, we need to carry out a more detailed computation than usual. Set $v = |x|^{-\tau-2}u$. Then (3.125) reduces to

$$||x|v|^2_{1,\tau|x|^{-1}} \leq c|P_\tau(x,D)v|^2 \quad \tau > \tau_0 \tag{3.87}$$

where P_τ is the conjugated operator, which in this case has the form

$$P_\tau(x,D) = (D_i - i\tau x_i|x|^{-1})^2 x^2$$

Split P_τ into its self-adjoint and its skew-adjoint part,

$$P_\tau^r = xD^2x - (\tau+1)^2$$

$$P_\tau^i = -i(\tau+1)(xD + Dx)$$

Compute

$$[P_\tau^r, P_\tau^i] \;=\; -i(\tau+1)xD^2x(xD+Dx) - (xD+Dx)xD^2x = 0$$

Then we have

$$|P_\tau(x,D)v|^2 = |P_\tau^r(x,D)v|^2 + |P_\tau^i(x,D)v|^2 \tag{3.88}$$

Now the L^2 estimate for v follows from the following simple computation

$$\begin{aligned} |(xD+Dx)v|^2 &= |(xD+Dx-2i|x|\rho^{-1})v|^2 - 4\rho^{-2}||x|v|^2 + 4\rho^{-1} < [xD+Dx, i|x|]v, v > \\ &= |(xD+Dx-2i|x|\rho^{-1})v|^2 - 4\rho^{-2}||x|v|^2 + 8\rho^{-1} < |x|v, v > \\ &\geq 4\rho^{-2}||x|v|^2 \end{aligned}$$

Remark 3.51 *A small change in the last estimate above allows on to obtain the following stronger inequality:*

$$(\tau+1)^2|\frac{1}{\log\rho - \log|x|}u|^2 \leq ||x|^{-\tau}\Delta u|^2 \quad \tau \in R, \quad \epsilon > 0 \tag{3.89}$$

Once we have obtained an estimate for u, we can easily take advantage of the elliptic part of the conjugated operator to estimate the gradient of u as well.

Corollary 3.6.1 *Under the same assumptions as in Theorem 3.59 the following estimate holds:*

$$\tau^2(||x|^{-\tau-1}u|^2 + ||x|^{-\tau}\nabla u|^2 \leq c\rho^2||x|^{-\tau}\Delta u|^2 \quad \tau \in R \tag{3.90}$$

Proof : With the same notations as in the proof of Theorem 3.59, we need to get the L^2 estimate for $\nabla x^2 v$. Compute

$$|\nabla x^2 v|^2 =< D^2 x^2 v, x^2 v >=< P_\tau^r v, x^2 v > + < [(\tau+1)^2 + n - 4]v, x^2 v >$$

therefore

$$|\nabla x^2 v|^2 \leq c(|P_\tau v||x^2 v| + \tau^2 ||x|v|^2) \leq c(\rho^2 |P_\tau v|^2 + \tau^2 ||x|v|^2)$$

which implies that

$$|\nabla x^2 v|^2 \leq c\rho^2 |P_\tau v|^2$$

q.e.d.

Based on (3.90) we obtain the following strong unique continuation result:

Theorem 3.52 *Let $x_0 \in R^n$ and let u be an H^2 function which satisfies the differential inequality*

$$|\Delta u(x)| \leq c(|u(x)| + |\nabla u(x)|) \tag{3.91}$$

near x_0. If u vanishes of infinite order at some $x_0 \in R^n$ then $u = 0$ near x_0.

Proof : Without any restriction in generality assume that $x_0 = 0$. If ρ is sufficiently small, then by cutting off u we reduce the problem to the case when u is supported in $B(0, 2\rho)$ and satisfies (3.91) in $B(0, \rho)$. Apply then (3.90) to u. We obtain

$$||x|^{-\tau} u|^2 + ||x|^{-\tau} \nabla u|^2 \leq c\rho^2 (||x|^{-\tau} u|^2 + ||x|^{-\tau} \nabla u|^2) + c\rho^{-\tau}$$

If ρ is sufficiently small this gives

$$|(\frac{\rho}{|x|})^\tau u|^2 + |(\frac{\rho}{|x|})^\tau \nabla u|^2 \leq c$$

which as $\tau \to \infty$ yields

$$u = 0 \quad \text{in} \quad B(0, \rho)$$

q.e.d.

One is tempted to infer that a similar result should hold in the variable coefficient case where $|x|$ is substituted by the Riemmanian distance. Unfortunately, this doesn't work in general; it appears that an additional condition on the curvature is required.

The obstacle is that the pseudoconvexity is degenerate for the function $\ln|x|$, and even a small perturbation may destroy it. However, we can fix that by adding a bit more convexity on ϕ.

Suppose that $\phi = \phi(-\ln|x|)$. Then let ψ be a function which satisfies

$$\psi'(y)e^{-2\psi(y))} = e^{-2(\phi(y)+y)}$$

Lemma 3.53 *Suppose that ϕ is increasing and convex. Then*

a) ψ is increasing and convex.

b) $\psi' > \phi' + 1$.

Proof : We prove (b) first. We have

$$e^{-2\psi(y)} = 2\int_y^\infty e^{-2(\phi(z)+z)}dz = \frac{1}{\phi'(y)+1}e^{-2(\phi(y)+y)} - \int_y^\infty \frac{\phi''(z)}{\phi'(z)^2}e^{-2(\phi(z)+z)}dz$$
$$\geq \frac{1}{\phi'(y)+1}e^{-2(\phi(y)+y)}$$

and (b) follows. For (a) compute

$$\psi'' = 2(\psi')^2 - 2\phi' e^{2(\psi-\phi(y)-y)} = 2\psi'(\psi'-\phi'-1)$$

Part (c) is straightforward.

Theorem 3.54 *Let ϕ be an increasing convex function, and ψ be as above. Then we have*

$$|(y^{-1}+\psi'')^{1/2}e^{2\psi(y)-\phi(y)}u|+ \leq |e^{\tau\phi}\Delta u| \tag{3.92}$$

(where $y = -\ln|x|$)

Proof : Set

$$v = e^{2\psi-\phi}u$$

and

$$P_\phi = e^{\phi}D^2e^{-2\psi+\phi}$$

To estimate $|P_\phi v|$ we split again P_τ into a selfadjoint and a skew-adjoint part. We have

$$\begin{aligned}
P_\phi &= e^{\phi-\psi}e^{\psi}D^2e^{-\psi}e^{\phi-\psi} \\
&= e^{\phi-\psi}D^2e^{\phi-\psi} - |\nabla\psi|^2e^{2(\phi-\psi)} + ie^{\phi-\psi}(\nabla\psi D + D\nabla\psi)e^{\phi-\psi} \\
&= e^{\psi-\phi}D^2e^{\psi-\phi} - (\psi')^2|x|^{-2}e^{2(\phi-\psi)} + ie^{\phi-\psi}(|x|^{-2}\psi' xD + Dx\psi'|x|^{-2})e^{\phi-\psi} \\
&= \frac{|x|}{\sqrt{\psi'}}D^2\frac{|x|}{\sqrt{\psi'}} - \psi' + i(xD+Dx)
\end{aligned}$$

Hence,

$$P_\phi^r = \frac{|x|}{\sqrt{\psi'}} D^2 \frac{|x|}{\sqrt{\psi'}} - \psi'$$
$$P_\phi^i = i\tau(xD + Dx)$$

Note that

$$ixD(\frac{|x|}{\sqrt{\psi'}}) = \frac{|x|}{\sqrt{\psi'}} - |x|(\frac{1}{\sqrt{\psi'}})' = \frac{|x|}{\sqrt{\psi'}}(1 + \frac{\psi''}{2\psi'})$$

Then we can compute the commutator

$$[P_\tau^r, P_\tau^i] = \frac{|x|}{\sqrt{\psi'}}(\frac{\psi''}{\psi'} D^2 + D^2 \frac{\psi''}{\psi'})\frac{|x|}{\sqrt{\psi'}} + 2\psi''$$

Hence,

$$\begin{aligned}
|P_\tau v|^2 &= |P_\tau^r v|^2 + |P_\tau^i v|^2 + < [P_\tau^r, P_\tau^i] w, w > \\
&= |(P_\tau^r + \frac{\psi''}{\psi'})v|^2 + |P_\tau^i v|^2 + 4\psi'' - (\frac{\psi''}{\psi'})^2
\end{aligned}$$

$$4\psi'' - (\frac{\psi''}{\psi'})^2 = \psi''(4 - 2(1 - (\phi' + 1)(\psi')^{-2} e^{2(\psi - \phi - y)}) = 2\psi''(1 + 4(\phi' + 1)(\psi')^{-1}) \geq 2\psi''$$

Thus,

$$2\psi'' e^{2(\psi - \phi)} = 2|x|^{-2}(\psi' - \phi' + 1)$$

Corollary 3.6.2 *Let ϕ be an increasing convex function, and ψ be as above. Then we have*

$$|(\tau^{-1/2} y^{-1} + (\psi')^{1/2}) e^{\tau(2\psi(y) - \phi(y))} u| + \leq |e^{\tau\phi} \Delta u| \tag{3.93}$$

Now we can tackle the variable coefficient case,

$$P(x, D) = D_i g^{ij} D_j$$

where the positive definite matrix g^{ij} is C^1.

Theorem 3.55 *Let ϕ be an increasing convex function, and ψ be as above. Assume that*

$$\phi'' > c\phi' e^{-y}$$

If ϵ is sufficiently small then we have

$$|(y^{-1} + \psi'')^{1/2} e^{2\psi(y) - \phi(y)} u| + \leq |e^{\tau\phi} P(\epsilon x, D) u| \tag{3.94}$$

Proof : The proof is easier if we use some special coordinates near the origin. Ideally, one would like to use the geodesic coordinates with respect to the origin. However, this is not possible for C^1 coefficients. What we can do, though, is to reduce the problem to the case when the balls $B(0,r)$ are the geodesic balls for the metric g. This can be easily achieved by multiplying P by the lipschitz function $|\nabla r|^2$. This leads to replacing g^{ij} by

$$\tilde{g}^{ij} = g^{ij}|\nabla_g r|^{-2}$$

Then

$$|\nabla_{\tilde{g}} r|^2 = 1$$

which concludes our reduction.

Set

$$v = e^{2\psi-\phi}u$$

and

$$P_\phi = e^{\phi}D^2 e^{-2\psi+\phi}$$

To estimate $|P_\phi v|$ we split again P_τ into a selfadjoint and a skew-adjoint part. We have

$$\begin{aligned} P_\phi &= e^{\phi-\psi}e^{\psi}P(x,D)e^{-\psi}e^{\phi-\psi} \\ &= e^{\phi-\psi}P(x,D)e^{\phi-\psi} - |\nabla\psi|^2 e^{2(\phi-\psi)} + ie^{\phi-\psi}(\nabla\psi\cdot D + D\cdot\nabla\psi)e^{\phi-\psi} \\ &= \frac{|x|}{\sqrt{\psi'}}P(x,D)\frac{|x|}{\sqrt{\psi'}} - \psi' + i(x\cdot D + D\cdot x) \end{aligned}$$

To accomodate C^1 coefficients, we need to break a bit the symmetry and set

$$P^r_\phi = \frac{|x|}{\sqrt{\psi'}}D_i g^{ij}D_j\frac{|x|}{\sqrt{\psi'}} - \psi' + (\partial_i g^{ij})x_j$$
$$P^i_\phi = ig^{ij}(x_iD_j + D_ix_j)$$

Note that P^r is still self-adjoint, but P^i is no longer skew adjoint. Again

$$ixD(\frac{|x|}{\sqrt{\psi'}}) = \frac{|x|}{\sqrt{\psi'}} - |x|(\frac{1}{\sqrt{\psi'}})' = \frac{|x|}{\sqrt{\psi'}}(1 + \frac{\psi''}{2\psi'}$$

but this time we need to compute $(P^i)^*P^r - P^rP^i$. This is quite similar to the constant coefficient case, except for the derivatives which fall on the metric g. We obtain

$$\begin{aligned} C &= P^rP^i - (P^i)^*P^r \\ &= \frac{|x|}{\sqrt{\psi'}}[(\frac{\psi''}{\psi'}P(x,D) + P(x,D)\frac{\psi''}{\psi'})\frac{|x|}{\sqrt{\psi'}} + 2\psi'' + \frac{|x|}{\sqrt{\psi'}}D_i(x_l\partial^j g^{il} \\ &+ x_l\partial^j g^{il} - x_l\partial^l g^{ij}D_j\frac{|x|}{\sqrt{\psi'}} \end{aligned}$$

Hence,

$$\begin{aligned} |P_\tau v|^2 &= |P_\tau^r v|^2 + |P_\tau^i v|^2 + < Cw, w > \\ &\geq |(P_\tau^r + \frac{\psi''}{\psi'})v|^2 + |P_\tau^i v|^2 + 2\psi''(1 + \frac{\phi'}{\psi'}) - c(||x|^{1/2}\nabla e^{\psi-\phi} v|^2 + ||x|^{-1/2} e^{\psi-\phi} v|^2 \end{aligned}$$

and further

$$|P_\tau v|^2 \geq |(P_\tau^r + \frac{\psi''}{\psi'})v|^2 + |P_\tau^i v|^2 + 2\psi''(1 + \frac{\phi'}{\psi'}) - c\epsilon(||x|^{3/2}(\psi')^{1/2}\nabla v|^2 + ||x|^{1/2}|\psi'|^{1/2} v|^2)$$

Suppose now that

$$\psi''(y) \geq c\psi'(y)e^{-y}$$

Then the negative RHS terms can be absorbed if ϵ is sufficiently small. This yields

$$c|P_\tau v|^2 \geq |(P_\tau^r + \frac{\psi''}{\psi'})v|^2 + |P_\tau^i v|^2 + 2\psi''(1 + \frac{\phi'}{\psi'})$$

q.e.d.

3.6.2 Parabolic operators

Consider the backwards heat operator,

$$P(D) = iD_t - D_x^2$$

Then

Theorem 3.56 *Let $u \in H^2$ be supported in a fixed neighbourhood of* 0*. Then*

$$|t^{-\tau} e^{-\frac{x^2}{8\tau}} u|_{1,\tau t^{-1}}^2 \leq c|t^{-\tau} e^{-\frac{x^2}{8\tau}} P(D)u|^2 \quad \tau > \tau_0 \tag{3.95}$$

Proof :

Proof : Since only a degenerate pseudoconvexity condition holds here, we need again to carry out a more detailed computation than usual. Set $v = t^{-\tau-1} e^{-\frac{x^2}{8t}} u$. Then (3.125) reduces to

$$|v|_{1,\tau t^{-1}}^2 \leq c|P_\tau(x, D)v|^2 \quad \tau > \tau_0 \tag{3.96}$$

where P_τ has the form

$$P_\tau(x, D) = [i(D_t - i\tau t^{-1} + \frac{x^2}{8t^2} - (D - i\tau\frac{x}{4t})^2]t$$

Split P_τ into its self-adjoint and its skew-adjoint part,

$$P^r_\tau = -tD_x^2 - \frac{x^2}{16t} + \tau + 1/2$$

$$P^i_\tau = i(\frac{tD_t + D_t t}{2} - \frac{(xD + Dx)}{4})$$

Then a simple computation gives

$$[P^r_\tau, P^i_\tau] = 0$$

Hence, we have

$$|P_\tau(x, D)v|^2 = |P^r_\tau(x, D)v|^2 + |P^i_\tau(x, D)v|^2$$

Now we estimate v as in the elliptic case, using

Lemma 3.57 *Suppose that v is supported in $B(0, \rho)$. Then*

$$\rho^{-1}||x|v| \leq |P^i_\tau v|$$

The proof is identical. The estimate on ∇v follows also as in the elliptic case.

The variable coefficient case can be studied using a modification of the above argument. Suppose $P(x, D)$ is an operator of the form

$$P(x, t, \partial) = \partial_t + \partial_i a^{ij}(x, t)\partial_j$$

where we assume that $P(0, 0, \partial) = \Delta$.

What we are looking for is an estimate of the following type

$$|e^{\tau\phi(x,\tau t)}u|_X < c|e^{\tau\phi(x,\tau t)}P(\epsilon x, \epsilon^2 t, \partial)u| \tag{3.97}$$

where ϕ is a modification of the weight in the constant coefficient case,

$$\phi_0(x, s) = -\ln s - \frac{x^2}{8s}$$

and X will be specified below.

Set

$$\ll (x,s) = fracx + s^{1/2}t$$

and introduce the following classes of functions

$$f \in \Phi^q \iff \quad |\partial_x^\alpha \partial_s^\beta f| \leq c_{\alpha,\beta} \ll^{q-|\alpha|-2\beta}), \quad 0 < |\alpha| + \beta, \quad q \in R$$

Note that $\phi_0 \in \Phi^0$.

Then we have

Theorem 3.58 *There exist $\phi \in \phi_0 + \Phi^{1/2}$ and $c, \tau_0 > 0$ so that (3.97) holds for $\tau > \tau_0$ provided that ϵ is small enough, with*

$$|v|_X^2 = \tau| \ll^{1/2} \nabla v|^2 + \tau^3| \ll^{3/2} v|^2$$

Proof : With the usual notation $v = e^{\tau\phi(x,\tau t)}u$ the estimate reduces to

$$|v|_X^2 \leq |P_\tau v|^2 \tag{3.98}$$

where the conjugated operator P_τ is

$$P_\tau = P_\tau^r + P_\tau^i + R$$

with

$$P_\tau^r = \partial_i a^{ij}(x,t)\partial_j + \tau^2(\phi_r^2 x_i a^{ij} x_j - \phi_s) + 2\tau\phi_r \tag{3.99}$$

$$P_\tau^i = \partial_t - a^{ij}(\phi_r x_i \partial_j + \partial_j x_i \phi_r) - 2\phi_r \tag{3.100}$$

and

$$R = \phi_r x_i a_j^{ij} \tag{3.101}$$

The last term R is negligible since it is controlled by the X norm. Hence, in the sequel we set it equal to 0. For the rest, we need the following

Lemma 3.59 *Assume that $\phi = \phi_0 + \psi$ with $\psi \in \Phi^{1/2}$. Then the following estimate holds:*

$$||P_\tau v|^2 - |P_\tau^r v|^2 - |P_\tau^i v|^2 - < 4\psi_{rr} x\partial v, x\partial v > -\tau^3 < fv, v > | \leq c(\epsilon + \tau^{-1})|v|_X$$

where

$$f = \frac{\psi}{4s^2} - \frac{4\psi_r}{s}$$

First we show how to conclude the proof of the Theorem using the Lemma. We need to make a choice for ψ. We claim that ψ can be chosen in Φ^1 so that

a) $\psi_{rr} > 0$

b) $f >\ll^3$

Such a choice is, for instance,

$$\psi(r,s) = \frac{(ds+r)^{3/2}}{s}$$

for sufficiently large d.

With this choice for ϕ the above lemma gives

$$\tau^3 |\ll^{3/2} v| + |P_\tau^r v|^2 + |P_\tau^i v|^2 \leq c(|P_\tau v| + (\epsilon\tau + 1)| \ll^{1/2} \nabla v|^2$$

Then the conclusion follows if we show that

$$| \ll^{1/2} \nabla v|^2 \leq c(\tau^2 | \ll^{3/2} v|^2 + \tau^{-2} | \ll^{-1/2} P_\tau^r v|^2)$$

Indeed,

$$| \ll^{1/2} \nabla v|^2 \leq <\ll a^{ij}\partial_i u, \partial_j u > \leq <\ll v, \partial_i a^{ij} \partial_j v > + | \ll^{1/2} \nabla v|| \ll^{3/2} v|$$

which gives

$$| \ll^{1/2} \nabla v|^2 \leq c(\tau^2 | \ll^{3/2} v|^2 + \tau^{-2} | \ll^{-1/2} \partial_i a^{ij} \partial_j v|^2) \leq c(\tau^2 | \ll^{3/2} v|^2 + \tau^{-2} | \ll^{-1/2} P_\tau^r v|^2)$$

q.e.d.

We conclude with

Proof of Lemma 3.59 : We have

$$|P_\tau v|^2 = |P_\tau^r v|^2 + |P_\tau^i v|^2 + 2\mathrm{Re}\ < P_\tau^r v, P_\tau^i v > = |P_\tau^r v|^2 + |P_\tau^i v|^2 + 2\mathrm{Re}\ < Rv, v >$$

where

$$R = P_\tau^r P_\tau^i - P_\tau^{i*} P_\tau^r$$

Hence, we need to estimate R. Recall that

$$P_\tau^r = \partial_i a^{ij}(x,t)\partial_j + \tau^2(\phi_r^2 x_i a^{ij} x_j - \phi_s) - 2\tau\phi_r \tag{3.102}$$

$$P_\tau^i = \partial_t - \tau a^{ij}(\phi_r x_i \partial_j + \partial_j x_i \phi_r) + 2\tau\phi_r \tag{3.103}$$

Then $R = R_1 + R_2 + R_3 + R_8$ where

$$R_3 = \tau^3(-\partial_s + 2\phi_r x_i a^{ij}\partial_j + 4\tau\phi_r)(\phi_r^2 x_i a^{ij} x_j - \phi_s)$$

$$R_2 = \tau^2(-8\phi_r^2 + 2(\partial_s - 2\phi_r x_i a^{ij}\partial_j)\phi_r = \tau^2(-8\phi_r^2 + 2\phi_{rs} - 2\phi_r\phi_{rr}x_i a^{ij} x_j)$$

$$R_1 = +2\tau(\phi_r\partial_i a^{ij}(x,t)\partial_j + \partial_i a^{ij}(x,t)\partial_j\phi_r) - \tau(\phi_r\partial_i a^{ik}a^{kj}\partial_j + \partial_i a^{ik}a^{kj}\partial_j\phi_r + 2\partial_i a^{ik}a^{kj}\partial_j)$$

$$R_0 = -\tau(x_j x_k\phi_{rr}\partial_i a^{ij}a^{kl}\partial_l\partial_i a^{ij}x_j x_k\phi_{rr}a^{kl}\partial_l\partial_i a^{ij}a^{kl}\partial_l x_j x_k\phi_{rr})$$

$$R_8 = \epsilon\tau(\nabla\Phi^{-1}O(1)\nabla + \Phi^{-1}\nabla O(1)\nabla\nabla O(1)\nabla\Phi^{-1}) = \epsilon\tau(\nabla O(\ll)\nabla + O(\ll^2)\nabla$$

Now we consider separately each term, taking into account our assumption on ϕ. For R_0 use the fact that ϕ_0 is linear in r, therefore $x_i x_j\phi_{rr} \in \Phi^{-1}$. Hence,

$$R_0 = -4\tau\partial_i a^{ij}x_j x_k\phi_{rr}a^{kl}\partial_l + \tau O(\ll^2)\nabla$$

For R_1 use the fact that $a^{ik}a^{kj} - a^{ij} = O(\epsilon x)$. Then

$$R_1 = \epsilon\tau(\nabla O(\ll)\nabla + O(\ll^2)\nabla) + O(\ll^2)\nabla$$

For R_2 observe first that it vanishes if $\phi = \phi_0$. The remainder consists of terms which contain at least one ψ therefore is one order higher. Hence,

$$R_2 \in \tau^2\Phi^{-3}$$

R_3 also vanishes if $\phi = \phi_0$. Then by the same token

$$R_3 = \text{ linearisation in } \psi \ + \tau^3\Phi^{-2}$$

The linearization in ψ is τ^3 times

$$-(\partial_s + \frac{r}{s}\partial_r + \frac{1}{s})(\frac{r}{s}\psi_r + \phi_s) - \frac{4}{s}\psi_r$$

Assuming that ψ is 1/2 homogeneous this gives

$$\frac{1}{4s^2}\psi - \frac{4}{s}\psi_r$$

therefore

$$R_3 = \tau^3(\frac{1}{4s^2}\psi - \frac{4}{s}\psi_r) + \tau^3\Phi^{-2}$$

Summing together the information above about each of the components of R we obtain the conclusion of the Lemma, q.e.d.

3.6.3 Notes

Eliptic:

Parabolic:

The symbolic calculus:

$$p^r_\tau = a^{ij}\xi_i\xi_j - \tau^2|\nabla\phi|^2$$

$$p^i_\tau = 2\tau a^{ij}\xi_i\phi_j$$

Their Poisson bracket is

$$\begin{aligned} \{p^r_\tau, p^i_\tau\} &= 4\tau\xi_i\phi^{ij}\xi_j + 2\tau^3\phi^i\partial_i|\nabla\phi|^2 && (3.104)\\ &= -2\tau p^r_\tau\phi^i|\nabla\phi|^{-2}\partial_i|\nabla\phi|^2 + 2\tau\xi_i(2\phi^{ij} - a^{ij}\phi^k|\nabla\phi|^{-2}\partial_k|\nabla\phi|^2)\xi_j && (3.105)\end{aligned}$$

3.7 Anisotropic operators

The Carleman estimates in Section 3.1 are valid as well for anisotropic operators, with the appropriate modifications of the meaning of the notations, as described in Section 1.3. The aim of this section is to describes some features specific to anisotropic operators.

3.7.1 Estimates with singular weight function

We start with the following result:

Theorem 3.60 *Assume that $\phi(t,x)$ is strongly pseudoconvex with respect to P. Let $f(t) \in C^\infty$, $f \geq 1$. Then*

$$\tau|e^{\tau f(\tau t)\phi}u|^2_{m-1,\tau} \leq c|e^{\tau f(\tau t)\phi}P(x,D)u|^2$$

for $\mu \geq 1$, $\tau > \tau_0$ and v with compact support.

Proof : As usual this follows from the stronger estimate

Lemma 3.61 *Under the assumptions of the theorem we have*

$$|v|^2_{m-1,\tau} \leq < P^r_\tau v, P^i_\tau v > + |P_\tau v|^2_{-1,\tau}$$

Proof : The above estimate is stable with respect to small C^1 changes in the coefficients. (the novelty here is that functions of the form $f(\tau t)$ map H^s_τ into itself, uniformly in τ.) Consequently, we assume in the sequel that the coefficients of P are smooth.

Theorem 3.62 *Assume that $\phi(t,x)$ is strongly pseudoconvex with respect to P. Let $g \geq 1$ be a function with the property that*

$$|g^{(j)}(x)| < c_j g^j(x), \qquad j = 1, m$$

Then

$$g\tau |e^{\tau g(t)\phi} u|^2_{m-1,\tau} \leq c|e^{\tau g(t)\phi} P(x,D)u|^2 \tag{3.106}$$

for $\tau > \tau_0$ and v with compact support. Furthermore, the constant c depends only on c_j.

Proof : W.a.r.g. assume that $c_1 = 1/2$. Let ψ be a smooth nonnegative function supported in $[1/2, 1]$ so that

$$\sum \psi(2^{-j}x) = 1$$

Set

$$\psi_j(x) = \psi(2^{-j}x), \quad \Psi_j(x) = \psi_{j-1}(x) + \psi_j(x) + \psi_{j+1}(x)$$

Now define

$$\ll_j = 2^j\tau$$

$$g_j(\ll_j t) = 2^{-j}\Psi_j(g(t))g(t) + (1 - \Psi_j(g(t)),$$

$$u_j = \psi_j(g(t))u$$

Then the following bounds are straightforward:

$$|g_j^{(k)}| < c\tau^{-j}$$

Consequently, if τ is sufficiently large we can apply Theorem 3.54 to the pair $\ll_j, g_j, u_j$ to get the uniform estimate

$$\ll_j |e^{\ll_j g_j(\ll_j t)\phi} u_j|^2_{m-1,\ll_j} \leq c|e^{\tau g_j(\ll_j t)\phi} P(x,D)u_j|^2$$

Observe, furthermore, that $2^j g_j(\ll_j t) = g(t) \in [2^{j-1}, 2^{j+1}]$ in supp u_j. Hence, we obtain

$$2^j\tau |e^{\tau g\phi} u_j|^2_{m-1,2^j\tau} \leq c|e^{\tau g\phi} P(x,D)u_j|^2 \tag{3.107}$$

On the other hand,

$$P(x,D)u_j = P(x,D)\psi_j u = \psi_j P(x,D)u + [P(x,D),\psi_j]\Psi_j u$$

therefore

$$|e^{\tau g\phi}P(x,D)u_j| \leq |e^{\tau g\phi}\psi_j P(x,D)u| + c2^j|e^{\tau g\phi}\Psi_j u|_{m-1,2^j\tau} \tag{3.108}$$

Combining (3.118) with (3.119) we obtain

$$2^j\tau|e^{\tau g\phi}\phi_j u|^2_{m-1,2^j\tau} \leq c(|e^{\tau g\phi}\phi_j P(x,D)u|^2 + |e^{\tau g\phi}\Phi_j u|^2_{m-1,2^j\tau} \tag{3.109}$$

For sufficiently large τ the summation in j gives

$$\tau\sum_j 2^j|e^{\tau g\phi}\phi_j u|^2_{m-1,2^j\tau} \leq c\sum_j |e^{\tau g\phi}\phi_j P(x,D)u|^2 \tag{3.110}$$

which implies (3.117), q.e.d.

The similar estimate with cutoff is also valid:

Theorem 3.63 *Assume that $\phi(t,x)$ is strongly pseudoconvex with respect to P. Let $g\geq 1$ be a function with the property that*

$$|g^{(j)}(x)| < c_j g^j(x), \qquad j = 1, m$$

Then

$$g\tau|\phi^2 e^{\tau g(t)\phi}u|^2_{m-1,\tau} \leq c|\phi^2 e^{\tau g(t)\phi}P(x,D)u|^2 + \tau^2|e^{\tau\phi(x)g(t)}u|^2_{m-2} \tag{3.111}$$

for $\tau > \tau_0$ and v with compact support. Furthermore, the constant c depends only on c_j.

3.7.2 The regularity of the coefficients

In the isotropic case, the minimal regularity of the coefficients required in order to obtain the Carleman estimates is C^1 (perhaps Lipschitz).

A natural question is what is the corresponding regularity of the coefficients in the anizotropic case.

The Carleman estimates follow from the stronger estimate

$$c|v|^2_{m-1,\tau} \leq < P^r_\tau v, P^i_\tau v > +d|P_\tau v|^2_{-1,\tau} \quad \tau > \tau_0$$

This estimate holds if the coefficients are smooth, and the constants c, d are stable with respect to smooth perturbations of the coefficients which are small in the C^1_x norm. However, τ_0 is not stable under such perturbations.

Let $X \subset C^1_x$ be a Banach space of coefficients so that C^∞ is dense in X and the estimate above is stable with respect to small perturbations of the coefficients.

For the second RHS term we need to know that

$$XH^{-1} \subset H^{-1} \tag{3.112}$$

For the first RHS term we use integration by parts, moving one derivative at a time alternatively from left to right and from right to left. The trouble is that the time derivative has order 2, therefore we can move it only half at a time. Consequently, in order to bound the first RHS term by the H^{m-1}_τ norm of v we need to know that

$$[X, D_x] : L^2 \to L^2 \qquad [X, D_t^{1/2}] : L^2 \to L^2 \tag{3.113}$$

Note that (3.123) follows from (3.124). The first part of (3.124) is equivalent to $X \subset C^1_x$. On the other hand, by Theorem 2.12 the second part of (3.124) holds if $D_t^{1/2} X \subset BMO$. This can be better understood if we look at the inclusion

$$C^{t^{1/2}|\log t|^{-1}} \subset BMO^{1/2} \subset C^{1/2}$$

Consequently, we obtain

Theorem 3.64 *The anisotropic Carleman estimates hold if the coefficients of the principal part of P are C^1 in x and $BMO^{1/2}$ in t.*

3.8 Notes

3.8.1 Degenerate weight functions

. One can ask whether the Carleman estimates are still valid if the weight function has gradient 0 at one point (see also 1.6.2). This question is largely irrelevant as far as unique continuation problems are concerned; however, it presents a certain interest when one is interested in having some nice global inequalities (as opposed to global inequalities obtained by patching local estimates).

Assuming that (1.28) holds one would like to have estimates of the form

$$\tau|e^{\tau\phi}u|^2_{m-1,\tau\nabla\phi} \leq c|e^{\tau\phi}P(x,D)u|^2 \quad \tau > \tau_0 \tag{3.114}$$

where we have done the appropriate modification in the weighted norm in the LHS term.

To keep things simple assume that $\nabla\phi(x_0) = 0$ and the Hessian $D^2_\phi(x_0)$ is nondegenerate. The inequality (3.114) would follow from the following analogue of (3.4):

$$B(v,v) = 2Im < P^r_\tau v, P^i_\tau v > +d|P_\tau v|^2_{-1,\tau|x-x_0|} \geq c|v|^2_{m-1,\tau|x-x_0|} \quad \tau > \tau_0 \tag{3.115}$$

This inequality is as usual stable with respect to small C^1 perturbations of the coefficients, therefore w.a.r.g. we can assume that P has smooth coefficients.

The principal symbol of F_P is

$$b(x,\xi,\tau) = \{\bar{p}(x,\xi - i\tau\nabla_x\phi), p(x,\xi + i\tau\nabla_x\phi)\}/\tau i + (\xi^2 + \tau^2 x^2)^{-1}|p_\tau(x,\xi + i\tau\nabla_x\phi)|$$

which is positive definite by (1.28), i.e.

$$b(x,\xi,\tau) > c(\xi^2 + \tau^2 x^2)^{m-2}$$

Now we would like to use Garding's inequality to get (3.115). A bit of care, though, is required in going from the symbol calculus to the estimates. Since we want to use symbols in $S((\xi^2 + \tau^2 x^2)^{m-2})$, Garding's inequality gives

$$B(v,v) \geq c|v|^2_{m-1,\tau|x-x_0|} - c_1\tau^{m-1}|v|^2$$

which further gives

$$B(v,v) \geq c|v|^2_{m-1,\tau|x-x_0|} - c_1\tau|v|^2_{m-2,\tau|x-x_0|} \tag{3.116}$$

Hence, instead of (3.114) we only get the weaker estimate

$$\tau|e^{\tau\phi}u|^2_{m-1,\tau\nabla\phi} \leq c(|e^{\tau\phi}P(x,D)u|^2 + \tau^{m-1}|e^{\tau\phi}u|^2) \quad \tau > \tau_0 \tag{3.117}$$

3.8.2 Products of operators

We discuss here Carleman estimates for operators $P(x,D)$ whose principal symbol $p(x,\xi)$ admits a factorization

$$p(x,\xi) = p^1(x,\xi)p^2(x,\xi)$$

Start with the relation

$$\{\overline{p_\tau}, p_\tau\} = \{\overline{p_\tau^1}, p_\tau^1\}|p_\tau^2|^2 + \{\overline{p_\tau^2}, p_\tau^2\}|p_\tau^1|^2 \qquad \text{on char } P_\tau$$

This implies that

Lemma 3.65 *a) A function ϕ is strongly pseudoconvex with respect to P iff it is strongly pseudoconvex with respect to P^1, P^2 and* char P_τ^1, char P_τ^2 *are disjoint.*

b) The same applies to strongly pseudoconvex surfaces.

The interesting question is what happens if a function ϕ is strongly pseudoconvex with respect to both P^1 and P^2, but char P_τ^1, char P_τ^2 are not disjoint. The answer is relatively simple if

$$P(x, D) = P_1(x, D)P_2(x, D)$$

Then one can apply succesively the Carleman estimates for P_1 and P_2 to obtain

$$\tau^2|e^{\tau\phi}u|_{m-2}^2 \leq c|e^{\tau\phi}Pu|^2 \tag{3.118}$$

This gets a little better if

$$\text{char } P_\tau^1 \cap \text{char } P_\tau^2 \subset \{\tau > 0\}$$

, when we get

$$|e^{\tau\phi}u|_{m-2}^2 \leq c|e^{\tau\phi}Pu|^2 \tag{3.119}$$

In the first case it is already clear from (3.118) that the subprincipal symbol of P should play a role in general. We contend ourselves with the following observation:

Theorem 3.66 *Assume that $\{p_1, p_2\} = 0$ on* char $P_1 \cap$ char P_2 *and that the subprincipal symbol of P vanishes on the same set. Then (3.118) holds provided that ϕ is strongly pseudoconvex with respect to both P_1 and P_2.*

In the second case the estimate is stable with respect to small perturbations of P of order $m - 1$. Hence, it is natural to conjecture that the conclusion holds regardless of the lower order terms. This leads to the natural question: can one modify the weight function ϕ without changing its level sets, in such a way that the (best) constant c in (3.119) is made arbitrarily large ?

First, note that the constant c in (3.1) can be (almost) taken, near some $(x, \xi, \tau) \in$ char P_τ, to be

$$c = \frac{\{\bar{p}(x_0, \xi - i\tau\nabla\phi), p(x_0, \xi + i\tau\nabla\phi)\}(x_0, \xi)}{i\tau|(\xi, \tau\nabla\phi)|^{2m-2}}$$

Can we improve it by substituting ϕ by $g(\phi)$? The new constant c we get at $(\xi, \tau g'(\phi)^{-1})$ is

$$c = \frac{\{\bar{p}(x_0, \xi - i\tau\nabla\phi), p(x_0, \xi + i\tau\nabla\phi)\}(x_0, \xi)}{i\tau|(\xi, \tau\nabla\phi)|^{2m-2}} + \frac{\{p(x_0, \xi + i\tau\nabla\phi), \phi\}^2 g''(\phi)}{|(\xi, \tau\nabla\phi)|^{2m-2} g'(\phi)}$$

Hence, it is clear that that c can be improved iff $\{p(x_0, \xi + i\tau\nabla\phi), \phi\} \neq 0$. All we need to do is to choose $\frac{g''}{g'}$ to be sufficiently large. This can be achieved e.g. by taking $g(x) = e^{\ll x}$ for sufficiently large $\ll$.

Consequently, we get

Theorem 3.67 *Suppose that ϕ is strongly pseudoconvex with respect to both P^1 and P^2. Assume in addition that* char $P^1_\tau \cap$char $P^2_\tau \subset \{\tau > 0\}$ *and* $\{p^1_\tau, \phi\}, \{p^2_\tau, \phi\}$ *do not simultaneously vanish there. Then*

$$\ll |e^{\tau\psi} u|^2_{m-1} \leq c |e^{\tau\psi} P u|^2 \tag{3.120}$$

where $\psi = e^{\ll}\phi$.

Chapter 4

Examples

4.1 The wave equation

Consider a second order hyperbolic operator

$$P(x,D) = \sum_{i,j=1}^{n} \partial_i a^{ij} \partial_j + b_j \partial_j + c$$

in a domain $K \subset R^n$, $n \geq 3$, with smooth time-like boundary ∂K. Assume that P is hyperbolic with respect to the level sets of one of the coordinates, say x_0. We call this coordinate time, and the others space coordinates. The principal symbol of the operator P is the quadratic form

$$p(x,\xi) = a^{ij}\xi_i\xi_j$$

of signature $(m-1,1)$.

4.1.1 An elementary proof of the Carleman estimates

The following result is a consequence of Theorem 3.1:

Theorem 4.1 *Assume that P has C^1 coefficients. Let ϕ be a strongly pseudoconvex function with respect to P in K. Then*

$$\tau|e^{\tau\phi}u|_{1,\tau}^2 \leq c|e^{\tau\phi}P(x,D)u|^2 \quad \tau > \tau_0 \tag{4.1}$$

whenever $u \in H^1$ is supported in K.

The aim of this section is to provide a simpler proof of it, more precisely, a proof which does not use pseudodifferential operators.

Proof : With the substitution $v = e^{\tau\phi}u$ (4.1) reduces as usual to

$$\tau|v|^2_{1,\tau} \leq c|P_\tau(x,D)v|^2 \quad \tau > \tau_0 \tag{4.2}$$

Now decompose P_τ into a part with real principal symbol and one with purely imaginary principal symbol,

$$P_\tau(x,D,\tau) = P^r_\tau(x,\partial,\tau) + 2\tau P^i_\tau(x,\partial) + R(x,\partial,\tau)$$

Here

$$P_1(x,\partial,\tau) = P(x,\partial) + \tau^2 p(x,\nabla\phi) = \partial_i a^{ij}\partial_j + \tau^2\phi_i a^{ij}\phi_j$$

with principal symbol

$$p_1(x,\xi,\tau) = p(x,\xi) - \tau^2 p(x,\nabla\phi) = \text{Re } p(x,\xi + i\tau\nabla\phi)$$

$$P_2(x,D) = \phi_i a^{ij}\partial_j$$

with principal symbol

$$p_2(x,\xi) = i\phi_i a^{ij}\xi_j = i(2\tau)^{-1}\text{Im } p(x,\xi + i\tau\nabla\phi)$$

The remainder R contains lower order terms and satisfies

$$|Rv| \leq c|v|_{1,\tau}$$

For τ large enough it does not affect (4.1) therefore we can neglect it in the sequel.

The inequality (4.2) is a consequence of the following

Proposition 4.2 *There exists a smooth function h such that for large enough τ,*

$$\text{Re } < P_\tau(x,\partial,\tau)v, (2P_2(x,\partial,\tau) + h(x))v > \geq |v|^2_{1,\tau} + \tau|P_2 v|^2) \tag{4.3}$$

whenever $u \in H^2$ is supported in K.

Proof : The inequality (4.3) reduces to

$$\begin{aligned} c|v|^2_{1,\tau} \leq B(v,v) &= d|P^i_\tau v|^2 + 2\text{Re } < P^r_\tau(x,\partial,\tau)v, P_2(x,\partial,\tau)v > \\ &+ \text{Re } < P^r_\tau(x,\partial,\tau)v, h(x))v > \end{aligned} \tag{4.4}$$

To prove this, we do a simple integration by parts. Neglecting the lower order terms (which we only get when we move a derivative in P^r_τ across h) we have

$$B(v,v) =< b^{ij}(x)\partial_i v, \partial_j v > + < b^i \partial_i v, \tau v > + b^0 < \tau v, \tau v >$$

(we keep τ inside the inner products to emphasize that it plays the same role as a derivative) The symbol of B is

$$b(x,\xi,\tau) = b^{ij}\xi_i\xi_j + b^i\xi_i\tau + b^0\tau^2$$

and is given by

$$b(x,\xi,\tau) = d|p^i_\tau|^2 + \frac{1}{i}\{p^r_\tau, \tilde{p}^i_\tau\} + p^r_\tau(h(x) + h_0(x))$$

where $h_0(x)$ is a C^0 function defined by

$$h_0 = P^i_\tau - (P^i_\tau)^*$$

In order to get (4.4) it suffices to choose d, h so that the symbol $b(x,\xi,\tau)$ is a positive definite quadratic form in ξ, τ.

The strong pseudoconvexity condition implies that

$$\frac{1}{i}\{p_1,p_2\}(x,\xi,\tau) > 0 \quad \text{whenever} \quad p_1(x,\xi,\tau) = \tau p_2(x,\xi,\tau) = 0, \quad \tau \geq 0, \quad (\xi,\tau) \neq 0 \quad (4.5)$$

The following algebraic Lemma shows the way we use this condition:

Lemma 4.3 *Assume that (4.5) above holds. Then there exists $d > 0$ and a smooth function h such that*

$$0 < \frac{1}{i}\{p_1,p_2\}(x,\xi,\tau) + d|p_2|^2(x,\xi,\tau) + h(x)p_1(x,\xi,\tau) \quad (4.6)$$

Proof : Note first that it suffices to prove (4.6) for fixed x; these local versions of (4.6) can then be put together using a partition of unit.

According to (4.5), if c is large enough then we have

$$q(x,\xi,\tau) = \frac{1}{i}\{p_1,p_2\}(x,\xi,\tau) + c|p_2|^2(x,\xi,\tau) > 0 \quad \text{whenever} \quad p_1(x,\xi,\tau) = 0, \quad (\xi,\tau) \neq 0$$

Look now at the zero set Z_λ for

$$q(x,\xi,\tau) + \lambda p_1(x,\xi,\tau)$$

If λ is small enough then Z_l is contained in $\{p_1 > 0\}$, while if λ is large enough then Z_l is contained in $\{p_1 < 0\}$. Then there are two posibilities.

a) There exists some λ such that $Z_\lambda = \emptyset$. Then the conclusion of the lemma follows.

b) There exists some λ such that Z_λ intersects both $\{p_1 < 0\}$ and $\{p_1 > 0\}$. Since Z_λ cannot intersect $\{p_1 = 0\}$, it follows that it is projectively disconnected. But this is impossible, for the zero set of a quadratic form in R^n is always projectively connected.

4.1.2 Unique continuation inside the domain

We start with Hormander's theorem:

Theorem 4.4 *Let*

$$P(x,D) = \sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j + b_j\partial_j + c \tag{4.7}$$

where a^{ij} are C^1, b^j are L^∞ and c is L^n. Let $\Sigma = \{\phi = 0\}$ be an oriented C^2 hypersurface, and $x_0 \in \Sigma$. Let $u \in H^1(K)$ solve $P(x,D)u = 0$ near x_0. If there exists a neighbourhood V of x_0 such that $u = 0$ in $V \cap \{\phi > 0\}$ then u vanishes in a neighbourhood of x_0.

Another version of this is

Theorem 4.5 *Let*

$$P(x,D) = \sum_{i,j=1}^{n} \partial_i a^{ij}\partial_j + \partial_j b^j + c \tag{4.8}$$

where a^{ij} are C^1, b^j are L^∞ and c is L^n. Let $\Sigma = \{\phi = 0\}$ be an oriented C^2 hypersurface which is strongly pseudoconvex with respect to P, and $x_0 \in \Sigma$. Let $u \in L^2(K)$ solve $P(x,D)u = 0$ near x_0. If there exists a neighbourhood V of x_0 such that $u = 0$ in $V \cap \{\phi > 0\}$ then u vanishes in a neighbourhood of x_0.

This follows from the Carleman estimate

Theorem 4.6 *Let P be as in the previous theorem. Then*

$$\tau|e^{\tau\phi}u|^2 \leq c|e^{\tau\phi}P(x,D)u|^2_{-1,\tau} \quad \tau > \tau_0 \tag{4.9}$$

whenever $u \in L^2$ has compact support in K.

Proof : The estimate is stable with respect to the lower order terms in P. Hence, w.a.r.g. we can assume that P has no lower order terms.

Now the argument in the proof of Theorem 3.2.1 allows one to shift down the energy level by one in (4.1) to get (4.2).

Yet another version of this result is

Theorem 4.7 *Let*

$$P(x,D) = \sum_{i,j=1}^{n} \partial_i \partial_j a^{ij} + \partial_j b^j + c \tag{4.10}$$

where a^{ij} are C^1, b^j are C^1 and c is $W^{1,n}$. Let $\Sigma = \{\phi = 0\}$ be an oriented C^2 hypersurface which is strongly pseudoconvex with respect to P, and $x_0 \in \Sigma$. Let $u \in H^{-1}(K)$ solve $P(x,D)u = 0$ near x_0. If there exists a neighbourhood V of x_0 such that $u = 0$ in $V \cap \{\phi > 0\}$ then u vanishes in a neighbourhood of x_0.

In all the above results we have limited ourselves to the case when the coefficients of the principal part of P are C^1. If instead we assume that the coefficients are in the Ξ_s space for some $s > 1$ then the range of the energy levels where the unique continuation result holds increases accordingly (see also Theorem 3.2.1).

The second type of unique continuation results we include here are those which follow from Theorem 3.32, i.e. when the coefficients of the principal part have some partial analiticity. To simplify the exposition, assume that we have a time coordinate, called t, so that its level sets $t = const$ are space-like. We denote the corresponding Fourier variable. We choose to write the operator P in the form (4.7). Corresponding results also hold if P is as in (4.8) or (4.10).

The first case we consider is when the coefficients are time-independent. Then we have

Theorem 4.8 *Assume that the coefficients of P are time independent and that a^{ij} are C^1, b^j are L^∞ and c is L^{n-1}. Let $\Sigma = \{\phi = 0\}$ be a noncharacteristic surface and $x_0 \in \Sigma$. Let $u \in H^1(K)$ solve $P(x,D)u = 0$ near x_0. If there exists a neighbourhood V of x_0 such that $u = 0$ in $V \cap \{\phi > 0\}$ then u vanishes in a neighbourhood of x_0.*

Theorem 3.44 implies that this holds if Σ is strongly pseudoconvex with respect to P in the set $s = 0$. By theorem 1.8 this reduces also to the case $\tau = 0$. But P_τ is elliptic in the region $\tau = s = 0$, q.e.d.

A second interesting case is when the coefficients are analytic in x. Then the following result holds:

Theorem 4.9 *Assume that the coefficients a^{ij}, b^j, c of P are analytic in x and C^0, respectively L^2, L^2 in t. Let $\Sigma = \{\phi = 0\}$ be a noncharacteristic time-like surface and $x_0 \in \Sigma$. Let $u \in H^1(K)$ solve $P(x,D)u = 0$ near x_0. If there exists a neighbourhood V of x_0 such that $u = 0$ in $V \cap \{\phi > 0\}$ then u vanishes in a neighbourhood of x_0.*

4.1.3 Continuation of regularity inside the domain

Again consider for example the second order hyperbolic operator in divergence form as in (4.8). Then

Theorem 4.10 *Assume that the coefficients a^{ij} are C^1, b^j are L^∞ and c is L^n. Let $\Sigma = \{\phi = 0\}$ be an oriented C^2 hypersurface which is strongly pseudoconvex with respect to P, and $x_0 \in \Sigma$. Let $u \in L^2(K)$ solve $P(x,D)u =\in L^2$ near x_0. If there exists a neighbourhood V of x_0 such that $u \in H^1_{loc}(V \cap \{\phi > 0\})$ then u is H^1 in a neighbourhood of x_0.*

Next we show a sample of the results one can get if the coefficients have better regularity.

Theorem 4.11 *Assume that the coefficients a^{ij} are Ξ_s. Let $\Sigma = \{\phi = 0\}$ be an oriented C^2 hypersurface which is strongly pseudoconvex with respect to P, and $x_0 \in \Sigma$. Let $u \in H^{-s}(K)$ solve $P(x,D)u =\in H^s$ near x_0. If there exists a neighbourhood V of x_0 such that $u \in H^{s+1}_{loc}(V \cap \{\phi > 0\})$ then u is H^{s+1} in a neighbourhood of x_0.*

4.1.4 Carleman estimates near the boundary

Let ψ be a smooth function vanishing simply on ∂K which is negative inside K. Denote by

$$\partial_\nu = p(\nabla\psi)^{-1/2}\ \psi_{x_i} a^{ij} \partial_j$$

the conormal derivative. The covector $N = p(\nabla\psi)^{-1/2}\nabla\psi$ lies in the conormal bundle of the boundary.

If coordinates are chosen near the boundary in such a way that $\partial K = \{x_n = 0\}$ and the principal symbol of P has the form

$$p(x,\xi) = \xi_n^2 - r(x,\xi')$$

then we can take $\psi(x) = x_n$ therefore $\partial_\nu = \partial_n$.

The Cauchy data of a function u with respect to P on the boundary is $(u_{|\partial K}, \partial_\nu u_{|\partial K})$. Consequently, Theorem 3.32 gives

$$\tau|e^{\tau\phi}u|_{1,\tau}^2 \leq c(|e^{\tau\phi}P(x,D)u|^2 + \tau|e^{\tau\phi}u|_{\partial,1,\tau}^2 + \tau|e^{\tau\phi}\partial_\nu u|_{\partial,\tau}^2) \quad \tau > \tau_0 \tag{4.11}$$

What if we only have some partial information on the Cauchy data ? Not everything is lost, provided that some Lopatinskii condition holds. To keep things simple, we just consider problems with Dirichlet and Neuman boundary condition.

In the case of the Dirichlet boundary condition it turns out that the strong Lopatinskii condition holds iff $\frac{\partial\phi}{\partial\nu} > 0$. Consequently, Theorem 3.22 gives

$$\tau(|e^{\tau\phi}u|_{1,\tau}^2 + |e^{\tau\phi}\partial_\nu u|_{\partial,\tau}^2) \leq c(|e^{\tau\phi}P(x,D)u|^2 + \tau|e^{\tau\phi}u|_{\partial,1,\tau}^2) \quad \tau > \tau_0 \tag{4.12}$$

For the Neuman boundary condition, on the other hand, there is no hope for the strong Lopatinkii condition to hold. It always fails when $\tau = 0$, $r(x,\xi') = 0$. However, the weak Lopatinskii condition holds when $\frac{\partial\phi}{\partial\nu} = 0$. Hence, by Theorem 3.27 we get

$$\tau|e^{\tau\phi}u|_{1,\tau}^2 \leq c|e^{\tau\phi}P(x,D)u|^2 \quad \tau > \tau_0 \tag{4.13}$$

whenever $\frac{\partial\phi}{\partial\nu} = 0$, $\frac{\partial u}{\partial\nu} = 0$.

We leave to the reader to verify that the above claims about the Lopatinskii condition are valid. Instead, we prove below the above estimates using the elementary proof of the Carleman estimates for the wave equation, given in 4.1.1. In the process, we shall obtain a slight improvement of these estimates.

In effect, the proof in 4.1.1 applies without any changes to all these estimates, except for one (crucial) point, which is the integration by parts in the quadratic form B. Hence, what we need to do is to carefully examine the boundary terms arising in this computation. It is convenient to represent P as

$$P = \partial_\nu^*\partial_\nu - R(x,\partial)$$

Then R is a tangential operator on the boundary. This leads to the decompositions:

$$P_\tau^r = \partial_\nu^*\partial_\nu + \tau^2(\partial_\nu\phi)^2 - R_\tau^r$$

$$P_\tau^i = -\phi_\nu\partial_\nu - R_\tau^i$$

What boundary terms do we get when we reduce 2Re $< P^r_\tau v, P^i_\tau v >$ to a first order quadratic form, integrating by parts ? Since R_τ is tangential to the boundary, we get contributions only from the terms involving conormal derivatives. If we take into account the simple formula

$$< \partial_\nu u, v > + < u, \partial_\nu^* v > = < u, v >_\partial$$

then we have

$$2\text{Re } < \partial_\nu^* \partial_\nu v, -\phi_\nu \partial_\nu v > = - < \phi_\nu \partial_\nu u, \partial_\nu u >_\partial + \text{ interior terms}$$

$$2\text{Re } < \partial_\nu^* \partial_\nu v, -R^i_\tau v > = - < \partial_\nu v, R^i_\tau v >_\partial + \text{ interior terms}$$

$$2\text{Re } < \tau^2(\partial_\nu\phi)^2 - R^r_\tau v, -\phi_\nu \partial_\nu v > = - < \tau^2(\partial_\nu\phi)^2 - R^r_\tau v, -\phi_\nu v >_\partial$$

In addition, when we move one derivative from P^r_τ onto h we get the contribution

$$< \partial_\nu v, hv >_\partial$$

Hence, integration by parts for the quadratic form B yields the boundary contribution

$$B_\partial(v,v) = - < \phi_\nu \partial_\nu v, \partial_\nu v >_\partial - < \partial_\nu v, R^i_\tau v >_\partial - < \tau^2(\partial_\nu\phi)^2 - R^r_\tau v, \phi_\nu v >_\partial + < \partial_\nu v, hv >_\partial \tag{4.14}$$

In general one can see that

$$B_\partial(v,v) > -c(|v|^2_{\partial,1,\tau} + |\partial_\nu v|^2_\partial)$$

which implies (4.11). If $\partial_\nu \phi < 0$ then we get

$$B_\partial(v,v) > c|\partial_\nu v|^2_\partial - d|v|^2_{\partial,1,\tau}$$

which gives (4.12). Finally, if $\partial_\nu\phi = 0$ and $\partial_\nu v = 0$ then

$$B_\partial = 0$$

which leads to (4.13).

We can obtain a small improvement of (4.11) in the particular case when $\partial_\nu u = 0$. This implies that $\partial_\nu v = \tau(\partial_\nu\phi)v$. Substituting this into B_∂ we get

$$B_\partial(v,v) \geq -c\tau^2|v|^2_\partial + < Rv, \phi_\nu v >_\partial$$

Hence, if we use the X^s_θ spaces associated to the operator R we get

$$B_\partial(v,v) \geq -c(\tau^2|v|^2_\partial + |v|^2_{X^{1/2}_{1/2}})$$

Consequently, the following variation of (4.11) holds:

$$\tau|e^{\tau\phi}u|^2_{1,\tau} \leq c(|e^{\tau\phi}P(x,D)u|^2 + \tau^3|e^{\tau\phi}u|^2_\partial + \tau|e^{\tau\phi}u|^2_{X^{1/2}_{1/2}}) \quad \tau > \tau_0 \tag{4.15}$$

if $\partial_\nu u = 0$. While this has no consequences as far as unique continuation is concerned, it is useful in problems involving regularity questions.

4.1.5 Unique continuation near the boundary

Neuman boundary condition From the Carleman estimate (4.13) it follows that UCP holds when Σ is strongly pseudoconvex with respect to P and and conormal to ∂K. However, it turns out that we have a bit more freedom:

Theorem 4.12 *Let $S = \{\phi = 0\}$ be an oriented C^2 hypersurface, and $x_0 \in S \cap \partial K$. Assume that $\dfrac{\partial\phi}{\partial\nu}(x_0) < 0$ and that $S \cap \partial K$ is strongly pseudoconvex with respect to R at x_0.*

$$\begin{cases} P(x,D)u = 0 & \text{in } K \\ \frac{\partial u}{\partial\nu} = 0 & \text{on } \partial K \end{cases} \tag{4.16}$$

If there exists a neighbourhood V of x_0 such that $u = 0$ in $V \cap \{\phi > 0\}$ then u vanishes in a neighbourhood of x_0.

Proof : The following Lemma achieves the reduction to the case when Σ is strongly pseudoconvex with respect to P and and conormal to ∂K.

Lemma 4.13 *Let $\Sigma = \{\phi = 0\}$ be an oriented surface and $x_0 \in S \cap \partial K$. Suppose that $\dfrac{\partial\phi}{\partial\nu}(x_0) < 0$ and that $\Sigma \cap \partial K$ is strongly pseudoconvex with respect to R at x_0.*

Then there exists another function ψ such that

(i) $\psi(x_0) = \phi(x_0)$, $\phi \geq \psi$ in K near x_0

(ii) $\dfrac{\partial\phi}{\partial\nu} = 0$ near x_0

(iii) the surface $\{\psi = 0\}$ is strongly pseudoconvex with respect to P at x_0.

Proof :

Let z be a function vanishing on ∂K so that $\frac{\partial z}{\partial \nu} = 1$ on ∂K.

Choose

$$\psi(x) = \phi(x) - z\frac{\partial \phi}{\partial \mu} + \lambda z^2$$

Clearly $\psi \leq \phi$ near x_0 and $\psi(x_0) = \phi(x_0)$. We claim that for large enough λ the pseudoconvexity condition is still fulfilled at x_0.

We have

$$\{p,\{p,\psi\}\}(x_0,\xi) = \{p,\{p,\phi\}\}(x_0,\xi) - \{p,\{p,z\}\}(x_0,\xi)\frac{\partial \phi}{\partial \mu} - \{p,z\}\{p,\frac{\partial \phi}{\partial \mu}\} + \lambda\{p,z\}^2$$

Since λ can be chosen arbitrarily large, to get the pseudoconvexity condition for ψ it suffices to require that ϕ satisfies

$$\{p,\{p,\phi\}\}(x_0,\xi) - \{p,\{p,z\}\}(x_0,\xi)\frac{\partial \phi}{\partial \mu} > 0$$

whenever

$$p(x_0,\xi) = \{p,\phi\}(x,\xi) = \{p,z\}(x,\xi) = 0$$

But this is equivalent to saying that $\Sigma \cap \partial K$ is strongly pseudoconvex with respect to R at x_0. q.e.d.

The corresponding result in the case of the Dirichlet boundary condition is a bit stronger:

Theorem 4.14 *Let $S = \{\phi = 0\}$ be an oriented C^2 hypersurface, and $x_0 \in S \cap \partial K$. Assume that*

(a) $\frac{\partial \phi}{\partial \nu}(x_0) < 0$

(b) There exists some $0 < \alpha < 1$ so that

$$\alpha\{r,\{r,\phi\} + (1-\alpha)\{p,\{p,\phi\} > 0 \quad \text{when } p = \{p,\phi\} = \{p,\psi\} = 0 \tag{4.17}$$

Let $u \in H^1$ solve

$$\begin{cases} P(x,D)u = 0 & \text{in } K \\ u = 0 & \text{on } \partial K \end{cases} \tag{4.18}$$

If there exists a neighbourhood V of x_0 such that $u = 0$ in $V \cap \{\phi > 0\}$ then u vanishes in a neighbourhood of x_0.

The result follows as in Theorem 4.12. The idea is that one tries to substitute the function ϕ by the function

$$\psi(x) = \phi(x) - \alpha\psi\frac{\partial\phi}{\partial\mu} + \lambda z^2$$

with $0 < \alpha < 1$ and λ arbitrarily large.

Remark 4.15 *Either (i) $S \cap \partial K$ is strongly pseudoconvex with respect to R at x_0 or (ii) S is strongly pseudoconvex with respect to P at x_0 along glancing rays.*

Namely, this says that we have unique continuation across S for solutions to (4.16) whenever the oriented "angle" between S and ∂K is less than $\pi/2$ (with respect to the pseudo-riemmanian metric generated by P) and $S \cap \partial K$ is strongly pseudoconvex with respect to $\tilde{P}$.

Another way one can think of the last condition is that ϕ is strongly pseudoconvex along the gliding vector field on $T^*\partial K$ associated to P. If $\frac{\partial\phi}{\partial\nu} = 0$ then this is the same with the strong pseudoconvexity of ϕ with respect to P on bicharacteristics of P which are tangent to the boundary.

However, if $\frac{\partial\phi}{\partial\nu} < 0$ then the interpretation of this condition depends on the convexity of the domain K. Let $\gamma \in T^*\partial K$ be a gliding point.

a) If ∂K is convex at γ, i.e. $\{p, \{p, z\}\}(\gamma) > 0$, then the condition (BSPC) is less restrictive than the strong pseudoconvexity of S at γ. Furthermore, in this case condition (BSPC) is clearly necessary since the singularities do propagate along gliding rays.

b) If ∂K is flat at γ, i.e. $\{p, \{p, z\}\}(\gamma) = 0$, then the condition (BSPC) is equivalent to the strong pseudoconvexity of S at γ. Again, the condition (BSPC) is necessary.

c) If ∂K is concave at γ, i.e. $\{p, \{p, z\}\}(\gamma) < 0$, then the condition (BSPC) is more restrictive than the strong pseudoconvexity of S at γ. However, in this case condition (BSPC) should not be necessary at least for the regularity result since the C^∞ singularities do not propagate along gliding rays. Nevertheless, analytic singularities do propagate along gliding rays, so (BSPC) might still be necessary for unique continuation.

4.1.6 Continuation of regularity near the boundary

The results in this section are based on the Carleman estimates which are similar to (4.11), (4.12), (4.13), (4.15) but in addition have a cutoff function inserted as in Theorem 3.10. An exercise we leave to the reader is to verify that these estimates can be obtained simply by substituting v by $\phi^2 v$ in (4.3) (or rather its analogue for boundary value problems) and performing a few computations.

No boundary condition Here we use the estimates

$$\tau|e^{\tau\phi}\phi_+^2 u|_{1,\tau}^2 \leq c(|e^{\tau\phi}\phi^2 P(x,D)u|^2 + \tau|e^{\tau\phi}u|_{\partial,1,\tau}^2 + \tau|e^{\tau\phi}\partial_\nu u|_{\partial,\tau}^2 + \tau|e^{\tau\phi}u|^2) \quad \tau > \tau_0 \quad (4.19)$$

respectively

$$\tau|e^{\tau\phi}\phi_+^2 u|_{1,\tau}^2 \leq c(|e^{\tau\phi}\phi^2 P(x,D)u|^2 + \tau|e^{\tau\phi}u|_{X_{1/2}^{1/2}}^2 + \tau^3|e^{\tau\phi}u|_{\partial,\tau}^2 + \tau|e^{\tau\phi}u|^2) \quad \tau > \tau_0 \quad (4.20)$$

when $\partial_\nu u = 0$.

Based on this, one can prove the following result on continuation of regularity:

Theorem 4.16 *Let $\Sigma = \{\phi = 0\}$ be an oriented C^2 hypersurface which is strongly pseudoconvex with respect to P, and $x_0 \in \Sigma$. a) Let $u \in L^2(K)$ such that near*

(i) $P(x,D)u = \in L^2$ near x_0.

(ii) $u \in H^1(\partial K)$ near x_0.

(iii) $\partial_\nu u \in L^2(\partial K)$ near x_0.

If there exists a neighbourhood V of x_0 such that $u \in H^1_{loc}(V \cap \{\phi > 0\})$ then u is H^1 in a neighbourhood of x_0.

b) The same conclusion holds if we substitute (i), (ii) by

(i)' $u \in X^{1/2}_{1/2}(\partial K)$ near x_0.

(ii)' $\partial_\nu u = 0$ near x_0.

The following consequence of the above result turns out to be useful in applications:

Theorem 4.17 *Let $x_0 \in \partial K$ so that ∂K is strongly pseudoconvex with respect to P at x_0. a) Let $u \in L^2(K)$ such that near*

(i) $P(x,D)u = \in L^2$ near x_0.

(ii) $u \in H^1(\partial K)$ near x_0.

(iii) $\partial_\nu u \in L^2(\partial K)$ near x_0.

Then u is H^1 in a neighbourhood of x_0.

b) The same conclusion holds if we substitute (i), (ii) by

(i)' $u \in X^{1/2}_{1/2}(\partial K)$ near x_0.

(ii)' $\partial_\nu u = 0$ near x_0.

4.1.7 Dirichlet boundary condition

Suppose we know u on ∂K, but have no information on the conormal derivative $\partial_\nu u$. The Carleman estimate with cutoff is in this case

$$\tau(|e^{\tau\phi}\phi^2 u|^2_{1,\tau}+|e^{\tau\phi}\phi^2\partial_\nu u|^2_{\partial,\tau}) \leq c(|e^{\tau\phi}\phi^2 P(x,D)u|^2+\tau|e^{\tau\phi}\phi^2 u|^2_{\partial,1,\tau}+\tau^2|e^{\tau\phi}u|^2) \quad \tau > \tau_0 \tag{4.21}$$

provided that $\frac{\partial\phi}{\partial\nu} > 0$.

Consequently, we obtain

Theorem 4.18 *Let $S = \{\phi = 0\}$ be an oriented C^2 hypersurface, and $x_0 \in S \cap \partial K$. Assume that*

(a) $\dfrac{\partial\phi}{\partial\nu}(x_0) < 0$

(b) There exists some $0 < \alpha < 1$ so that

$$\alpha\{r,\{r,\phi\} + (1-\alpha)\{p,\{p,\phi\} > 0 \qquad \text{when } p = \{p,\phi\} = \{p,\psi\} = 0 \tag{4.22}$$

Let $u \in L^2$ such that

(i) $P(x,D)u =\in L^2$ near x_0.

(ii) $u \in H^1(\partial K)$ near x_0.

(iii) $\partial_\nu u \in L^2(\partial K)$ near x_0.

If there exists a neighbourhood V of x_0 such that $u \in H^1_{loc}(V \cap \{\phi > 0\})$ then u is H^1 and $\partial_\nu u$ is $L^2(\partial K)$ in a neighbourhood of x_0.

4.1.8 Neuman boundary condition

The Carleman estimate with cutoff is

$$\tau|e^{\tau\phi}\phi^2 u|^2_{1,\tau} \leq c(|e^{\tau\phi}\phi^2 P(x,D)u|^2 + \tau^2|e^{\tau\phi}u|^2) \quad \tau > \tau_0 \tag{4.23}$$

whenever $\frac{\partial\phi}{\partial\nu} = 0$, $\frac{\partial u}{\partial\nu} = 0$.

4.2 The Schroedinger equation

To keep things simple we consider the Schroedinger operator with the principal part in divergence form and C^1 coefficients in the principal part. Thus, let

$$P(x,\partial) = i\partial_t - \sum_{i,j=1}^{n} \partial_i a^{ij}\partial_j + b^i\partial_i + c \tag{4.24}$$

4.2.1 Pseudoconvex surfaces

A surface $\Sigma = \{\phi = 0\}$ is called noncharacteristic iff $\phi_x \neq 0$. A noncharacteristic surface intersects the hyperplanes $t = const.$ on smooth $n-$dimensional surfaces.

The strong pseudoconvexity condition (for noncharacteristic surfaces) is uncoupled on the hyperplanes $t = const.$ We have

$$p(t, x, s, \xi) = s + a(t, x, \xi)$$

Hence

$$p_\tau(t, x, s, \xi) = s + a(t, x, \xi + i\tau\nabla\phi)$$

and

$$p^r_\tau = s + a^r_\tau = s + a(t, x, \xi) - \tau^2 a(t, x, \nabla\phi)$$
$$p^i_\tau = a^i_\tau = 2\tau a(x, t)(\xi, \nabla\phi)$$

Consequently,

$$\{p^r_\tau, p^i_\tau\} = \{a^r_\tau, a^i_\tau\}$$

By Theorem 1.8, the strong pseudoconvexity condition for a noncharacteristic surface reduces to the case $\tau = 0$, in which case it reads

$$\{a, \{a, \phi\}\} > 0 \quad \text{when } s + a = \{a, \phi\} = 0$$

which can be rewritten as

$$\{a, \{a, \phi\}\} > 0 \quad \text{when } \{a, \phi\} = 0 \tag{4.25}$$

Similarly, the strong pseudoconvexity condition for a function ϕ reduces to

$$\{a, \{a, \phi\}\} > 0, \{\{a, \phi\}, a(\nabla\phi)\} > 0 \quad \text{when } \{a, \phi\} = 0 \tag{4.26}$$

4.2.2 The Carleman estimates

The simplest Carleman estimates for the Schroedinger equation have the form

Theorem 4.19 *Let ϕ be a strongly pseudoconvex function with respect to P in K. Then*

$$\tau|e^{\tau\phi}u|^2_{1,\tau} \leq c|e^{\tau\phi}P(x, D)u|^2 \quad \tau > \tau_0 \tag{4.27}$$

whenever $u \in H^1$ is supported in K.

Note that the $|.|_{1,\tau}$ norm means one derivative in x, one "derivative" in τ but only 1/2 derivative in t. While this result is a consequence of Theorem 3.1, we again take advantage of the simple form of the operator to give a simpler proof. This time, however, since we do not want to use microlocal analysis, we need to make the additional assumption that the coefficients are C^2.

With the substitution $v = e^{\tau\phi}u$ (4.27) to

$$\tau|v|_{1,\tau}^2 \leq c|P_\tau(x, D)v|^2 \quad \tau > \tau_0 \tag{4.28}$$

Neglecting the lower order terms we can decompose P_τ as usual into a part with real principal symbol and one with purely imaginary principal symbol,

$$P_\tau(x, D, \tau) = P_\tau^r(x, \partial, \tau) + 2\tau P_\tau^i(x, \partial)$$

Here

$$P_\tau^r(x, \partial, \tau) = i\partial_t - A(t, x, \partial_x) - \tau^2 a(t, x, \nabla\phi) = i\partial_t - \partial_i a^{ij}\partial_j - \tau^2\phi_i a^{ij}\phi_j$$

with principal symbol

$$p_\tau^r(x, \xi, \tau) = s + a(t, x, \xi) - \tau^2 a(t, x, \nabla_x\phi) = s + \text{Re } a(x, \xi + i\tau\nabla\phi)$$

$$P_\tau^i(x, D) = \phi_i a^{ij}\partial_j + \partial_i a^{ij}\phi_j$$

with principal symbol

$$p_\tau^i(x, \xi) = i\phi_i a^{ij}\xi_j = i(2\tau)^{-1}\text{Im } a(x, \xi + i\tau\nabla\phi)$$

The inequality (4.27) is a consequence of the following

Proposition 4.20 *For large enough τ,*

$$\text{Re } < P_\tau(x, \partial, \tau)v, 2P_2(x, \partial, \tau)v > \geq |(\tau, \partial_x)v|^2 + \tau|P_2v|^2) \tag{4.29}$$

whenever $u \in H^2$ is supported in K.

Proof : The inequality (4.29) reduces to

$$c|(\tau, \partial_x)v|^2 \leq B(v, v) = d|P_\tau^i v|^2 + 2\text{Re } < P_\tau^r(x, \partial, \tau)v, P_2(x, \partial, \tau)v > \tag{4.30}$$

To prove this, we do a simple integration by parts. Modulo lower order terms we get

$$B(v,v) =< b^{ij}(x)\partial_i v, \partial_j v > + < b^i \partial_i v, \tau v > + b^0 < \tau v, \tau v >$$

The symbol of B is

$$b(x,\xi,\tau) = b^{ij}\xi_i\xi_j + b^i\xi_i\tau + b^0\tau^2$$

and is given by

$$b(x,\xi,\tau) = d|a_\tau^i|^2 + \frac{1}{i}\{a_\tau^r, a_\tau^i\}$$

In order to get (4.30) it suffices to choose d so that the symbol $b(x,\xi,\tau)$ is a positive definite quadratic form in ξ,τ. The strong pseudoconvexity condition for ϕ (4.26) insures that this can be done.

The estimate (4.19), however, does not take advantage of the anisotropic structure of the operator P. An additional anisotropic feature is that the weight function can be chosen to be a cutoff function in time. Consequently, the corresponding Carleman estimates are localized in time.

To give an example, let $\phi(x)$ be a negative weight function which is strongly pseudoconvex with respect to P at time $t=0$. Then choose ϵ small and

$$g(t) = \frac{1}{1 - t^2\epsilon^{-2}}$$

Then Theorem 3.56 implies that the following estimate holds

$$\tau|e^{\tau\phi(x)g(t)}u|_{1,\tau}^2 \leq c|e^{\tau\phi(x)g(t)}P(x,D)u|^2 \quad \tau > \tau_0 \tag{4.31}$$

whenever $u \in H^1(K)$.

If ϕ above is allowed to be positive then the weight function could blow up. For instance, the following estimate holds:

$$\tau|e^{\frac{\tau\phi}{|t|}}u|_{1,\tau}^2 \leq c|e^{\frac{\tau\phi}{|t|}}P(x,D)u|^2 \quad \tau > \tau_0 \tag{4.32}$$

The important feature of this estimate is that it holds for all $u \in H^1$ with compact support in K for which the RHS is finite. This in turns implies that the LHS is finite and in particular that u vanishes when $t=0$, $\phi > 0$.

Introducing a cutoff function in this estimate, as in Theorem 3.57, yields

$$\tau|\phi^2 e^{\tau\phi(x)g(t)}u|_{1,\tau}^2 \leq c(|\phi^2 e^{\tau\phi(x)g(t)}P(x,D)u|^2 + \tau^2|e^{\tau\phi(x)g(t)}u|^2 \quad \tau > \tau_0 \tag{4.33}$$

Assume now that K is a domain with boundary. For simplicity take K to be a cilynder $K = \Omega \times [-\epsilon, \epsilon]$ with boundary $\partial K = \times[-\epsilon, \epsilon]$.

Then the following estimates hold for solutions to boundary value problems:

a) No boundary conditions, no restriction on ϕ:

$$\tau|e^{\tau\phi(x)g(t)}u|_{1,\tau}^2 \leq c(|e^{\tau\phi(x)g(t)}P(x,D)u|^2 + \tau|e^{\tau\phi(x)g(t)}u|_{\partial,1,\tau}^2 + \tau|e^{\tau\phi(x)g(t)}\partial_\nu u|_\partial^2) \quad \tau > \tau_0 \tag{4.34}$$

b) No boundary conditions, $\partial_\nu\phi < 0$:

$$\tau|e^{\tau\phi(x)g(t)}u|_{1,\tau}^2 + \tau|e^{\tau\phi(x)g(t)}\partial_\nu u|_\partial^2 \leq c(|e^{\tau\phi(x)g(t)}P(x,D)u|^2 + \tau|e^{\tau\phi(x)g(t)}u|_{\partial,1,\tau}^2) \quad \tau > \tau_0 \tag{4.35}$$

c) Neuman boundary condition $\partial_\nu u = 0$, no restriction on ϕ:

$$\tau|e^{\tau\phi(x)g(t)}u|_{1,\tau}^2 \leq c(|e^{\tau\phi(x)g(t)}P(x,D)u|^2 + \tau|e^{\tau\phi(x)g(t)}u|_{X_{1/2}^{1/2}}^2 + \tau^3|e^{\tau\phi(x)g(t)}u|_{\partial,1,\tau}^2) \quad \tau > \tau_0 \tag{4.36}$$

d) Neuman boundary condition $\partial_\nu u = 0$, $\partial_\nu\phi < 0$:

$$\tau|e^{\tau\phi(x)g(t)}u|_{1,\tau}^2 \leq c|e^{\tau\phi(x)g(t)}P(x,D)u|^2 \tag{4.37}$$

e) All these estimates remain valid if one inserts the cutoff function as in (4.31).

4.2.3 Unique continuation

The classical form of the unique continuation result is:

Theorem 4.21 *a) Assume that $\Sigma \subset K$ is strongly pseudoconvex with respect to P. Then UCP holds for H^1 solutions u to $P(x,D)u = 0$.*

b) Assume in addition that $b^i \in W^{1,n}$. Then UCP holds for L^2 solutions u to $P(x,D)u = 0$.

The anisotropic version of the unique continuation result is localized at time $t = 0$:

Theorem 4.22 *a) Assume that $\Sigma \subset K \cap \{t = 0\}$ is strongly pseudoconvex with respect to P. Then AUCP holds for H^1 solutions u to $P(x, D)u = 0$.*

b) Assume in addition that $b^i \in W^{1,n}$. Then AUCP holds for L^2 solutions u to $P(x, D)u = 0$.

If the coefficients have some partial analyticity then the results are better. The following example is perhaps the most important:

Theorem 4.23 *Assume that the coefficients of P are time-independent. Let $\Sigma \subset K \cap \{t = 0\}$ be noncharacteristic.*

a)Then UCP holds for H^1 solutions u to $P(x, D)u = 0$.

b) Assume in addition that $b^i \in W^{1,n}$. Then UCP holds for L^2 solutions u to $P(x, D)u = 0$.

The above results lead one to the following

Conjecture 4.2.1 *Assume that the coefficients of P are time-independent. Then a weak anisotropic SUCP holds, i.e. if u solves $Pu = 0$ and vanishes of sufficiently high order at some (x_0, t_0) then $u = 0$ at $t = t_0$.*

Clearly it is insufficient to assume only that u vanishes of infinite order at a point. For instance if one considers the Cauchy problem for the Schroedinger equation with initial data at $t = 0$ supported away from $x = 0$ and as smooth as possible then the corresponding solutions could vanish near 0 of order $e^{-|t|^{-\alpha}}$ for any $\alpha < 1$. One is thus led to a minimal assumption that u vanishes at least like $e^{1/(|t|+x^2)}$ at $(0, 0)$ in order to hope to get SUCP.

Now look at boundary value problems. Suppose $K = \Omega \times [-\epsilon, \epsilon]$. Then $\partial K = \times[-\epsilon, \epsilon]$. The main result for the Neuman problem is

Theorem 4.24 *Let Σ be an oriented noncharacteristic surface.*

a) Assume that $S \cap \partial K$ is strongly pseudoconvex with respect to R (on ∂K) and that $\partial_\nu \phi < 0$. Then AUCP holds for H^1 solutions u to $P(x, D)u = 0$, $\partial_\nu u = 0$.

b) Assume in addition that $b^i \in W^{1,n}$. Then AUCP holds for L^2 solutions u to $P(x, D)u = 0$, $\partial_\nu u = 0$.

For the Dirichlet problem we have

Theorem 4.25 *Let Σ be an oriented noncharacteristic surface.*

a) Assume that $S \cap \partial K$ is strongly pseudoconvex with respect to R (on ∂K) and that $\partial_\nu \phi < 0$. Then AUCP holds for H^1 solutions u to $P(x, D)u = 0$, $\partial_\nu u = 0$.

b) Assume in addition that $b^i \in W^{1,n}$. Then AUCP holds for L^2 solutions u to $P(x, D)u = 0$, $\partial_\nu u = 0$.

4.2.4 Continuation of regularity

The result inside the domain is

Theorem 4.26 *Assume that the coefficients a^{ij} are C^1, b^j are $L^\infty \cap W^{1,n}$ and c is L^n. Let $\Sigma = \{\phi = 0\}$ be an oriented C^2 hypersurface which is strongly pseudoconvex with respect to P, and $x_0 \in \Sigma$. Let $u \in L^2(K)$ solve $P(x, D)u = \in L^2$ near x_0. If there exists a neighbourhood V of x_0 such that $u \in H^1_{loc}(V \cap \{\phi > 0\})$ then u is H^1 in a neighbourhood of x_0.*

4.3 The plate equation

Consider the plate operator

$$P = \partial_t^2 - (\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j)^2 + l.o.t.$$

with C^1 coefficients in a set $K \subset R^n \times R$ so that the boundary dk is not "time-like", i.e. $dt \not\in N(\partial K)$. For most applications, the case when K is a cylinder $K = \Omega \times R$ is sufficient.

4.3.1 Carleman estimates

The principal symbol of P is

$$p(t, x, s, \xi) = -s^2 + l^2(t, x, \xi))$$

We can factor it as

$$p(t, x, s, \xi) = -(s + l(t, x, \xi))(s - l(t, x, \xi)) = -p_1 p_2$$

The trouble is that for any function ϕ the symbols of the two conjugated operators

$$p_\tau^1 = s + l(t, x, \xi + i\tau\nabla_x\phi), \quad p_\tau^2 = s - l(t, x, \xi + i\tau\nabla_x\phi)$$

can vanish simultaneously. The good news is that this can only happen when $s = 0$ and $\tau > 0$. Furthermore, in that region we have

$$\{p^1_\tau, \phi\} \neq 0$$

Consequently, Theorem 3.67 implies the Carleman estimate

$$|e^{\tau\phi}u|^2_{3,\tau} \leq c|e^{\tau\phi}Pu|^2 \tag{4.38}$$

where the constant c can be made arbitrarily small by appropriately modifying ϕ.

The plate equation is also anisotropic, therefore we can use weight functions which are cutoff in time:

$$|e^{\tau\phi(x)g(t)}u|^2_{3,\tau} \leq c|e^{\tau\phi(x)g(t)}Pu|^2 \tag{4.39}$$

Since the corresponding estimates with cutoff function are used only to study regularity questions, we neglect altogether what happens in the region $\tau > 0$. When $\tau = 0$ there are no multiple characteristics, therefore we get

$$\tau|\phi^2 e^{\tau\phi}u|^2_{3,\tau} \leq c(|\phi^2 e^{\tau\phi}Pu|^2 + \tau^3|u|^2_{2,\tau}) \tag{4.40}$$

Next we discuss the types of boundary condition which satisfy the strong Lopatinskii condition.

A. $\partial_\nu\phi \geq 0$. Then p_0 could have degree 4 therefore in order for the strong Lopatinskii condition to be fulfilled one needs to use all the Cauchy data on the boundary.

Things get a little better when we are only concerned with the regularity issue. Then the strong Lopatinskii condition needs to be fulfilled only when $\tau = 0$.

If $\tau = 0$ then we have

$$\tau_{1,2,3,4} = \pm(i|\tilde{\xi}'|^2 + |s|)^{1/2}, \quad \pm(-|\tilde{\xi}'|^2 + |s|)^{1/2}, \tag{4.41}$$

Hence

$$p^0 = \begin{cases} (\xi_n - \tau_1)(|\tilde{\xi}|^2 - |s|) & if\ |s| \geq |\tilde{\xi}'|^2 \\ (\xi_n - \tau_1)(\xi_n - \tau_3) & if\ |s| < |\tilde{\xi}'|^2 \end{cases}$$

Consequently, at least three boundary conditions are necessary. A simple computation shows that

Theorem 4.27 *any three of the four standard boundary conditions*

$$B_{0,1,2,3}u = u,\ \frac{\partial u}{\partial \nu},\ \Delta u,\ \frac{\partial \Delta u}{\partial \nu}$$

will do. Other boundary conditions are left to the reader.

B. $\partial_\nu \phi > 0$.

Then p_0 has degree at most 2.

On the boundary consider several types of boundary conditions.

a) $B_1 = I,\ B_2 = \partial_n u$.

b) $B_1 = I,\ B_2 = L$

c) $B_1 = I,\ B_2 = \partial_\nu L$

d) $B_1 = \partial_\nu,\ B_2 = L$

where $L = \sum_{i,j=1}^n a^{ij}\partial_i \partial_j$.

Then we have

Theorem 4.28 *Consider the plate equation in K with boundary conditions as in (a),(b),(c) or (d) on ∂K. Then the strong Lopatinskii condition with respect to $d\phi$ is fulfilled at some $x \in \partial K$ iff $\partial_\nu \phi < 0$.*

Other types of boundary conditions are left for the interested reader.

Proof : Since we have only two boundary conditions, for the SLC to be fulfilled it is necessary for p_0 to have degree at most 2. Then the necessity of the condition $\partial_\nu \phi < 0$ follows as in the similar proof for the wave equation.

For the sufficiency, we proceed in a similar manner. We can find local coordinates near the boundary so that $\partial K = \{x_n = 0\}$ and $l(x,\xi) = \xi_n^2 + r(x,\xi')$ where r is a second order elliptic tangential symbol.

A. The case $\tau = 0$. The symbol of P in the new coordinates is

$$p(x,\xi) = s^2 - (\xi_n^2 + r)^2$$

where s is the time dual variable.

Since p_0 has degree at most 2, it divides a polynomial of the form

$$p_1(x,\xi) = (\xi_n - \alpha)(\xi_n - \beta)$$

where α, β are the two roots of $p(x, \xi)$, as a polynomial in ξ_n, with positive imaginary part if $|s| < r$, and one real and one imaginary root of p otherwise. To set the notations, assume that α is a root for $\xi_n^2 + |s| + r(x, \xi')$ and β for $\xi_n^2 - |s| + r(x, \xi')$.

It suffices to prove that b_1, b_2 are complete modulo p_1. This is straightforward in case (a). In case (b) this is equivalent to $\alpha + \beta \neq 0$ which is clear since $\alpha + \beta$ has positive imaginary part. In case (c), the same condition leads to $r - \alpha\beta + (\alpha + \beta)^2 \neq 0$. But $r = r/2 + r/2 = -(\alpha^2 + |s|)/2 - (\beta^2 - |s|)/2$ therefore this is is equivalent to $(\alpha + \beta)^2 \neq 0$ which is clear from case (b). Finally, in case (d) we are led again to the condition $(\alpha+\beta)^2 \neq 0$. The same arguments apply in the case $\tau > 0$, q.e.d.

C. The weak Lopatinskii condition. As usual, if $\partial_\nu \phi = 0$ then some weak Lopatinskii condition could hold. As it turns out, this happens for instance for the boundary condition (B_0, B_2).

If we restrict ourselves to $\tau = 0$ then we also get the boundary condition (B_0, B_1).

4.3.2 Unique continuation and continuation of regularity

Same results as usual.

4.4 Parabolic equations

4.5 Coupled hyperbolic equations

Let P_1, P_2 be two second order hyperbolic operators, and Q_1, Q_2 be first order operators. We now want to investigate the unique continuation problem for the weakly coupled system

$$\begin{cases} P_1(x, D)u_1 = Q_1 u_2 \\ P_2(x, D)u_2 = Q_2 u_1 \end{cases} \tag{4.42}$$

Start with the Carleman estimates for the corresponding inhomogeneous system

$$\begin{cases} P_1(x, D)u_1 = Q_1 u_2 + f_1 \\ P_2(x, D)u_2 = Q_2 u_1 + f_2 \end{cases} \tag{4.43}$$

Suppose ϕ is a strongly pseudoconvex function with respect to both P^1 and P^2. Then the Carleman estimates applied to each of the equations yield

$$\tau |e^{\tau\phi} u_1|_{1,\tau}^2 \leq c(|e^{\tau\phi} f_1|^2 + |u_2|_{1,\tau}^2)$$

$$\tau|e^{\tau\phi}u_2|^2_{1,\tau} \leq c(|e^{\tau\phi}f_2|^2 + |u_1|^2_{1,\tau})$$

Summing up these relations yields, for sufficiently large τ,

$$\tau|e^{\tau\phi}u_1|^2_{1,\tau} + \tau|e^{\tau\phi}u_2|^2_{1,\tau} \leq c(|e^{\tau\phi}f_1|^2 + |e^{\tau\phi}f_2|^2)$$

This leads to the following unique continuation result:

Theorem 4.29 *Let Σ be an oriented surface which is strongly pseudoconvex with respect to both P_1 and P_2. Then UCP holds for H^1 solutions u to (4.42).*

The corresponding result on continuation of regularity holds as well:

Theorem 4.30 *Let $\Sigma = \{\phi = 0\}$ be an oriented surface which is strongly pseudoconvex with respect to both P_1 and P_2. Let $x_0 \in \Sigma$. Suppose $u \in L^2$ solves (4.43), with $f \in L^2$.*

If u is H^1 in $\{\phi > 0\}$ near x_0 then u is H^1 near x_0.

Next, consider the case when the coefficients of P^1, P^2, Q^1, Q^2 are time independent. Suppose Σ is noncharacteristic with respect to both P^1 and P^2. Then we can represent it as $\Sigma = \{\phi = 0\}$ where ϕ is strongly pseudoconvex with respect to both P^1 and P^2 in the set $\{s = 0\}$ (here s is the time Fourier variable). Consequently, we can apply Theorem 3.32 to each of the equations to get

$$\tau|Q^\phi_\theta(D,x)u_1|^2_{1,\tau} \leq c(|Q^\phi_\theta(D,x)f_1|^2_0 + |Q^\phi_\theta(D,x)u_2|^2_{1,\tau} + |e^{\tau(\phi-\delta)}u|^2_{1,\tau} \tag{4.44}$$

$$\tau|Q^\phi_\theta(D,x)u_2|^2_{1,\tau} \leq c(|Q^\phi_\theta(D,x)f_2|^2_0 + |Q^\phi_\theta(D,x)u_1|^2_{1,\tau} + |e^{\tau(\phi-\delta)}u_2|^2_{1,\tau} \tag{4.45}$$

Summing them we get, for sufficiently large τ

$$\tau|Q^\phi_\theta(D,x)u|^2_{1,\tau} \leq c(|Q^\phi_\theta(D,x)f|^2_0 + |e^{\tau(\phi-\delta)}u|^2_{1,\tau}) \tag{4.46}$$

This estimate leads to the following unique continuation result:

Theorem 4.31 *Assume that the coefficients of P^1, P^2, Q^1, Q^2 are time independent. Let Σ be an oriented surface which is noncharacteristic with respect to both P_1 and P_2. Then UCP holds for H^1 solutions u to (4.42).*

4.6 Maxwell's equations

We consider here Maxwell's equations in an isotropic but possibly inhomogeneous medium,

$$\begin{cases} \epsilon\frac{\partial E}{\partial t} = \text{curl } H \\ \mu\frac{\partial H}{\partial t} = \text{curl } E \\ \div\epsilon E = 0 \\ \div\mu H = 0 \end{cases} \tag{4.47}$$

Here $\epsilon(x), \mu(x)$ are the permittivity, respectively the magnetic permeabillity of the medium.

Consider also the corresponding inhomogeneous equations

$$\begin{cases} \epsilon\frac{\partial E}{\partial t} = \text{curl } H + f \\ \mu\frac{\partial H}{\partial t} = \text{curl } E + g \\ \div\epsilon E = f_1 \\ \div\mu H = g_1 \end{cases} \tag{4.48}$$

Since the coefficients are inherently time independent, we are particularly interested in Holmgreen type unique continuation results.

First uncouple the equations for E and H. For E we have

$$\epsilon E_{tt} = \text{curl } \mu^{-1}\text{curl } E + f_t + \text{curl } \mu^{-1}g$$

$$\div\epsilon E = f_1$$

which further gives

$$\epsilon\mu E_\theta = \Delta E - \nabla(\ln\mu) \times \text{curl } E + \nabla(\nabla(\ln\epsilon)\cdot E)\mu f_t + \mu\text{curl } \mu^{-1}g + \nabla\epsilon^{-1}g_1$$

and a similar equation for H.

But this is a system of weakly coupled hyperbolic equations, therefore we can argue as in the previous section, namely use Carleman estimates on each component and then sum them up.

Thus, if ϕ is a function which is strongly pseudoconvex with respect to the operator

$$P(x, D) = \epsilon\mu\partial_t^2 - \Delta$$

then the following estimate holds:

$$\tau(|e^{\tau\phi}E|^2_{1,\tau} + |e^{\tau\phi}E|^2_{1,\tau}) \le c(|e^{\tau\phi}f|^2_{1,\tau} + |e^{\tau\phi}g|^2_{1,\tau} + |e^{\tau\phi}f_1|^2_{1,\tau} + |e^{\tau\phi}g_1|^2_{1,\tau}) \tag{4.49}$$

The similar estimate one energy level lower is perhaps simpler:

$$\tau(|e^{\tau\phi}E|^2 + |e^{\tau\phi}E|^2) \leq c(|e^{\tau\phi}f|^2 + |e^{\tau\phi}g|^2 + |e^{\tau\phi}f_1|^2 + |e^{\tau\phi}g_1|^2) \tag{4.50}$$

Consequently, we obtain the following unique continuation result:

Theorem 4.32 *Let Σ be an oriented surface which is strongly pseudoconvex with respect to P. Then UCP holds for L^2 solutions u to (4.47).*

The corresponding result on continuation of regularity follows from the analogue Carleman estimates with cutoff:

Theorem 4.33 *Let $\Sigma = \{\phi = 0\}$ be an oriented surface which is strongly pseudoconvex with respect to P. Let $x_0 \in \Sigma$. Suppose $u \in L^2$ solves (4.43), with $f, g, f_1, g_1 \in H^1$.*

If u is H^1 in $\{\phi > 0\}$ near x_0 then u is H^1 near x_0.

Let now ϕ be a strongly pseudoconvex with respect to both P^1 and P^2 in the set $\{s = 0\}$ (here s is the time Fourier variable). Since the coefficients are time independents, we can apply Theorem 3.32 to each of the equations and then sum them up to get

$$\tau(|Q_\theta^\phi E|^2 + |Q_\theta^\phi E|^2) \leq c(|Q_\theta^\phi f|^2 + |Q_\theta^\phi g|^2 + |Q_\theta^\phi f_1|^2 + |Q_\theta^\phi g_1|^2) + (|e^{\tau(\phi-\delta)}E|^2 + |e^{\tau(\phi-\delta)}E|^2) \tag{4.51}$$

This estimate leads to the following unique continuation result:

Theorem 4.34 *Let Σ be a noncharacteristic surface. Then UCP holds for L^2 solutions u to (4.42).*

Bibliography

[1] S. Alihnac, M. Baouendi *A non uniqueness result for operators of principal type*, preprint

[2] C. Bardos, G.Lebeau and R.Rauch *Controle et stabilisation de l'equation des ondes.* SIAM J. Control and Optimization, to appear

[3] R. Beals, C. Fefferman *Spatially inhomogeneous pseudo-differential operators* Comm. Pure Appl. Math. 27 (1974), 1-24

[4] T. Carleman *Sur un probleme d'unicite pour le systemes d'equations aux derivees partielles a deux variables independantes* Ark.Mat.Astr.Fys. 26 B, 17(1939), pp. 1-9

[5] A. Favini, I. Lasiecka, M.A. Horn and D. Tataru, *Global existence, uniqueness and regularity of solutions to a von Kármán system with nonlinear boundary dissipation*, to appear in Diff. and Int. Eq.

[6] Garding,L., *Le problem de la derivee oblique* C.R. Acad. Sci. Paris 285, 1977

[7] A. Greenleaf, A. Seeger *Fourier integral operators with fold singularities* preprint

[8] Y. Guo, W. Littman *The null controllability for the semilinear heat equation* IMA preprint 1128

[9] L.Hormander *Linear Partial Differential Operators* Springer Verlag, 1963

[10] L.Hormander *The Analysis of Linear Partial Differential Operators* Springer Verlag, 1985

[11] L. Hormander *A uniqueness theorem for second order hyperbolic differential equations* Comm. PDE, 17 (1992), 699-714

[12] V. Isakov *Carleman type estimates in an anisotropic case and applications.* J. Diff. Eq. 105 (1993) 217-238

[13] C.E. Kenig, *Carleman estimates, uniform Sobolev inequalities for second order differential operators, and unique continuation theorems* ICM Proceedings, Berkeley, California, 1986

[14] C.E.Kenig, G. Ponce, and Luis Vega *The Cauchy problem for the KdV equation in Sobolev spaces of negative indices* Duke Math. J. Vol. 71, No. 1 1993

[15] C.E.Kenig, A.Ruiz, C.D.Sogge *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators* Duke Math. J. 55(2) 1987, 329-347

[16] K. Kurata *On a backward estimate for solutions of parabolic differential equations and its application to unique continuation*, Spectral and scattering theory and applications, Adv. Stud. Pure Math. 23, 1994

[17] J.E.Lagnese *Boundary Stabilisation of Thin Plates.* SIAM 1989

[18] J.E. Lagnese and J.L.Lions *Modeling Analysis and Control of Thin Plates.* Masson, Paris 1988

[19] I. Lasiecka, R. Triggiani, *Sharp regularity theory for second order hyperbolic equations of Neuman type* Annali di Matematica Pura Applicata,(IV)vol CLVII (1990), 285-367.

[20] G. Lebeau *Equation des ondes amorties*, to appear in "Algebraic and Geometric Methods in Math Physics", Math-Physics Studies Book Series, Kluwers

[21] G. Lebeau, L. Robbiano *Controle exact de l'equation de chaleur*, preprint

[22] Fang Hua Lin *A uniqueness theorem for parabolic equations* CPAM 43 (1990) 127-136

[23] H. Lindblad, C.D. Sogge *On existence and scattering with minimal regularity for semi-linear wave equations*, preprint

[24] J.L.Lions *Controlabilite Exacte. Perturbations et Stabilisation des Systemes Distribue Vol 1 et 2* Masson, Paris (1988)

[25] W. Littman *Remarks on boundary control for polyhedral domains and related results*, Proc. Conf. On boundary Control, Sophia Antipolis, oct. 1990

[26] R.B.Melrose *Transformation of Boundary Problems* Acta Math. 147, 1982, 149-255

[27] R.B. Melrose *Equivalence of gliding hypersurfaces*, Invent. Math. 37, 165-191 (1976)

[28] R.B.Melrose, J.Sjostrand *Singularities of Boundary Value Problems I-II* Comm. Pure Applied Math. 31, 1978, pp. 593-617 and 35, 1982, pp. 129-168

[29] S. Miyatake, *Mixed problems for hyperbolic equations of second order with first order complex boundary operators* Japan J. Math., 1, 1975, pp 111-158

[30] C.Parenti, F.Segala, *Propagation and reflection of singularities for a class of evolution equations,* Comm. Partial Differential Equations 1981,741-782

[31] J.P. Puel and M. Tucsnak, *Stabilisation frontière pour les équations de von Kármán*, CRAS, **314**, série I, p.609-612, 1992.

[32] M. Renardy *On the linear stability of hyperbolic PDE's*, preprint

[33] J. Rauch, M.E. Taylor, *Decay of solutions to nondissipative hyperbolic equations in bounded domains* Comm. Pure Appl. Math. 28(1975), 501-523

[34] L.Robbiano *Theoreme d'unicite adapte au controle des solutions des problemes hyperboliques*, Comm. Partial Differential Equations 16 (1991), 789-800

[35] R. Sakamoto *Mixed problems for Hyperbolic equations I-II* J. Math. Kyoto Univ. 10, 375-401(1970), 10, 403-417(1970)

[36] J.C. Saut, B. Scheurer *Unique continuation for some evolution equations* J. Diff. Equ. 66 (1987) 118-139

[37] C.D. Sogge *Oscillatory integrals, Carleman inequalities and unique continuation for second order elliptic differential equations* J. Amer. Math. Soc. 2 (1989), 491-516

[38] C.D. Sogge *Uniqueness in Cauchy problems for hyperbolic differential operators* Trans. AMS 333, no. 2, 1992 821-833

[39] C.D. Sogge *A unique continuation theorem for second order parabolic differential operators*, Ark. Mat. 28 (1990), no. 1, 159-182

[40] E.M. Stein *Harmonic Analysis*, Princeton Mathematical Series, 43, 1993

[41] D. Tataru *A-priori estimates of Carleman's type in domains with boundary* Journal de Mathematiques Pure et Appl., 73, 1994, 355-387

[42] D. Tataru *Boundary controllability for conservative P.D.E.* , Applied Math. and Optimization, 31, 1995, 257-295

[43] D. Tataru *Uniform decay rates and attractors for evolution P.D.E.'s with boundary dissipation* , Journal of Differential Equations, to appear

[44] D. Tataru *Carleman estimates and unique continuation for the Schroedinger equation* Integral and Diff. Equ. 8, 1994, 901-905

[45] D. Tataru *Carleman estimates and unique continuation near the boundary for P.D.E.'s* to appear, Journal de Math. Pure et Appl.

[46] D. Tataru *Unique continuation for P.D.E's: between Holmgren's theorem and Hormander's theorem*, Comm. P.D.E. 20 (1995), 855-884

[47] D. Tataru *On the regularity of the boundary traces for the wave equation*, submitted

[48] D. Tataru *Carleman estimates and unique continuation for the wave equation with Neuman boundary condition*, preprint

[49] D. Tataru *On the* X^s_θ *spaces and unique continuation for semilinear hyperbolic equations*, preprint

[50] D. Tataru . *Unique continuation along hyperplanes for anisotropic pde's*, in preparation

[51] D.Tataru . *Unique continuation for anisotropic pde's with nonsmooth coefficients*, in preparation

[52] D. Tataru, M. Tucsnak *Well-posedness for the von Karman system* (joint with M. Tucsnak), in preparation

[53] M.E. Taylor *Pseudodifferential Operators.* Princeton University Press, 1981

[54] M.E. Taylor, *Pseudodifferential operators and nonlinear PDE's*, 1991, Birkhauser

[55] C. Zuily, *Uniqueness and non uniqueness in the Cauchy problem*, Progress in Math, 33, Birkhauser, Boston, 1983