ON l-ADIC REPRESENTATIONS ATTACHED TO MODULAR FORMS II

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To Robert Rankin on the occasion of his 70th birthday

0. Introduction. Suppose that $f = \sum_{n=1}^{\infty} a_n q^n$ is a newform of weight k on $\Gamma_1(N)$. Thus f is in particular a cusp form on $\Gamma_1(N)$, satisfying

$$f \mid T_n = a_n \cdot f$$

for all $n \geq 1$. Associated with f is a Dirichlet character

$$\varepsilon_0 : (\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}$$

such that

$$f \mid \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \varepsilon_0(d) f$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$.

Let E be the subfield of $\mathbf{C}$ generated by the coefficients a_n of f; then E is a finite extension of $\mathbf{Q}$. Let O be the integer ring of this number field. For each prime l, we let

$$O_l = O \otimes_{\mathbf{Z}} \mathbf{Z}_l, \qquad E_l = E \otimes_{\mathbf{Q}} \mathbf{Q}_l.$$

Thus O_l is the product of the completions O_λ of O at the primes λ of O dividing l, and similarly E_l is the product of its analogous completions E_λ. According to Deligne [**3**] (for $k > 1$) and to Deligne–Serre [**7**] (for $k = 1$), we can find a continuous representation

$$\rho_l : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}(2, O_l) \subset \mathbf{GL}(2, E_l),$$

unramified at all primes $p \nmid lN$, with the following property: for all primes $p \nmid lN$, if $\phi_p \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ is an arithmetic Frobenius element for p, we have

$$\mathrm{trace}\, \rho_l(\phi_p) = a_p, \qquad \det \rho_l(\phi_p) = \varepsilon_0(p) p^{k-1}. \tag{0.1}$$

One can show that these properties characterize ρ_l uniquely (up to isomorphism) as an E_l-representation of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, but not necessarily as an O_l-representation. For this, and other basic facts about the ρ_l, the reader may wish to consult [**13**].

Let $G = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, and for each prime l, let $G_l = \rho_l(G)$. Then G_l is a closed subgroup of $\mathbf{GL}(2, O_l)$, and therefore an l-adic Lie group. Let

$$\mathfrak{g}_l = \mathrm{Lie}(G_l)$$

be the Lie algebra of G_l.

† Research partially supported by the National Science Foundation.

Glasgow Math. J. **27** (1985) 185–194.

If $k=1$, then by construction G_l is a finite group independent of l; the Lie algebra $\mathfrak{g}_l$ is thus 0. Assume, then, that k is greater than 1. We are interested in the general problem of "identifying" the Lie algebras $\mathfrak{g}_l$ (for all l) and the groups G_l (for almost all l). This problem has a relatively simple and explicit solution in the case when f is a form with "complex multiplication" (in the sense of [**13**, p. 34]). For instance, the Lie algebra $\mathfrak{g}_l$ is just the abelian Lie algebra $K\otimes_{\mathbf{Q}}\mathbf{Q}_l$ if K is the field of "complex multiplication". Thus we shall make the further assumption that f is not such a form.

In the early 1970's, Serre and Swinnerton-Dyer considered the problem of identifying the $\mathfrak{g}_l$ and the G_l in the case of a form on $\mathbf{SL}(2,\mathbf{Z})$ (i.e. such that $N=1$) having integral coefficients (i.e. such that $E=\mathbf{Q}$). They proved the following two statements in this case.

(1) We have $\mathfrak{g}_l=\mathfrak{gl}(2,\mathbf{Q}_l)$ for each l: i.e. G_l is open in $\mathbf{GL}(2,\mathbf{Z}_l)$ for all l.

(2) For almost all l (i.e. all sufficiently large l) we have

$$G_l=\{M\in\mathbf{GL}(2,\mathbf{Z}_l)\mid\det(M)\in\mathbf{Z}_l^{*(k-1)}\}.$$

(In (2), $\mathbf{Z}_l^{*(k-1)}$ denotes the group of $(k-1)$th powers in $\mathbf{Z}_l^*$.) As interpreted by Serre and Swinnerton-Dyer, (2) states that the coefficients a_p of f satisfy no mod l congruences. (See [**15**] and [**18**].)

Subsequently, generalizations of these results were obtained by the author and by F. Momose. First, in [**12**], the author removed the restriction $E=\mathbf{Q}$, but he again assumed $N=1$. He next treated the case of arbitrary N, but he considered only the Lie algebras $\mathfrak{g}_l$, and even for them obtained only a partial result [**13**]. Finally, Momose [**11**] calculated the Lie algebras $\mathfrak{g}_l$ completely; he also obtained a result for the groups G_l, generalizing (2), under the extra assumption $k=2$.

This assumption was imposed by a technical problem: our poor understanding of the behavior of the representations ρ_l at the primes $p\neq l$ which divide N. The natural conjecture concerning this behavior was proved by Deligne [**4**] and Langlands [**10**] in the vast majority of cases, but was proved in general only recently, by H. Carayol [**1**], [**2**]. Now that Carayol's work is available, we are able to provide a satisfactory generalization of (2). (See (3.1) below.) The purpose of this article is twofold: we describe the results of Momose and Carayol, and we outline the arguments (for the most part quite standard) which lead to this generalization.

1. For each prime number l, the decompositions

$$O_l=\prod_{\lambda\mid l}O_\lambda,\qquad E_l=\prod_{\lambda\mid l}E_\lambda$$

lead to a decomposition of ρ_l as a direct sum of representations

$$\rho_\lambda:G\to\mathbf{GL}(2,O_\lambda)\subset\mathbf{GL}(2,E_\lambda).$$

In this section, our idea is to consider a fixed prime number p and to discuss the family of representations $(\rho_\lambda)_{\lambda\nmid p}$ locally at p. For $p\nmid N$, this local behavior is essentially given by the equations (0.1); therefore, we are especially interested in the case $p\mid N$.

Fix a place $\mathfrak{p}$ dividing p in $\bar{\mathbf{Q}}$. Let $D\subset G$ be the decomposition group associated to $\mathfrak{p}$, i.e. the stabilizer of $\mathfrak{p}$ in G. Then D is naturally the Galois group of an algebraic closure $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. We let I be the inertia subgroup of D, so that the quotient D/I is the Galois group of an algebraic closure of $\mathbf{F}_p$. Thus D/I is a "cyclic" profinite group isomorphic to $\hat{\mathbf{Z}}$ and generated by the Frobenius automorphism $x \mapsto x^p$ of $\bar{\mathbf{F}}_p$. The Weil group associated to $\mathfrak{p}$ is that subgroup W of D which contains I and is such that W/I consists of all integral powers of this Frobenius substitution.

For each λ, we define the representation $\rho_\lambda^\vee$ contragredient to ρ_λ by the formula

$$\rho_\lambda^\vee(g) = (\rho_\lambda(g)^{-1})^t.$$

Its restriction to W is a continuous λ-adic representation of W. By a construction of Grothendieck, as reformulated by Deligne, we may interpret this representation as a representation $(\sigma_\lambda, N_\lambda)$ of the Weil–Deligne group

$$W' = W'(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$$

(see [**6**, §8.4] or [**19**, §4]). This representation is well defined, up to isomorphism, as a representation of W' over E.

Now let $\mathbf{A}$ be the adèle ring of $\mathbf{Q}$, and let π be the automorphic representation of $\mathbf{GL}(2, \mathbf{A})$ which is associated to f in the following way: as explained by Rogawski and Tunnell in [**14**, §2 (especially pp. 21–24)], we can view f as a complex-valued function ϕ on $\mathbf{GL}(2, \mathbf{Q})\backslash\mathbf{GL}(2, \mathbf{A})$. We take π to be the representation of $\mathbf{GL}(2, \mathbf{A})$ which is generated by this function in the space of functions on $\mathbf{GL}(2, \mathbf{Q})\backslash\mathbf{GL}(2, \mathbf{A})$, viewed as a representation of $\mathbf{GL}(2, \mathbf{A})$ by right translation. The central character of π is then the product $\alpha^{-k}\varepsilon^{-1}$, where

$$\alpha : \mathbf{A}^* \to \mathbf{C}^*$$

is the normalized absolute value character, while ε is the Dirichlet character ε_0 suitably interpreted: if ω is a local uniformizing parameter at a prime $q \nmid N$, then

$$\varepsilon(\omega)^{-1} = \varepsilon_0(q).$$

The representation π is defined over the field E.

Now π is a tensor product of admissible irreducible representations of the various local components $\mathbf{GL}(2, \mathbf{Q}_v)$ of $\mathbf{GL}(2, \mathbf{A})$. Let π_p be the component of π at the prime p: thus π_p is a representation of $\mathbf{GL}(2, \mathbf{Q}_p)$ which is defined over E. Via the local Langlands correspondence of $\mathbf{GL}(2)$ (see [**9**]), we next associate to π_p a Φ-semisimple representation (σ, N) of W', whose isomorphism class is well defined. We do this "à la Hecke" [**5**, 3.2.6], so that this isomorphism class is defined over E.

THEOREM 1.1. *The isomorphism classes of (σ, N) and $(\sigma_\lambda, N_\lambda)$ are equal over E_λ, for each prime λ of E which is prime to p.*

This theorem was proved by Deligne and Langlands (cf. [**4**] and [**10**]) in various contexts which together embrace all cases in which π_p is not an extraordinary cuspidal representation of $\mathbf{GL}(2, \mathbf{Q}_p)$. (An *extraordinary* cuspidal representation is one which does not arise from a Grossencharacter of a quadratic extension of $\mathbf{Q}_p$. There are no such representations for $p > 2$.) Recent work of Carayol ([**1**], [**2**]) gives the general case.

COROLLARY 1.2. (a) *For* $\lambda, \mu \nmid p$, *the representations* $(\sigma_\lambda, N_\lambda)$ *and* (σ_μ, N_μ) *are compatible* [**19**, p. 23].

(b) *There is an open subgroup* U *of* I *such that* $\rho_\lambda(g)$ *is a unipotent matrix for all* $g \in U$ *and all* $\lambda \nmid p$.

Proof. The first part follows immediately from the theorem. For (b), one takes U to be the kernel of the restriction of some σ_λ to I; this kernel is independent of λ because of the theorem. The construction of the σ_λ is such that ρ_λ is unipotent where σ_λ is trivial.

2. *For the rest of this article, we will assume that the weight* k *is greater than* 1 *and that* f *is not a form with complex multiplication in the sense of* [**13**].

For each place λ of E, we let $\bar\rho_\lambda$ denote the "naive" reduction of ρ_λ: the composite of ρ_λ and the reduction map

$$\mathbf{GL}(2, O_\lambda) \to \mathbf{GL}(2, \mathbf{F}_\lambda),$$

where $\mathbf{F}_\lambda$ is the residue field of λ. We write $l(\lambda)$, or simply l, for the residue characteristic of λ, i.e. the characteristic of the field $\mathbf{F}_\lambda$. We let $\bar G_\lambda$ be the image of the representation $\bar\rho_\lambda$.

THEOREM 2.1. *Let* H *be an open subgroup of* G. *Then for almost all* λ, *the following assertions are true.*

(a) *The representation* $\bar\rho_{\lambda|H}$ *is an irreducible* 2*-dimensional representation of* H *over* $\mathbf{F}_\lambda$, *i.e. the group* $\bar H_\lambda = \bar\rho_\lambda(H)$ *is an irreducible subgroup of* $\mathbf{GL}(2, \mathbf{F}_\lambda)$.

(b) *The order of the group* $\bar H_\lambda$ *is divisible by* $l(\lambda)$.

Proof. It suffices to prove the theorem in the special case $H = G$. Indeed, suppose that (a) and (b) are almost always true for the groups $\bar G_\lambda$. Then by a well-known theorem [**8**, Theorem 2.8.4], $\bar G_\lambda$ contains a conjugate of the group $\mathbf{SL}(2, \mathbf{F}_l)$, for almost all λ. A calculation shows that all orbits of the projective line $\mathbf{P}^1(\mathbf{F}_\lambda)$, under the action of $\bar G_\lambda$, have cardinality at least l in this case. For $l > (G:H)$, this forces the action of $\bar H_\lambda$ to be irreducible, so that (a) is true for H. On the other hand, (b) is true for H whenever (b) is true for G and the prime l is prime to the index of H in G.

Now in the case $G = H$, we first show that (a) is almost always true. Suppose that $\bar\rho_\lambda$ is reducible. Its semisimplification is the direct sum of two 1-dimensional representations, given by characters which we may write $\varepsilon_1\chi_l^n$, $\varepsilon_2\chi_l^m$, where the ε_i are Dirichlet characters unramified outside N (viewed as characters of G in the obvious way), χ_l is the mod l cyclotomic character, and n and m are integers modulo $(l-1)$. Using the argument of [**18**, Lemma 8], we find, when l is sufficiently large, that one of n, m is 0 and the other is $k-1$. By (1.2), the characters ε_i are trivial on some open normal subgroup of G which we can specify in advance (independently of λ). For all primes p which split completely in the corresponding Galois extension of $\mathbf{Q}$, we have the mod λ congruence

$$a_p \equiv 1 + p^{k-1} \qquad (p \nmid lN).$$

Thus if $\bar\rho_\lambda$ is infinitely often reducible, the equality

$$a_p = 1 + p^{k-1}$$

holds for an infinite number of primes p. This obviously contradicts known estimates for the complex absolute values of the a_p, since we have $k>1$.

To prove that (b) is almost always true, we consider for each λ the image P_λ of $\bar{G}_\lambda$ in $\mathbf{PGL}(2,\mathbf{F}_\lambda)$. By [**16**, Proposition 16], if $\bar{G}_\lambda$ has order prime to l, then P_λ is cyclic, dihedral, or isomorphic to one of the three "exceptional" groups $\mathbf{A}_4, \mathbf{S}_4, \mathbf{A}_5$. We must show that each of these possibilities occurs only finitely many times.

Suppose first that P_λ is cyclic. Then $\bar{G}_\lambda$ is contained in a Cartan subgroup C_λ of $\mathbf{GL}(2,\mathbf{F}_\lambda)$. Assuming that l is odd, we see that C_λ contains an element with distinct rational eigenvalues: the image under $\bar{\rho}_\lambda$ of a complex conjugation in G. Accordingly, C_λ is a *split* Cartan subgroup of $\mathbf{GL}(2,\mathbf{F}_\lambda)$, and $\bar{\rho}_\lambda$ is therefore reducible. Since (a) is almost always true, this case occurs only a finite number of times.

Next, suppose that there are infinitely many λ for which P_λ is isomorphic to one of the groups $\mathbf{S}_4, \mathbf{A}_4, \mathbf{A}_5$. For all $p\nmid N$, set

$$r_p = a_p^2/(\varepsilon_0(p)p^{k-1}) \in E.$$

Then we find easily that r_p is one of the four numbers $4, 0, 1, 2$, or else a root of the quadratic equation

$$r^2-3r+1=0,$$

cf. [**12**, p. 264] or [**16**, §§2.5, 2.6]. On the other hand, when λ is a place of E for which $E_\lambda=\mathbf{Q}_l$, one knows that the image of ρ_λ is an open subgroup of $\mathbf{GL}(2, E_\lambda)$ because of the results of [**13**]. Let

$$F: G \to \mathbf{Z}_l$$

be the function

$$F(g) = (\operatorname{trace} \rho_\lambda(g))^2/(\det \rho_\lambda(g)).$$

Its image is then certainly an infinite subset of $\mathbf{Z}_l$. By the Cebotarev density theorem, the quantities $F(\phi_p)=r_p$ for $p\nmid lN$ are dense in this image. (Here ϕ_p is again a Frobenius element for p in G.) In particular, they cannot be finite in number, so we get a contradiction.

It remains to discuss the case where infinitely many P_λ are dihedral. Let Λ be the set of λ for which P_λ has this property. For each $\lambda\in\Lambda$, we can find a Cartan subgroup C_λ of $\mathbf{GL}(2,\mathbf{F}_\lambda)$ such that $\bar{G}_\lambda$ is contained in the normalizer N_λ of C_λ in $\mathbf{GL}(2,\mathbf{F}_\lambda)$, but such that $\bar{G}_\lambda$ is not contained in C_λ. The inverse image of $\bar{G}_\lambda\cap C_\lambda$ in G is then an open subgroup H_λ of G of index 2 in G; it corresponds to a quadratic field $K_\lambda\subset\bar{\mathbf{Q}}$ which is unramified *a priori* at all primes not dividing Nl. However, the argument on pp. 29–30 of [**18**] shows that K_λ is in fact unramified at l when l is sufficiently large. This implies that the quadratic fields K_λ (for $\lambda\in\Lambda$) are finite in number. After replacing Λ by an infinite subset of Λ, we can assume that the K_λ all coincide with a fixed quadratic extension K of $\mathbf{Q}$ in $\bar{\mathbf{Q}}$.

Let α be the Dirichlet character which corresponds to K. We have

$$a_p \equiv \alpha(p)a_p \pmod{\lambda}$$

for all $p \nmid lN$, whenever λ belongs to Λ. The infinity of Λ then implies that we have the equality

$$a_p = \alpha(p)a_p$$

for almost all p, contrary to the hypothesis that f is not a form with complex multiplication. This completes the proof of Theorem 2.1.

REMARK. In two places in our argument we have invoked the Serre/Swinnerton-Dyer theory of [**18**] and [**15**] in order to have control over the local behavior of the mod λ representations $\bar{\rho}_\lambda$ at the prime l. A method due to Fontaine and Serre (cf. [**17**]) should furnish a complete description of the semisimplifications of the representations $\bar{\rho}_{\lambda|I}$, where $I \subset G$ is an inertia group at l, provided that the prime l is larger than $k+1$ and prime to N. Such a description would be an alternate way of obtaining the information we need.

3. In this section, we propose to prove an analogue of the statement (2) given in the introduction. For this, we again assume $k>1$ and that f has no complex multiplication. We begin by sketching some results of Momose [**11**].

For each automorphism γ of the field E, we consider the newform

$$\gamma f = \sum_{n=1}^{\infty} \gamma(a_n)q^n.$$

There may or may not exist a Dirichlet character χ such that

$$\gamma(a_p) = \chi(p)a_p$$

for almost all p. If the character χ exists, it is unique; we will then call it χ_γ. Let Γ be the subset of those γ such that χ_γ exists; then, as it turns out, Γ is an abelian subgroup of the group of automorphisms of E. Let $F = E^\Gamma$ be the fixed field of Γ. Let $n = [E:F]$.

We note that each character χ_γ may be regarded as an E^*-valued character on the Galois group G; its kernel is then an open subgroup H_γ of G. We let H be the intersection of the H_γ and let K be the corresponding Galois extension of Q. (Thus $H = \mathrm{Gal}(\bar{\mathbf{Q}}/K)$.)

Finally, for $\gamma, \delta \in \Gamma$, we can use the characters χ_γ and χ_δ to define a certain Jacobi sum $c(\gamma, \delta) \in E$. Then $c(\gamma, \delta)$, viewed as a function of γ and δ, is a 2-cocycle on Γ with values in E^*. In a well-known way, we may use this cocycle to define a central simple algebra $\mathfrak{X}$ over F such that E is a maximal commutative semisimple subalgebra of $\mathfrak{X}$. It turns out that $\mathfrak{X}$ has order at most 2 in the Brauer group of F; this means that we have either

$$\mathfrak{X} = M(n, F)$$

or else

$$\mathfrak{X} = M(\tfrac{1}{2}n, D),$$

where D is a quaternion division algebra over F. In the former case, we let $D = M(2, F)$; thus D is in either case a quaternion algebra over F whose image in the Brauer group of F is the same as that of $\mathfrak{X}$.

Now the construction of the representations ρ_l of G may be very quickly summarized in the following schematic way: there is a certain vector space V of dimension 2 over E such that ρ_l gives the action of G on

$$V_l = V \otimes_{\mathbf{Q}} \mathbf{Q}_l,$$

which is then a free rank-2 module over E_l. Momose showed that there is a natural action of $\mathfrak{X}$ on V which is such that the actions of $\mathfrak{X}$ and of H on V_l commute for each l. The commutant of $\mathfrak{X}$ in $\mathrm{End}_{\mathbf{Q}}(V)$ is naturally isomorphic to D; therefore for each prime number l, the restriction of ρ_l to H may be considered as a map

$$\rho_l : H \to (D \otimes_{\mathbf{Q}} \mathbf{Q}_l)^*.$$

If we let $\mathfrak{n}$ denote the reduced norm map in the algebra D (and its l-adic completions), we find easily that the composition $\mathfrak{n} \circ \rho_l$ is just the $(k-1)$th power of the cyclotomic character

$$\chi_l : H \to \mathbf{Z}_l^*.$$

For each l, let us now set $H_l = \rho_l(H)$. It follows from the above description that H_l is contained in the group

$$\{x \in (D \otimes_{\mathbf{Q}} \mathbf{Q}_l)^* \mid \mathfrak{n}x \in \mathbf{Q}_l^*\}.$$

Momose proved that H_l is an open subgroup of this latter group for each l, generalizing statement (1) of the introduction.

We now turn to statement (2). We consider only those prime numbers such that

$$D \otimes_{\mathbf{Q}} \mathbf{Q}_l \cong M(2, F \otimes_{\mathbf{Q}} \mathbf{Q}_l). \tag{$*$}$$

All but finitely many primes have this property. Let R be the integer ring of F, and for each l let $R_l = R \otimes_{\mathbf{Z}} \mathbf{Z}_l$ be the completion of R at l. Replacing ρ_l by an isomorphic representation, we may suppose for almost all l that ρ_l sends H to the group

$$A_l = \{x \in \mathbf{GL}(2, R_l) \mid \det(x) \in \mathbf{Z}_l^{*(k-1)}\}.$$

Then H_l is a subgroup of A_l for almost all l.

THEOREM 3.1. *We have $H_l = A_l$ for almost all l.*

Proof. For each finite place $v \nmid N$ of the field K, let a_v be that element of F which is equal to the trace of $\rho_l(\phi_v)$ for l prime to v; ϕ_v denotes a Frobenius element for v in H. Our first task is to show that there is a place v such that a_v^2 generates the field F over $\mathbf{Q}$. For this, let us take at random a prime number l for which $(*)$ is satisfied; we will find a v prime to lN such that a_v^2 generates $F \otimes \mathbf{Q}_l$ as a $\mathbf{Q}_l$-algebra.

Consider

$$U = \{x \in H_l \mid (\text{trace } x)^2 \text{ generates } F \otimes \mathbf{Q}_l\}.$$

This is obviously an open subset of H_l. Also, using the fact that H_l is open in A_l, it is not hard to show that U is non-empty. By the Cebotarev density theorem, we may find

Frobenius elements ϕ_v such that $\rho_l(\phi_v)$ belongs to U; the corresponding numbers a_v then satisfy our requirement.

Now a_v is an algebraic integer because the coefficients of f are algebraic integers; thus $a_v \in R$. Since a_v^2 generates F over $\mathbf{Q}$, it generates R_l as a $\mathbf{Z}_l$-algebra for almost all l. Taking $x_l = \rho_l(\phi_v)$, we find for almost all l that H_l contains an element x_l such that $(\text{trace } x_l)^2$ generates R_l over $\mathbf{Z}_l$.

Using (2.2) and (3.1) of [**12**], we find that the equality $H_l = A_l$ holds whenever the following conditions hold:

(i) the determinant map $H_l \to \mathbf{Z}_l^{*(k-1)}$ is surjective;

(ii) $l \geqslant 5$;

(iii) H_l contains an element x_l as above;

(iv) for each $\lambda \mid l$, the group $\bar{\rho}_\lambda(H)$ is an irreducible subgroup of $\mathbf{GL}(2, \mathbf{F}_\lambda)$ whose order is divisible by l.

In view of the above discussion and Theorem 2.1, conditions (ii), (iii) and (iv) are satisfied for almost all l. On the other hand, we know that $\det \circ \rho_l$ coincides on H with the $(k-1)$th power of the l-adic cyclotomic character χ_l. Since the map

$$\chi_l : H \to \mathbf{Z}_l^*$$

is surjective for almost all l, condition (i) holds also for almost all l.

4. In this section we present a variant of Theorem 3.1, found by E. Papier, concerning the full group

$$G_l = \rho_l(G).$$

For large enough l, we can (and will) view ρ_l as a continuous homomorphism

$$G \to \mathbf{GL}(2, O \otimes \mathbf{Z}_l)$$

whose restriction to H takes values in $A_l \subseteq \mathbf{GL}(2, R_l)$. Given $\gamma \in \Gamma$, we consider the representations $\gamma\rho_l$ and $\rho_l \otimes \chi_\gamma$. These are isomorphic, being semisimple and having the same character. Thus there is a matrix $X \in \mathbf{GL}(2, E_l)$ such that we have

$$X(\gamma\rho_l)X^{-1} = \rho_l \otimes \chi_\gamma.$$

Since χ_γ is trivial on H and since $\rho_l(H) \subseteq A_l$, the matrix X commutes with the image H_l of H. This implies that X is a scalar matrix, so that we have the equality of matrices

$$\gamma(\rho_l(g)) = \rho_l(g)\chi_\gamma(g)$$

for all $g \in G$ and $\gamma \in \Gamma$.

Meanwhile, for $g \in G$, let $\alpha(g) \in E^*$ be such that

$$\gamma(\alpha(g)) = \chi_\gamma(g)\alpha(g)$$

for all $\gamma \in \Gamma$; such an element exists by Hilbert's theorem 90. We may certainly choose $\alpha(g)$ independently of l (the prime l does not appear in the definition of $\alpha(g)$) and such that $\alpha(g)$ depends only on the image of g in G/H. Thus there are only finitely many numbers $\alpha(g)$, and we may assume that they are all elements of O_l^* by choosing l large enough.

The matrices $\rho_l(g)\alpha(g)^{-1}$ are then elements of $\mathbf{GL}(2, O_l)$ which are invariant by Γ; they are consequently in $\mathbf{GL}(2, R_l)$. In the equality

$$\rho_l(g) = \begin{bmatrix} \alpha(g) & 0 \\ 0 & \varepsilon(g)/\alpha(g) \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \alpha(g)^2/\varepsilon(g) \end{bmatrix} (\alpha(g)^{-1}\rho_l(g)) \right\}$$

the product enclosed in curly brackets is an element of A_l, since it belongs to $\mathbf{GL}(2, R_l)$ and has determinant equal to $\chi_l(g)^{k-1}$. By Theorem 3.1 we obtain the following result.

THEOREM 4.1. (E. Papier). *For all but finitely many primes l, the image of ρ_l is the subgroup of $\mathbf{GL}(2, O_l)$ generated by the group A_l, together with the finite set of matrices*

$$\begin{bmatrix} \alpha(g) & 0 \\ 0 & \varepsilon(g)/\alpha(g) \end{bmatrix},$$

where $g \in G/H$.

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