Raising the Levels of Modular Representations

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1 Introduction

Let ℓ be a prime number, and let $\mathbf{F}$ be an algebraic closure of the prime field $\mathbf{F}_\ell$. Suppose that

$$\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}(2, \mathbf{F})$$

is an irreducible (continuous) representation. We say that ρ is *modular of level* N, for an integer $N \geq 1$, if ρ arises from cusp forms of weight 2 and trivial character on $\Gamma_o(N)$.

The term "arises from" may be interpreted in several equivalent ways. For our present purposes, it is simplest to work with maximal ideals of the Hecke algebra for weight-2 cusp forms on $\Gamma_o(N)$. Namely, let $S(N)$ be the $\mathbf{C}$-vector space consisting of such forms, and for each $n \geq 1$ let $T_n \in \mathrm{End}\, S(N)$ be the n^{th} Hecke operator. Let $\mathbf{T} = \mathbf{T}_N$ be the subring of $\mathrm{End}\, S(N)$ generated by these operators. As is well known ([3], Th. 6.7 and [7], §5), for each maximal ideal m of $\mathbf{T}$, there is a semisimple representation

$$\rho_{\mathsf{m}} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}(2, \mathbf{T}/\mathsf{m}),$$

unique up to isomorphism, satisfying

$$\mathrm{tr}\, \rho_{\mathsf{m}}(\mathsf{Frob}_r) = T_r \pmod{\mathsf{m}}, \qquad \det \rho_{\mathsf{m}}(\mathsf{Frob}_r) = r \pmod{\mathsf{m}}$$

for almost all primes r. (Here Frob_r is a Frobenius element in $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ for the prime r.) This representation is in fact unramified at every prime r prime to ℓN, and the indicated relations hold for all such primes. We understand that ρ is modular of level N if there is a maximal ideal m of $\mathbf{T}$, together with an inclusion $\omega\colon \mathbf{T}/\mathsf{m} \hookrightarrow \mathbf{F}$, so that the representations ρ and $\rho_{\mathsf{m}} \otimes_\omega \mathbf{F}$ are isomorphic. (Cf. [7], §5.)

The representations ρ_{m} are nothing other than the Galois representations attached to mod ℓ eigenforms of weight 2 on $\Gamma_o(N)$. Indeed, let $\mathcal{L}$ be the space of forms in $S(N)$ which have rational integral q-expansions. As is well known, $\mathcal{L}$ is a lattice in $S(N)$, cf. [3], Proposition 2.7. The space $\overline{\mathcal{L}} = \mathcal{L}/\ell\mathcal{L}$ is the space of mod ℓ cusp forms on $\Gamma_o(N)$. The $\mathbf{F}_\ell$-algebra $\mathcal{A}$ generated by the Hecke operators T_n in $\operatorname{End}\overline{\mathcal{L}}$ may be identified with $\mathbf{T}/\ell\mathbf{T}$ (see, for example, [7], §5). To give a pair (m, ω) as above is to give a character (i.e., homomorphism)

$$\epsilon : \mathcal{A} \to \mathbf{F}.$$

If f is a non-zero element of $\overline{\mathcal{L}} \otimes_{\mathbf{F}_\ell} \mathbf{F}$ which is an eigenvector for all T_n, the action of $\mathcal{A}$ on the line generated by f defines such a character ϵ. It is an elementary fact that all characters ϵ arise in this manner.

Assume now that ρ is modular of level Np, where p is a prime number not dividing N. We say that ρ is p-new (of level pN) if ρ arises in a similar manner from the p-new subspace $S(pN)_{p-\mathrm{new}}$ of $S(pN)$. Recall that there are two natural inclusions (or degeneracy maps) $S(N) \rightrightarrows S(pN)$ and dually two trace maps $S(pN) \rightrightarrows S(N)$. (See [1] for the former maps.) The two maps $S(N) \rightrightarrows S(pN)$ combine to give an inclusion $S(N) \oplus S(N) \hookrightarrow S(pN)$, whose image is known as the p-old subspace $S(pN)_{p-\mathrm{old}}$ of $S(pN)$. The space $S(pN)_{p-\mathrm{new}}$ is defined as the orthogonal complement to $S(pN)_{p-\mathrm{old}}$ in $S(pN)$, under the Petersson inner product on $S(pN)$. It may also be characterized algebraically as the intersection of the kernels of the two trace maps; this definition is due to Serre. The space $S(pN)_{p-\mathrm{new}}$ is $\mathbf{T}_{pN}$-stable.

The image of $\mathbf{T}_{pN}$ in $\operatorname{End} S(pN)_{p-\mathrm{new}}$ is the *p-new quotient*

$$\overline{\mathbf{T}}_{pN} = \mathbf{T}_{pN/p-\mathrm{new}}$$

of $\mathbf{T}_{pN}$. We say that ρ is p-new if $\mathsf{m} \subset \mathbf{T}_{pN}$ and ω may be found, as above, in such a way that the maximal ideal m of $\mathbf{T}_{pN}$ is the inverse image of a maximal ideal of $\overline{\mathbf{T}}_{pN}$, under the canonical quotient map $\mathbf{T}_{pN} \to \overline{\mathbf{T}}_{pN}$. On a concrete level, this means that the character

$$\epsilon : \mathbf{T}_{pN} \to \mathbf{F}$$

coming from (m, ω) is defined by an eigenform in the mod ℓ reduction of the space

$S(pN)_{p-\mathrm{new}}$, i.e., in the $\mathbf{F}$-vector space $\Lambda \otimes_{\mathbf{Z}} \mathbf{F}$, where Λ is the lattice in $S(pN)_{p-\mathrm{new}}$ consisting of forms with rational integral coefficients.

THEOREM **1** *Let ρ be modular of level N. Let $p \nmid \ell N$ be a prime satisfying one or both of the identities*

$$\operatorname{tr} \rho(\mathsf{Frob}_p) = \pm(p+1) \pmod{\ell}. \tag{1}$$

Then ρ is p-new of level pN.

Remarks.

1. In the Theorem, and in the discussion below, we assume that ρ is irreducible, as above.

2. A slightly stronger conclusion may be obtained if one assumes that ρ is q-new of level N, where q is a prime number which divides N, but not N/q. Under this hypothesis, plus the hypothesis of Theorem 1, one may show that ρ is pq-new of level pN, in a sense which is easy to make precise as above. (See [7], §7, where a theorem to this effect is proved, under the superfluous additional hypothesis $p \equiv -1 \pmod{\ell}$.) The interest of Theorem 1 is that no hypothesis is made about the existence of a prime number q.

3. The case $p = \ell$ can be included in the Theorem if its hypothesis (1) is reformulated. Namely, (1) tacitly relies on the fact that ρ is unramified outside the primes dividing ℓN. Choose a maximal ideal m for ρ as in the definition of "modular of level N." Then (1) may be re-written as the congruence

$$T_p \equiv \pm(p+1) \pmod{\mathsf{m}}.$$

Assuming simply that p is prime to N, but permitting the case $p = \ell$, one proves that ρ is p-new of level pN if this congruence is satisfied (with at least one choice of $\pm$).

COROLLARY *Let ρ be modular of level N. Then there are infinitely many primes p, prime to ℓN, such that ρ is p-new of level pN.*

Indeed, suppose that p is prime to ℓN. Then p is unramified in ρ, so that a Frobenius element Frob_p is well defined, up to conjugation, in the image of ρ. By the Cebotarev Density Theorem, there are infinitely many such p such that Frob_p is conjugate to $\rho(c)$, where c is a complex conjugation in $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Both sides of the congruence (1) are then 0, so that (1) is satisfied. (Cf. [7], Lemma 7.1.)

Our corollary is stated (in terms of mod ℓ eigenforms) as "Théorème (A)" in a recent preprint of Carayol [2]. Carayol describes his Théorème (A) as having been proved in preliminary versions of [7], as an application of results in [6]. In later versions of [7], Théorème (A) was replaced by a theorem involving pq-new forms (alluded to above), which is proved by methods involving Shimura curves. The aim of this present note is to resurrect Théorème (A).

Our derivation of Theorem 1 is based on the results of [6]. Although we couch our results in the language of Jacobians of modular curves, it should be clear to the reader that we use no fine arithmetic properties of these Jacobians: the argument is entirely cohomological. As F. Diamond has recently shown [4], an elaboration of these methods leads to results for cusp forms of weight $k \geq 2$.

2 Summary of [6]

First let N be a positive integer, and consider the modular curve $X_o(N)_{\mathbf{C}}$, along with its Jacobian $J_o(N) = \mathrm{Pic}^o(X_o(N))$. The curve $X_o(N)$ comes equipped with standard Hecke correspondences T_n, which induce endomorphisms of $J_o(N)$ by Pic functoriality (cf. [7], §3). These endomorphisms, in turn, act on the space of holomorphic differentials on the abelian variety dual to $J_o(N)$, which is the Albanese variety of $X_o(N)$. This space of differentials is canonically identified with $S(N)$, and via this identification the endomorphism T_n of $J_o(N)$ acts on the space of differentials as the usual Hecke operator T_n of $S(N)$. Since the action of $\mathrm{End}(J_o(N))$ on $S(N)$ is faithful, it follows that the subring of $\mathrm{End}(J_o(N))$ generated by the T_n is "nothing other" than the ring $\mathbf{T}_N$.

We now choose a prime p prime to N and consider $X_o(pN)$ and $J_o(pN)$, to which the same remarks apply. The two curves $X_o(pN)$ and $X_o(N)$ are linked by a pair of natural degeneracy maps $\delta_1, \delta_p\colon X_o(pN) \rightrightarrows X_o(N)$, with the following (naive) modular interpretation. The curve $X_o(pN)$ is associated to the moduli problem of classifying elliptic curves E which are furnished with cyclic subgroups C_N and C_p of order N and p, respectively. Similarly, $X_o(N)$ classifies elliptic curves with cyclic subgroups of order N. The degeneracy map δ_1 maps (E, C_N, C_p) to (E, C_N), while δ_p maps (E, C_N, C_p) to $(E/C_p, C'_N)$, where C'_N is the image of C_N on E/C_p.

In a similar vein, we recall the modular interpretation of the correspondences T_p on $X_o(N)$ and on $X_o(pN)$. First, for $X_o(N)$ we have

$$T_p : (E, C_N) \mapsto \sum_D (E/D, (C_N \oplus D)/D),$$

where the sum is taken over the $(p+1)$ different subgroups D of order p in E. For $X_o(pN)$, we have a sum of p terms

$$T_p : (E, C_N, C_p) \mapsto \sum_{D \neq C_p} (E/D, (C_N \oplus D)/D, E[p]/D),$$

where $E[p]$ is the group of p-division points on E. (This latter group is the direct sum $C_p \oplus D$.) These formulas lead immediately to the relations among correspondences

$$\delta_1 \circ T_p = T_p \circ \delta_1 - \delta_p, \qquad \delta_p \circ T_p = p \cdot \delta_1. \tag{2}$$

The maps δ_1 and δ_p combine to induce a map on Jacobians

$$\alpha : J_o(N) \times J_o(N) \to J_o(pN), \qquad (x, y) \mapsto \delta_1^*(x) + \delta_p^*(y).$$

The image of this map is by definition the p-old subvariety A of $J_o(pN)$; the kernel of α is a certain finite group which is calculated in [6].

Namely, let Sh be the Shimura subgroup of $J_o(N)$, i.e., the kernel of the map $J_o(N) \to J_1(N)$ which is induced by the covering of modular curves $X_1(N) \to X_o(N)$. The group Sh is a finite group which may be calculated in the following way: Consider the maximal unramified subcovering $X \to X_o(N)$ of $X_1(N) \to X_o(N)$, and let $\mathcal{G}$ be the covering group of this subcovering. Then $\mathcal{G}$ and Sh are canonically $\mathbf{G}_{\mathrm{m}}$-dual.

Let $\Sigma \subset J_o(N) \times J_o(N)$ be the image of Sh under the antidiagonal embedding

$$J_o(N) \to J_o(N) \times J_o(N), \qquad x \mapsto (x, -x).$$

According to [6], Theorem 4.3, we have

PROPOSITION **1** *The kernel of α is the group Σ.*

The map α is equivariant with respect to Hecke operators T_n with $(n, p) = 1$. Namely, we have $\alpha \circ T_n = T_n \circ \alpha$ for all n prime to p, with the understanding that the endomorphism T_n of $J_o(N)$ acts diagonally on the product $J_o(N) \times J_o(N)$. On the other hand, this formula must be modified when n is replaced by p, as one sees from (2).

Before recording the correct formula for T_p, we introduce the notational device of reserving the symbol T_p for the p^{th} Hecke operator at level N, and the symbol U_p for the p^{th} Hecke operator at level pN. With this notation, we have (as a consequence of (2)) the formula

$$U_p \circ \alpha = \alpha \circ \begin{pmatrix} T_p & p \\ -1 & 0 \end{pmatrix}, \tag{3}$$

in which the matrix refers to the natural left action of $\mathrm{M}(2, \mathbf{T}_N)$ on the product $J_o(N) \times J_o(N)$.

Concerning the behavior of Sh and Σ under Hecke operators, the following (easy) result is noted briefly in [6] and proved in detail in [8].

PROPOSITION **2** *The Shimura subgroup Sh of $J_o(N)$ is annihilated by the endomorphisms*

$$\eta_r = T_r - (r + 1)$$

of $J_o(N)$ for all primes $r \nmid N$.

Corollary *The subgroup Σ of $J_o(N) \times J_o(N)$ lies in the kernel of the endomorphism* $\begin{pmatrix} 1+p & T_p \\ T_p & 1+p \end{pmatrix}$ *of $J_o(N) \times J_o(N)$. It is annihilated by the operators $T_r - (r+1)$ for all prime numbers r not dividing pN.*

The significance of the endomorphism introduced in the corollary appears when we note the formula $\beta \circ \alpha = \begin{pmatrix} 1+p & T_p \\ T_p & 1+p \end{pmatrix}$, in which $\beta : J_o(Np) \to J_o(N) \times J_o(N)$ is the map induced by the two degeneracy maps $X_o(Np) \rightrightarrows X_o(N)$ and Albanese functoriality of the Jacobian. (The map β becomes the dual of α when we use "autoduality of the Jacobian" to identify the Jacobians with their own duals.) The formula results from the fact that the two degeneracy maps are each of degree $p+1$, and from the usual definition of T_p as a correspondence in terms of degeneracy maps.

Let $\Delta \subset J_o(N) \times J_o(N)$ be the kernel of $\begin{pmatrix} 1+p & T_p \\ T_p & 1+p \end{pmatrix}$. Then Δ is a finite subgroup of $J_o(N) \times J_o(N)$. Indeed, Δ differs only by 2-torsion from the direct sums of the kernels of $T_p \pm (p+1)$ on $J_o(N)$. These latter kernels are finite because neither number $\pm(p+1)$ can be an eigenvalue of T_p on $S(N)$, in view of Weil's Riemann hypothesis, which bounds T_p's eigenvalues by $2\sqrt{p}$. Further, the group Δ comes equipped with a perfect $\mathbf{G}_{\mathrm{m}}$-valued skew-symmetric pairing, in view of its interpretation as the kernel $K(L)$ of a polarization map

$$\phi_L : J_o(N) \times J_o(N) \to (J_o(N) \times J_o(N))\check{}.$$

(One takes L to be the pullback by α of the "theta divisor" on the Jacobian $J_o(pN)$.)

The subgroup Σ of Δ is self-orthogonal under the pairing on Δ. In other words, if we let $\Sigma^\perp$ be the annihilator of Σ in the pairing, we have a chain of groups

$$\Delta \supset \Sigma^\perp \supset \Sigma.$$

Note also that Δ/Σ is naturally a subgroup of the abelian variety A, since A and Σ are the image and kernel of α, respectively. Thus the subquotient $\Sigma^\perp/\Sigma$ of Δ is in particular a subgroup of A.

On the other hand, the quotient $\Delta/\Sigma^{\perp}$ is canonically the Cartier (i.e., $\mathbf{G}_{\mathrm{m}}$) dual Σ^* of Σ. It is naturally a subgroup of $A^{\vee}$. Indeed, Σ is the kernel of the isogeny $J_o(N) \times J_o(N) \to A$ induced by α. The kernel of the dual homomorphism $A^{\vee} \to (J_o(N) \times J_o(N))^{\vee}$ may be identified with Σ^*.

To state the final result that we need, we introduce the p-new abelian subvariety B of $J_o(pN)$. To define it, consider the map

$$J_o(pN)^{\vee} \to A^{\vee}$$

which is dual to the inclusion $A \hookrightarrow J_o(pN)$. Its kernel is an abelian subvariety Z of $J_o(pN)^{\vee}$. Using the autoduality of $J_o(pN)$ to transport Z back to $J_o(pN)$, we obtain B. This subvariety of $J_o(pN)$ is a complement to A in the sense that $J_o(pN) = A+B$ and $A \cap B$ is finite. It is p-new in that $\mathbf{T}_{pN}$ stabilizes B and acts on B through its p-new quotient $\overline{\mathbf{T}}_{pN}$ (which acts faithfully on B). The following main result of [6] is a formal consequence of Proposition 2:

THEOREM **2** *The finite groups $A \cap B$ and $\Sigma^{\perp}/\Sigma$ are equal.*

In the notation of [6], $A \cap B$ is the group Ω, which can be described directly in terms of Δ and the kernel of α ([6], pp. 508–509). Once this kernel is identified, the description of Theorem 2 is immediate.

3 Proof of Theorem 1

We assume from now on that ρ is modular of level N, and choose an ideal m of $\mathbf{T}_N$, plus an embedding $\omega\colon \mathbf{T}_N/\mathsf{m} \hookrightarrow \mathbf{F}$ as in the definition of "modular of level N." Assuming that one of the two congruences (1) is satisfied, we will construct

1. A maximal ideal $\mathcal{M}$ of $\overline{\mathbf{T}}_{pN}$, and

2. An isomorphism $\mathbf{T}_N/\mathsf{m} \approx \overline{\mathbf{T}}_{pN}/\mathcal{M}$ which takes T_r to T_r for all primes $r \neq p$.

This is enough to prove the theorem, since the representations ρ_{m} and $\rho_{\mathcal{M}}$ will necessarily be isomorphic, in view of the T_r-compatible isomorphism between the residue fields of m and $\mathcal{M}$. Our procedure is to construct $\mathcal{M}$ first as a maximal ideal of $\mathbf{T}_{Np}$ and then to verify that $\mathcal{M}$ in fact arises by pullback from a maximal ideal of $\overline{\mathbf{T}}_{pN}$.

It might be worth pointing out explicitly that our construction of $\mathcal{M}$ depends on the sign $\pm$ in (1). If $p \not\equiv -1 \pmod{\ell}$, then there is a unique sign $\pm$ which makes (1) true, under the hypothesis of the theorem, and our construction proceeds in a mechanical way. In case $p \equiv -1 \pmod{\ell}$, both congruences (1) are satisfied under the hypothesis of the theorem, and the construction requires us to decide whether (1) should read $0 \equiv +0$ or $0 \equiv -0$. The two choices of sign lead to different ideals $\mathcal{M}$, at least when ℓ is odd, since our construction shows that $U_p \equiv \pm 1 \pmod{\mathcal{M}}$, with the same sign $\pm$ as in (1).

Before beginning the construction, we introduce the following abbreviations:

$$R = \mathbf{T}_N, \qquad k = \mathbf{T}_N/\mathsf{m}, \qquad \mathbf{T} = \mathbf{T}_{pN}, \qquad \overline{\mathbf{T}} = \overline{\mathbf{T}}_{pN}.$$

Also, let

$$V = J_o(N)[\mathsf{m}]$$

be the kernel of m on $J_o(N)$, i.e., the intersection of the kernels on $J_o(N)$ of the various elements of m. This group is a finite k-vector space which is easily seen to be non-zero (cf. [5], or [7], Theorem 5.2). The group $V \times V$ is then a finite subgroup of $J_o(N) \times J_o(N)$. This subgroup has zero intersection with $\mathsf{Sh} \times \mathsf{Sh}$, in view of the irreducibility of ρ_m, Proposition 2 above, and [7], Theorem 5.2(c). In particular, α maps $V \times V$ isomorphically into A. Therefore, we can (and will) regard $V \times V$ as a subgroup of that abelian variety.

We now assume that one of the two congruences (1) is satisfied. To fix ideas we will treat only the case

$$\operatorname{tr} \rho(\mathsf{Frob}_p) \equiv -(p+1) \pmod{\ell}.$$

Using the isomorphism between ρ and $\rho_{\mathsf{m}} \otimes_{\omega} \mathbf{F}$, we restate this congruence in the form

$$T_p \equiv -(p+1) \pmod{\mathsf{m}}. \tag{4}$$

(The left-hand side of (4) is the trace of $\rho_{\mathsf{m}}(\mathsf{Frob}_p)$.) We embed V in $V \times V$ via the *diagonal* embedding; the antidiagonal embedding would be used instead if T_p were $p+1$ modulo m. We have

$$V \hookrightarrow V \times V \hookrightarrow A.$$

LEMMA **1** *The subgroup V of A is stable under $\mathbf{T}$. The action of $\mathbf{T}$ on V is summarized by a homomorphism $\gamma : \mathbf{T} \to k$ which takes T_n to T_n modulo m for $(n,p) = 1$ and takes U_p to -1.*

Proof. That $T_n \in \mathbf{T}$ acts on V in the indicated way, for n prime to p, follows from the equivariance of α with respect to such T_n. The statement relative to U_p then follows from (3) and (4). ∎

Define $\mathcal{M} = \ker \gamma$, so that we have an inclusion $\mathbf{T}/\mathcal{M} \hookrightarrow k = R/\mathsf{m}$. This map is in fact an *isomorphism* since k is generated by the images of the T_n with n *prime to* p. Indeed, T_p lies in the prime field $\mathbf{F}_\ell$ of k because of (4).

To conclude our proof of Theorem 1, we must show that the maximal ideal $\mathcal{M}$ of $\mathbf{T}$ arises by pullback from $\overline{\mathbf{T}}$. For this, it suffices to show that $\mathbf{T}$ acts on V through its quotient $\overline{\mathbf{T}}$. This fact follows from

LEMMA **2** *The subgroup V of A lies in the intersection $A \cap B$.*

Proof. We first note that V, considered diagonally as a subgroup of $J_o(N) \times J_o(N)$, lies in the group Δ. Indeed, $V \subset J_o(N)$ is killed by $T_p + p + 1$ by virtue of (4). The isomorphic image of V in $J_o(pN)$ therefore lies in Δ/Σ. To prove the lemma, we must show that this image lies in the subgroup $A \cap B = \Sigma^{\perp}/\Sigma$ of Δ/Σ. In other words, we must show that the image of V in $\Delta/\Sigma^{\perp}$ is 0.

A somewhat painless way to see this is to view the varieties $J_o(N)$, $J_o(Np)$, A, ... as being defined over $\mathbf{Q}$. The group $\Delta/\Sigma^{\perp}$ is canonically the $\mathbf{G}_{\mathrm{m}}$-dual of Σ, which may identified $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-equivariantly with the Shimura subgroup Sh of $J_o(N)$. This latter group is in turn the $\mathbf{G}_{\mathrm{m}}$-dual of the covering group $\mathcal{G}$ introduced above. It follows that the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $\Delta/\Sigma^{\perp}$ is trivial. (We note in passing that the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on Sh is given by the cyclotomic character $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \hat{\mathbf{Z}}^*$.) Hence if V maps non-trivially to $\Delta/\Sigma^{\perp}$, the semisimplification of V (as a $\mathbf{F}_\ell[\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})]$-module) contains the trivial representation. This semisimplification may be constructed by the following recipe: find the semisimplification W on V as a $k[\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})]$-module, and consider W as an $\mathbf{F}_\ell$-module. (A simple representation over k remains semisimple after "restriction of scalars" from k to $\mathbf{F}_\ell$.) Hence W contains $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-invariant vectors, if V maps non-trivially to $\Delta/\Sigma^{\perp}$. This conclusion is absurd, since W is the direct sum of a number of copies of the k-simple 2-dimensional representation ρ_{m} ([5], Chapter II, Proposition 14.2). ∎

Bibliography

[1] Atkin, A.O.L. and Lehner, J. Hecke operators on $\Gamma_o(m)$. Math. Ann. **185**, 134–160 (1970)

[2] Carayol, H. Sur les représentations Galoisiennes modulo ℓ attachées aux formes modulaires. Preprint

[3] Deligne, P. and Serre, J-P. Formes modulaires de poids 1. Ann. Sci. Ec. Norm. Sup. **7**, 507–530 (1974)

[4] Diamond, F. I. Congruence primes for cusp forms of weight $k \geq 2$. To appear

[5] Mazur, B. Modular curves and the Eisenstein ideal. Publ. Math. IHES **47**, 33–186 (1977)

[6] Ribet, K. Congruence relations between modular forms. Proc. International Congress of Mathematicians 1983, 503–514

[7] Ribet, K. On modular representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ arising from modular forms. Preprint

[8] Ribet, K. On the component groups and the Shimura subgroup of $J_o(N)$. Séminaire de Théorie des Nombres, Université de Bordeaux, 1987/88

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