Math 55 (discrete mathematics) Spring, 1996 Professor K. Ribet

First Midterm Exam—February 26 1996

1a *(5 points)*. True or false: If a function f maps a set to itself, then f is one-to-one if and only if f is onto.

1b *(5 points)*. True or false: If p and q are propositions, then $\neg p \to q$ and $p \wedge q$ are equivalent.

2a *(5 points)*. In the sleepy town of Cognac, 33 cars have license plates containing `TI`, while 109 have license plates containing the string `NY`. Suppose that 115 cars have license plates containing either `TI` or `NY`. How many cars have plates with `TI` or `NY`, but not both?

2b *(8 points)*. Use the Euclidean algorithm to solve the congruence $5x \equiv 1 \pmod{13}$.

3 *(10 points)*. The celebrated Ribonacci numbers R_n are defined as follows:

$$R_0 = 0; \quad R_1 = 1; \quad R_n = 3R_{n-1} - 2R_{n-2} \text{ for } n \geq 2.$$

Prove that $R_n = 2^n - 1$ for all $n \geq 0$.

4a *(5 points)*. True or false: If $f(x) = \log_\pi x$ and $g(x) = \log_{10} x$, then $f(x) = O\big(g(x)\big)$.

4b *(7 points)*. Find a non-negative integer x satisfying the two congruences

$$\begin{cases} x \equiv 15 \pmod{18} \\ x \equiv 47 \pmod{50}. \end{cases}$$

[Hint: first find a solution with x negative.]

Second midterm exam—April 8, 1996

1 *(9 points)*. A hand of three cards is drawn from an ordinary deck. If it is known that two of the cards are the ace of spades and the ace of clubs, what is the probability that all three cards are aces?

2a *(7 points)*. Stan wears either a bathing suit alone, or else a bathing suit, a T-shirt, and a hat. He has 6 bathing suits, 9 T-shirts and 7 hats. In how many different ways can he get dressed?

2b *(9 points)***.** In how many ways can I divide a class of 15 students into five groups of three students?

3 *(11 points)***.** Find the number of solutions to $x + y + z = 10$ in non-negative integers which satisfy $x \leq 5$ and $y \leq 3$.

4 *(9 points)***.** A hand of three cards is drawn from an ordinary deck. If it is known that at least two of the cards are aces, what is the probability that all three cards are aces?

Final exam—May 18, 1996

1a *(6 points)***.** How many onto functions are there from a set with 7 elements to one with 3 elements?

1b *(5 points)***.** What is the coefficient of a^7b^5 in the expansion of $(2a-b)^{12}$?

2 *(8 points)***.** Pokerhontas drops the King of ♣, the King of ♡ and the Ace of ♠ into a large bag. She shakes the bag vigorously and then removes two of the cards without looking at them. (a) If it is known that one of the two cards is a king, what is the probability that both cards are kings? (b) If it is known that one of the two cards is the King of ♡, what is the probability that both cards are kings?

3 *(6 points)***.** Let G be the multigraph with adjacency matrix $\begin{pmatrix} 0 & 3 & 3 & 2 \\ 3 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}$. How many vertices does G have? How many edges? Does G have an Euler circuit?

4 *(9 points)***.** Each box of C^{++} Cereal comes packed with a small plastic toy, which is shaped either like a "C" or like a "+." A given box contains a "+" with probability 2/3 and a "C" with probability 1/3. Suppose that I buy n boxes of cereal ($n \geq 3$). What is the probability that I have at least one "C" and at least two "+"s?

5 *(9 points)***.** How many ways can n books be placed on k distinguishable shelves (a) if the books are indistinguishable copies of the same title? (b) if no two books are the same and the positions of the books on the shelves matter?

6 *(8 points)***.** Consider the first 250 Fibonacci numbers: $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2, \ldots,$ $f_{249} \approx 5 \times 10^{51}$. (a) Show that there are at least 84 of them which have the same remainder when divided by 3. (b) How many of them are divisible by 2?

7 *(7 points)***.** Find an integer d such that $(M^{11})^d \equiv M \bmod 55$ for all M prime to 55.

8a *(3 points)***.** Show that $\dfrac{1}{\sqrt{n+1}} \geq 2\left(\sqrt{n+2} - \sqrt{n+1}\right)$ for all integers $n \geq 1$. (Multiply both sides by the positive quantity $\sqrt{n+2} + \sqrt{n+1}$.)

8b *(7 points)***.** Use mathematical induction and the result of part (a) to show:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \;\geq\; 2(\sqrt{n+1} - 1)$$

for $n \geq 1$.

9 *(7 points)***.** The diagram below, copied from a famous number theory book, shows the logical relation among chapters—to understand chapter 12, for instance, you have to have read the first five chapters as well as chapters 7 and 10. The diagram defines a partial ordering on the set $S := \{1, 2, \ldots, 15\}$: $a \prec b$ if and only if a must be read before b. With respect to this partial ordering: (a) What is the least upper bound of 2 and 8? (b) What are the maximal and minimal elements of S? Finally, (c) describe a total ordering on S which is compatible with the given partial ordering.