Math 55 Last Midterm
April 4, 2013
Sketchy solutions provided by K. A. Ribet

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences*; this is even more important than it was on the first midterm. Explain what you are doing: the paper you hand in will be your only representative when your work is graded. Do not worry about simplifying or evaluating expressions with decimal numbers, factorials, binomial coefficients and the like.

Problem	1	2	3	4	5	*Total*
Max. points	5	8	6	6	5	30

1. Numbers a_n $(n \geq 0)$ are defined as follows: $a_0 = 0$; $a_1 = 1$; $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 0$. Prove that $a_n = 3^n - 2^n$ for $n \geq 0$.

Let $P(n)$ be the statement that a_n is $3^n - 2^n$. The truth of $P(0)$ and $(P1)$ is clear by inspection. The proof proceeds by a (strong) induction: we prove that $P(k)$ and $P(k+1)$ imply $P(k+2)$, which is a statement about a_{k+2}. Assuming $P(k)$ and $P(k+1)$, we have

$$a_{k+2} = 5a_{k+1} - 6a_k = 5(3^{k+1} - 2^{k+1}) - 6(3^k - 2^k).$$

The right-hand side of the equation may be written as $(15-6)3^k - (10-6)2^k$ or $9 \cdot 3^k - 4 \cdot 2^k$. This latter expression is $3^{k+2} - 2^{k+2}$, as desired.

2. `http://opinionator.blogs.nytimes.com/2010/04/25/chances-are/`
Before going on vacation for a week, you ask a forgetful friend to water your ailing plant. Without water, the plant has a 90 percent chance of dying. Even with proper watering, it has a 20 percent chance of dying. The probability that your friend will forget to water the plant is 30 percent. (The link at the beginning of the problem is to a column that I flashed up on the screen a few weeks ago.)

a. What's the probability that your plant will survive the week?

Let E be the event that your friend forgets to water the plant and let F be the event that the plant becomes an ex-plant. Then F is the disjoint union of $F \cap E$ and $F \cap \overline{E}$. Therefore

$$p(F) = p(F \cap E) + p(F \cap \overline{E}) = p(F|E)p(E) + p(F|\overline{E})p(\overline{E}) = 0.9 \cdot 0.3 + 0.2 \cdot 0.7.$$

This arithmetic expression, which you were not required to evaluate, comes out to be 0.41. Accordingly, the probability that the plant survives, which is $1 - p(F)$, is $0.59 = 59\%$.

b. If it's dead when you return, what's the probability that your friend forgot to water it?

This is a Bayes Rule problem. The aim is to compute

$$p(E|F) = \frac{p(F|E)p(E)}{p(F)}.$$

Usually, one proceeds to calculate the denominator as the sum

$$p(F|E)p(E) + p(F|\overline{E})p(\overline{E}).$$

However we have already calculated the denominator in part (a)! The numerator is $0.9 \cdot 0.3 = 0.27$ and the denominator is $0.27 + 0.14 = 0.41$. Hence the desired probability is $27/41 \approx 0.66$. In other words, the answer to the question is "about 66%."

3. a. If n is a positive integer, show that

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n}.$$

One way to do this problem is to expand $(1-1)^n$ by the binomial theorem; you'll get the displayed alternating sum of binomial coefficients. Since $(1-1)^n = 0^n = 0$, the result follows.

Alternatively: When n is odd, you'll see that the terms in the sum with a minus sign are the same as the terms with a plus sign, so there's an immediate cancellation. For n is even, the situation is apparently more complicated; for example, if $n = 4$, the sum is $1 - 4 + 6 - 4 + 1 = 0$. You can treat the case of n even by relating the terms to the various $\binom{n-1}{k}$, using Pascal's identity. You'll again get a cancellation.

b. Show that a nonempty set has the same number of subsets with an odd number of elements as it does subsets with an even number of elements.

(Note that the nonempty set was assumed to be finite; this was announced during the exam.)

This is just a restatement of part (a) because the sum of the $\binom{n}{k}$ for k even is the number of subsets of $\{1, \ldots, n\}$ with an even number of elements and the corresponding sum for k odd is the number of subsets of $\{1, \ldots, n\}$ with an odd number of elements.

4. a. Twenty-six identical bagels are to be distributed to 10 students at the conclusion of an examination. In how many ways can the bagels be distributed?

This is a "bagel problem." Number the students. If the ith student gets x_i bagels, then $x_1 + x_2 + \cdots + x_{10} = 26$. The number of solutions to this equation in non-negative integers is $\binom{26+10-1}{26}$. The numerical value here is 70607460, but I hope that you didn't calculate it!

b. In how many ways can the bagels be distributed if Noah (one of the students) gets three or more bagels?

Give three bagels to Noah (whoever he is) and distribute the remaining 23 bagels to the 10 students. The number of ways to do that is $\binom{32}{9} =$ 28048800.

c. In how many ways can the bagels be distributed if Noah gets no more than two bagels and Dolly (another student) gets no more than three bagels?

In how many ways can Noah (the first student, let's say) get no more than two bagels? If he gets no bagel, we calculate the number of solutions to $x_2 + \cdots + x_{10} = 26$. If he gets one bagel, we calculate the number of solutions to $x_2 + \cdots + x_{10} = 25$. If he gets two bagels, we calculate the number of solutions to $x_2 + \cdots + x_{10} = 24$. These numbers are respectively $\binom{34}{8}$, $\binom{33}{8}$, $\binom{32}{8}$. In each of the three situations, we can see what happens if Dolly gets more than three bagels; she gets at least four. The calculation here is like that in part (b), and there are respectively $\binom{30}{8}$, $\binom{29}{8}$, $\binom{28}{8}$ ways of giving 0, 1 or 2 bagels to Noah and at least four to Dolly. The number of ways of giving zip to Noah and at most three to Dolly is then $\binom{34}{8} - \binom{30}{8}$, and so on for the two other Noah possibilities. The answer to the question is the sum of these three differences:

$$\binom{34}{8} + \binom{33}{8} + \binom{32}{8} - \binom{30}{8} - \binom{29}{8} - \binom{28}{8} = 29305485.$$

You know where the name "Noah" comes from: Noah's Bagels. How about Dolly? There's a Dolly's Donuts in the Temescal area of Oakland. I've never tried the donuts, but I met Dolly once.

5. Bob and Alice toss a fair coin repeatedly until either two tails have come up in a row or the sequence heads-tails has come up. Bob wins the game in the first case; Alice wins in the second case. What is the probability that Alice wins the game?

Consider the possible outcomes of the first two tosses: `TT`, `HT`, `HH`, `TH`. They each occur with probability 1/4. In the first case, Bob wins and the game stops. In the second case, Alice wins and the game stops. In the third and fourth cases, the game goes on, but Alice wins — eventually. Once a head comes up, you just toss and toss until a tail comes up and then Alice is the victor. There's a theoretical possibility that the tosses could remain heads forever, but this possibility naturally has the probability 0. Accordingly, Alice wins the game with probability 3/4.

I put this problem on the exam because it was presented to me at a conference as a probability situation that most non-mathematicians find ridiculously paradoxical. In two tosses of a coin, the sequence tails-tails is just as likely as the sequence heads-tails. Nonetheless, the game is completely unfair!