Math 55 Third Midterm
May 17, 2013 at 7PM
Sketchy solutions provided by Ken Ribet

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write clearly and carefully in *complete sentences*; this is the final exam, so can we take this seriously, please? Explain what you are doing: the paper you hand in will be your only representative when your work is graded. Do not worry about simplifying or evaluating expressions with decimal numbers, factorials, binomial coefficients and the like.

Problem point values: 5, 6, 6, 7, 7, 7, 7, 7, 8. There were nine problems; the total point count is 60.

1. At a farmers' market somewhere in mathland, a bicycle rolls over a vendor's basket and crushes her eggs. The rider offers to pay for the damages and asks the vendor how many eggs she had brought. She does not remember the exact number, but when she had taken them out two at a time, there was one egg left. The same happened when she picked them out three, four, five, and six at a time, but when she took them seven at a time they came out even. What is the smallest number of eggs she could have had?

This problem comes from `http://www.chinapage.com/math/crt.html`; I paraphrased one of the problems at the top of the page. The number of eggs is given as being congruent to 1 mod 2, 3, 4, 5 and 6; equivalently, it's 1 (mod 60). It's supposed to be a multiple of 7. To figure out which multiple of 7 actually works, you probably used some trial and error technique. For example, we can look at expressions of the form $60m + 1$ until we get one that's divisible by 7. The number that works is 301. Alternatively, one can use the extended Euclidean algorithm, but I bet you didn't.

2. To each positive integer n with at least two digits, we associate the number r gotten by stripping away the units digit of n and then subtracting twice the units digit of n from the stripped integer. For example, if $n = 41283$ then $r = 4128 - 2 \cdot 3 = 4122$. Similarly, if n is 4122, then $r = 412 - 4 = 408$.

Prove that n is divisible by 7 if and only if r is divisible by 7.

If u is the units digit of n and x is the number gotten by stripping away the units digit from n, then $n = 10x + u$ and $r = x - 2u$. Mod 7, we have $n \equiv 3x + u$,

$r \equiv x + 5u$. Then $3r \equiv 3x + 15u \equiv 3x + u \equiv n$. Hence 7 divides n if and only if 7 divides $3r$. Since $\gcd(7, 3) = 1$, 7 divides $3r$ if and only if 7 divides r by Euclid's lemma (Lemma 2 on page 271 of the book).

3. Let A be the set of bit strings $a = a_1 a_2 \cdots a_9$ of length 9. Let $R \subseteq A \times A$ be the set of pairs (a, b) such that $a_1 = b_1$ or $a_2 = b_2$. Decide whether or not the relation R is: (i) reflexive, (ii) transitive, (iii) symmetric, (iv) antisymmetric, (v) an equivalence relation.

I'll be brief here: the relation is obviously symmetric and reflexive. It's also obviously *not* antisymmetric. The main wrinkle to the problem is that it isn't transitive: if $a_1 = b_1$ and $b_c = c_2$, then there's no reason for a_1 to be c_1 or for a_2 to be c_2. I'm sure that you can make up examples that show definitively that the relation is not transitive, and I'm hoping that you did make up such examples.

4a. Nine students decide to form three study groups, each with three students. In how many ways can these groups be formed?

This is a fairly standard problem, so I hope that most of you got it. There are $\binom{9}{3}$ ways of forming the "first" group and then $\binom{6}{3}$ of forming the "second" group. Once those groups are selected, there is only $1 = \binom{3}{3}$ way to form the "third" group. An issue here is that the groups are not numbered, so we have to take $\binom{9}{3}\binom{6}{3}$ and divide this number by $3! = 6$, which is the number of ways of numbering three groups. My answer is 280.

b. In how many ways can the study groups be formed if Mick and Keith (two of the students) are in the same group?

One way to do this problem is to decide that Mick and Keith will be in the first group. Then there are 7 ways instead of $\binom{9}{3}$ ways to fill out the first group. There will still be $\binom{6}{3}$ ways to choose the second group. We take the product $7\binom{6}{3}$ and divide it by 2 (instead of by 6) because the second and third groups are indistinguishable. This leads to 70 as the answer.

5. Alice buys a bag of 12 biased coins. Six of these are top-heavy coins that come up heads 2/3 of the time, while the other six are bottom-heavy coins that come up tails 2/3 of the time. Bob reaches into the bag, pulls out a coin at random and tosses it. The coin comes up heads. What's the probability that he pulled out a top-heavy coin?

Alice and Bob are really full of tricks, aren't they? This is a Bayes Rule kinda problem. Let A be the event that the coin was top-heavy, and let B be the event that pulled-out coin flips to heads. We want to calculate

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}.$$

Let C be the event that the coin was bottom-heavy; thus C is the complement of A. Then $p(A) = p(C) = \frac{1}{2}$ because there are an equal number of coins of each type in the bag. We have

$$p(B) = p(B \cap A) + p(B \cap C) = p(B|A)p(A) + p(B|C)p(C).$$

Thus

$$p(A|B) = \frac{p(B|A)p(A)}{p(B|A)p(A) + p(B|C)p(C)} = \frac{p(B|A)}{p(B|A) + p(B|C)}.$$

Since $p(B|A) = 2/3$ and $p(B|C) = 1/3$, the fraction works out to be 2/3.

6a. Let n be a positive integer. By applying the binomial theorem to $(1+x)^n$ and taking the derivative, establish the identity

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}.$$

The binomial theorem gives

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

Differentiate with respect to x:

$$n(1+x)^{n-1} = n + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}.$$

Now plug in $x = 1$; this gives the desired formula.

b. Prove the more general formula $\binom{n}{d} 2^{n-d} = \sum_{k=d}^{n} \binom{k}{d} \binom{n}{k}$ for $d = 0, 1, \ldots, n$.

Keep differentiating. Let's say we differentiate d times. On the left-hand side, we get $n(n-1)(n-2)\cdots(n-d+1)x^{n-d}$. The coefficient of x^{n-d} is $n!/(n-d)! = d!\binom{n}{d}$. Similarly, on the right-hand side we are differentiating $\sum_{k=0}^{n} \binom{n}{k} x^k$ and get $\sum_{k=d}^{n} \binom{n}{k} d! \binom{k}{d} x^{k-d}$. Notice that we've removed the terms on the right-hand side for which the powers of x were initially smaller than d because differentiating d times eliminates them. Once again we set $x = 1$ and we find the desired identity after dividing the two sides of the equation by $d!$.

7. Let $a_1 = 1$, $a_2 = 5$, $a_3 = 19$, $a_4 = 65$, $a_5 = 211$, $a_6 = 665$.

a. Find a linear homogeneous recurrence relation of degree 2 with constant coefficients that is satisfied by $a_1, a_2, \ldots, a_6$.

We are looking for an equation of the form $a_{n+2} = Ba_{n+1} + Aa_n$ with A and B to be determined. Set $n = 1$ and $n = 2$; these give the two equations $19 = 5B + A$, $65 = 19B + 5A$. By high school algebra, we get $B = 5$, $A = -6$. Thus the recurrence is

$$a_{n+2} - 5a_{n+1} + 6a_n = 0.$$

Of course, we should check that $211 = 5 \cdot 65 - 6 \cdot 19$ and $665 = 5 \cdot 211 - 6 \cdot 65$; both equations are true. (The problem is over-determined, which is annoying, but I took this problem off some guy's Facebook page and decided that I needed to adhere to his formulation for complete authenticity.)

b. Solve this recurrence relation and compute a_7 and a_8.

The characteristic equation for the problem is $\lambda^2 - 5\lambda + 6 = 0$, whose roots are 3 and 2. We find that $a_n = C3^n + C'2^n$ for some values of C and C'. Plugging in the values of a_1 and a_2, we get two equations for C and C' and learn ultimately that $a_n = 3^n - 2^n$ is the solution. In particular, $a_7 = 2059$ and $a_8 = 6305$. It was not necessary to compute these values explicitly; fine answers would have been $3^7 - 2^7$ and $3^8 - 2^8$. (See `http://oeis.org/A001047`.)

8. The integers 1–9 are listed in random order. We circle all numbers on the random list that are greater than the numbers to their left (if any). For example, if the list were $5, 7, 1, 2, 6, 3, 8, 4, 9$, we would circle 5, 7, 8 and 9.

a. Show that $\dfrac{1}{i}$ is the probability that we circle the number in the ith position on the random list.

Let's say that $i = 4$ to fix ideas. The "random list" that we write down amounts to a permutation of the set of integers from 1 to 9. The list of the first four of them corresponds to an injection $\{1, 2, 3, 4\} \hookrightarrow \{1, 2, \ldots, 9\}$. Let's say that the image of that injection is $\{1, 2, 5, 7\}$, as in the example. We are asking for the probability that 4 maps to 7. The number of maps $\{1, 2, 3, 4\} \xrightarrow{\sim} \{1, 2, 5, 7\}$ is $4!$, and the number of those maps that take 4 to 7 is $3!$. Hence the probability that the fourth number is largest, given that the first four numbers are 1,2, 5 and 7, is $3!/4! = 1/4$. Since this probability is the same for each possible image of the injection, the probability that we seek is $1/4!$.

For the general case, run the same argument with 4 replaced by i.

b. What is the expected number of circled numbers?

Let X_i be the random variable whose value is 1 if the ith number is circled and 0 otherwise. Then $E(X_i) = 1/i$ by part (a). The expected number of circled numbers is $E(X_1 + \cdots + X_9) = 1 + 1/2 + 1/3 + \cdots + 1/9$. I hope that you didn't bother to calculate this as a fraction. If you did, you might have gotten 7129/2520. For what it's worth, a decimal value of this fraction is 2.82896825396825, according to Sage.

9. Let G be the simple graph whose vertices are the bit strings of length 4, two bit strings being connected by an edge if and only if they differ in exactly one place.

a. Show that G is bipartite and that G has no circuits of length three.

It's bipartite because we can put in one camp the vertices corresponding to bit strings with an even number of `0`s and in the other those that correspond to bit strings with an odd number of `0`s. Edges connect vertices that are off by exactly one bit, so they connect vertices with different parities.

A circuit in a bipartite graph has to have even length, so there is no circuit of length 3.

b. Does G have an Euler circuit?

Yes because each vertex has even degree; the degree of each vertex is 4.

c. Is G planar?

No. The number of edges (32) is twice the number of vertices (e.g., by the handshaking theorem). For planar graphs with at least 3 vertices, the number of edges is at most twice the number of vertices minus 4 (Cor. 3, p. 723). This question is probably the "hardest" of the exam because it requires familiarity with one relatively arcane fact, namely the corollary in question.

Note: in Sage, one can construct this graph by typing `G=graphs.CubeGraph(4)`. You can view it and invesitgate its basic properties once it has been introduced.

Thanks everybody for your participation in the lectures, the discussion sessions, the office hours, the lunches and the breakfasts. You've been a great class! Please keep in touch. We'll definitely have a few reunion events in August.