Mathematics 55 — Professor K. A. Ribet
First Midterm Exam — September 25, 1997

F295 Haas and 1 Leconte
3:40–5:00 PM

1a *(4 points)*. Find the remainder when 2^{55} is divided by the prime number 53.

1b *(6 points)*. Suppose that f and g are functions $S \to S$, where S is the set of positive integers less than 10^3. If the composition $f \circ g$ is 1-1 and onto, must f and g be 1-1 and onto? (Give a short proof or a counterexample.)

In the following problems, it may be useful to know that $203 \cdot 83 - 39 \cdot 432 = 1$.

2a *(5 points)*. Find an integer x such that $83x \equiv 1 \bmod 432$.

2b *(5 points)*. Find an integer y such that $39y \equiv 4 \bmod 203$.

2c *(6 points)*. Find an integer z such that $z \equiv 2 \bmod 39$ and $z \equiv 3 \bmod 203$.

3 *(5 points)*. Can you conclude that $A = B$ if A, B and C are sets such that $A \cap C = B \cap C$ and $A \cup C = B \cup C$? (Explain why or why not.)

4 *(10 points)*. Numbers A_n are defined as follows:

$$A_0 = 0; \quad A_1 = 1; \quad A_n = 5A_{n-1} - 6A_{n-2} \text{ for } n \geq 2.$$

Prove that $A_n = 3^n - 2^n$ for all $n \geq 0$.

5 *(8 points)*. Suppose that $f(x) = 5^x$ and $g(x) = 10^x$. Decide whether each of the following statements is true. (Logarithms are to the base e.)

(a) $f(x) = O\big(g(x)\big)$.
(b) $g(x) = O\big(f(x)\big)$.
(c) $\log g(x) = O\big(\log f(x)\big)$.
(d) $f(x) = O\big(\log g(x)\big)$.

Explain your reasoning.

6a *(6 points)*. Find an integer d such that $(M^{11})^d \equiv M \bmod 55$ for all integers M such that $\gcd(M, 55) = 1$.

6b *(5 points)*. Determine whether $(\neg q \wedge (p \to q)) \to \neg p$ is a tautology.

Mathematics 55 — Professor K. A. Ribet

Last Midterm Exam — October 30, 1997

F295 Haas and 10 Evans

3:40–5:00 PM

1a *(5 points)*. How many poker hands contain no ace, exactly one king, and at least one heart? (A poker hand contains five cards.)

1b *(5 points)*. How many different strings of length ten can one make out of the letters in INDISCRETE?

2 *(8 points)*. How many solutions to $x + y + z + w = 1097$ in non-negative integers x, y, z and w satisfy at least one of the inequalities $x \geq 100$, $y \geq 100$, $z \geq 100$? [Hint: use inclusion-exclusion.]

3 *(6 points)*. In how many ways can a class of 15 be divided into 5 groups of 3 students in such a way that the two students named Ken are in the same group?

4a *(5 points)*. Persi's crooked penny comes up "heads" 2/3 of the time when it is tossed. What is the probability that exactly four heads come up when it is tossed six times?

4b *(6 points)*. A fair nickel and a fair penny are tossed simultaneously until both come up "tails." What is the expected number of tosses? (One "toss" is a flip of the two coins.)

5a *(4 points)*. What is the coefficient of $x^{30}y^{29}$ in the expansion of $(x - 2y)^{59}$?

5b *(6 points)*. How many integers are needed to guarantee that two of them leave the same remainders on division both by 15 and by 21? [Hint: note that 15 and 21 have a common factor.]

Mathematics 55 | Professor K. A. Ribet
Final Examination | December 17, 1997

International House 12:30–3:30 PM

1a *(5 points)*. How many subsets of the set $\{1, 2, \ldots, 19\}$ contain at least one odd integer?

1b *(5 points)*. Find the last (rightmost) decimal digit of 3^{1024}.

2a *(5 points)*. If $P(A) = 1/3$, $P(B) = 1/2$ and $P(A \cup B) = 2/3$, find $P\big((A \cap B)|A\big)$.

2b *(5 points)*. Mumford has a trick quarter with two heads, a trick quarter with two tails, and a standard quarter with one head and one tail. He chooses one of the three coins at random, tosses it in the air, and slaps it on the table. A head is showing. Find the probability that the selected coin has two heads.

3 *(9 points)*. For each positive integer n, let $f(n)$ be the number of subsets of $\{1, 2, \ldots, n\}$ which contain no two consecutive integers.

(a) Calculate $f(1)$, $f(2)$ and $f(3)$.

(b) Find a recursive formula for $f(n)$ and use it to calculate $f(8)$. [Justify your recursive formula.]

4 *(8 points)*. According to `math.berkeley.edu`, the number of Math 55 students who visited the class Web page at least once in November is divisible by 1, 2, 3, 4, 5 and 6, but leaves remainder 1 on division by 7. What is this number?

5 *(9 points)*. Show that

$$\sum_{i=1}^{2n} (-1)^{i+1} \frac{1}{i} = \sum_{i=n+1}^{2n} \frac{1}{i}$$

for all positive integers n. (For example, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{1999} - \frac{1}{2000} = \frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{2000}$.)

6 *(9 points)*. A box contains three red socks, three blue socks, and four white socks. (Socks of the same color are indistinguishable.) Eight socks are pulled out of the box, one at a time. In how many ways can this be done?

7 *(10 points)*. Let G be the simple graph whose vertices are the bit strings of length 6, two bit strings being connected by an edge if and only if they differ in exactly one place.

(a) Does G have an Euler circuit?

(b) Is the graph G planar?

8 *(8 points)*. Suppose that A is a finite set with at least two elements and that R is an equivalence relation on A (for instance, the relation of acquaintance among guests at a party). Show that there are distinct elements a and a' in A whose equivalence classes $[a]_R$ and $[a']_R$ have the same number of elements.

9 *(9 points)*. I roll a single die repeatedly, until three different faces have come up at least once. What is the expected number of times that I need to roll the die?