Math 54 Professor K. A. Ribet
Second Midterm Exam November 2, 2005

This exam was a 50-minute exam, which began at 2:10PM. There were 5 problems, worth 6, 7, 5, 5, and 7 points, respectively. The maximum possible score was 30 points—same as for the first midterm. I expect that the scores on this exam will be substantially higher than on the first midterm. Students who knew the material very well will probably be able to finish the exam and get essentially full credit on all the problems. These problems are not tricky or theoretical, as far as I can tell. (I am writing this enroute to NY and won't see students' reactions to the exam until after I've finished writing the solutions.)

> *Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Your paper is your ambassador when it is graded. Correct answers without appropriate supporting work will be regarded skeptically. Incorrect answers without appropriate supporting work will receive no partial credit. This exam has six pages. Please write your name on each page. At the conclusion of the exam, please hand in your paper to your GSI.*

1. *Let* $R = \begin{bmatrix} 2 & 8 & -1 & 1 & 0 \\ 0 & 6 & -1 & 2 & -3 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. *Exhibit bases for the following three spaces: the row space of* R, *the column space of* R, *the null space of* R.

Row space: take the top three rows, i.e., the non-zero ones. Column space: take the three left-hand rows, i.e., the ones with pivots. Null space: This space consists of the tuples $x = (a, b, c, d, e)$ such that $Ax = 0$. (In the equation, we view x vertically.) The fourth and fifth variables are "free variables"; given any d and e, you can solve for a, b and c uniquely by back substitution. There is a fine basis of the null space consisting of two vectors of the form $(?, ?, -3, 1, 0)$ and $(?, ?, ?, 0, 1)$. Unfortunately, some of the "question mark" entries have (small) denomiators. Scaling the basis vectors to clear denominators, I came up with $(-2, -7, -60, 0, 6)$ and $(8, -5, -18, 6, 0)$.

2. *Find three linearly independent eigenvectors for the matrix* $\begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$, *whose characteristic polynomial is* $(\lambda - 4)(\lambda - 2)^2$. *Is this matrix diagonalizable?*

The matrix will be diagonalizable because the three independent eigenvectors will form a basis for $\mathbf{R}^3$. There are two eigenspaces here, W_2 and W_4 in the notation that we've been using. Now W_4, which is the null space of $A - 4I$, will be exactly one-dimensional because $\lambda - 4$ occurs only to the first power in the characteristic polynomial. A basis for this 1-dimensional space is $(-1, 0, 1)$. The space W_2 is at least one-dimensional, and it will be two-dimensional if and only if A is diagonalizable. It turns out to be 2-dimensional, and one possible basis for this space is given by the pair of vectors $(1, 0, 1)$ and $(0, 1, 0)$.

3. *Let W be the span of the three vectors $v_1 = (1, -1, 3, -2)$, $v_2 = (1, 9, 1, -10)$ and $v_3 = 2v_1 - v_2$ in $\mathbf{R}^4$. What is the dimension of W? Find an orthogonal basis for W.*

There are three vectors generating W, so one might forget that W is only two-dimensional were it not for the first question. Looking at the first question, one recognizes that W is spanned already by v_1 and v_2, since v_3 is in the span of v_1 and v_2. Therefore, W is at most two-dimensional. It is exactly two-dimensional because v_1 and v_2 are clearly not multiples of each other. To get an *orthogonal* basis of W, we follow the G–Schmidt process and solve for $a \in \mathbf{R}$ so that v_1 and $v_2 + av_1$ are orthogonal. Now $v_1 \cdot (v_2 + av_1) = v_1 \cdot v_2 + a\|v_1\|^2 = 15 + 15a$, so it looks as if $a = -1$, here at 37,000 feet (with light chop). Note that $v_2 - v_1 = (0, 10, -2, -8)$ and $(1, -1, 3, -2) \cdot (0, 10, -2, -8) = -10 - 6 + 16 = 0$, as expected.

4. *Evaluate the determinant of the matrix* $\begin{bmatrix} 1 & 0 & 0 & 2 & 4 & 6 & 8 \\ 0 & 1 & 0 & 5 & 12 & 13 & 9 \\ 0 & 0 & 1 & -1 & 31 & 5 & 23 \\ 0 & 0 & 0 & 4 & 2 & 7 & 1 \\ 0 & 0 & 0 & -2 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 5 & 3 \end{bmatrix}$.

If you do column expansion along the first column (which conveniently starts with a 1 and then has only 0s), you can erase the first row and column. Do it; do it again; do it again. We now have to evaluate the determinant of the 4×4 matrix $\begin{bmatrix} 4 & 2 & 7 & 1 \\ -2 & 1 & 3 & -2 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & 5 & 3 \end{bmatrix}$. Expand now along the 3rd row, which again has all 0s except for a single 1. Our original $7\times$ determinant is equal to the negative of the determinant of the 3×3 matrix $\begin{bmatrix} 4 & 7 & 1 \\ -2 & 3 & -2 \\ -1 & 5 & 3 \end{bmatrix}$, i.e., to -125.

5. *Let A be an $n \times n$ (square) matrix. Suppose that $A^2 = A$. Show that $Ay = y$ for all y in the column space of A. If the null space of A is $\{0\}$, show that A is the identity matrix of size n.*

If y is in the column space of A, then $y = Ax$ for some x in $\mathbf{R}^n$. We thus have $Ay = A(Ax) = A^2x = Ax = y$. If the null space of A is $\{0\}$, then A is non-singular and therefore invertible. We can multiply the equation $A^2 = A$ by A^{-1} and get the desired equation $A = I$. (Alternatively, we could say that the column space of A is all of $\mathbf{R}^n$ because the null space has dimension 0, so that the rank of A is n. Since A is the identity on the column space of A, it is the identity on $\mathbf{R}^n$.)