Math 250A — Professor K. A. Ribet

First Midterm Exam — September 27, 2001

This is an 80-minute exam. Please hand in your blue books and papers promptly at 3:30PM. Although this is a "closed book" exam, you may consult a page of notes that you prepared in advance.

1 *(3 points). Find the number of elements of order 7 in a simple group of order 168.*

The 7-Sylow can't be normal because the group is simple. The number of 7-Sylows divides 24 and must be 1 mod 7, so it's 8. There are 6 elements of order 7 in each Sylow, and two Sylows have no common elements except for the identity. Hence the number of elements of order 7 is $6 \cdot 8 = 48$.

2 *(3 points). Use the solvability of groups of order 12 to prove that groups of order $588 = 2^2 \cdot 3 \cdot 7^2$ are solvable.*

The 7-Sylow here is normal because the number of 7-Sylows is 1 mod 7 and is a divisor of 12. The 7-Sylow is abelian, and therefore solvable in particular. The quotient of the group by the normal 7-Sylow is also solvable because it has order 12. Since the group is an extension of one solvable group by another, it's solvable.

3a *(3 points). If X and Y are objects of a category $\mathcal{C}$, explain succinctly (but precisely) what is meant by the product of X and Y.*

See page 58 of Lang. What's important to me is that the product is not just an object of $\mathcal{C}$; it's an object that comes equipped with projection maps to X and Y. These are the maps called f and g on page 58.

3b *(5 points). Let $\mathcal{C}$ be the following category:*

- *The objects of $\mathcal{C}$ are the positive integers 1, 2, 3,. . . .*
- $\mathrm{Mor}(n, m)$ *is the set of $m \times n$ matrices (m rows and n columns) with real coefficients.*
- *The composition law* $\mathrm{Mor}(n, m) \times \mathrm{Mor}(l, n) \to \mathrm{Mor}(l, m)$ *is ordinary matrix multiplication.*

Do products exist in this category? If so, what is the product of n and m in $\mathcal{C}$?

The category that I described in this question is secretly equivalent to the category of real vector spaces of the form $\mathbf{R}^n$ with $n \geq 1$. The product of $\mathbf{R}^n$ and $\mathbf{R}^m$

would be $\mathbf{R}^{n+m}$. This suggests that $n+m$ is the product of n and m in $\mathcal{C}$. To verify that $n+m$ works as the product, we have to give maps $n{+}m \to n$ and $n{+}m \to m$ in the category and verify that mapping to the purported product is the "same" as mapping to both n and m. A map f from $n+m$ to n is a matrix with $n+m$ columns and n rows; we take $f = \begin{pmatrix} I_n & 0 \end{pmatrix}$, where I_n is the $n \times n$ identity matrix and the "0" is a matrix of 0s with n rows and m columns. Similarly, $g : n{+}m \to m$ should be $\begin{pmatrix} 0 & I_m \end{pmatrix}$, where 0 now stands for a matrix with n columns and m rows. Now we have to check that this works: Suppose that we are given a map $\ell \to n+m$, where ℓ is an arbitrary positive integer. This is a matrix h with ℓ columns and $n+m$ rows; it's natural to write $h = \begin{pmatrix} F \\ G \end{pmatrix}$, where F and G both have ℓ columns, but where F has n rows and G has m of them. The product $f \circ h$ is the matrix product $\begin{pmatrix} I_n & 0 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}$, which comes out to be the matrix F of size $n \times \ell$. Similarly, $g \circ h = G$. The map $\begin{pmatrix} F \\ G \end{pmatrix} \mapsto (F, G)$ is a bijection from the space of maps $\ell \to n{+}m$ to the set of pairs of maps (F, G), in which the first entry is a map $\ell \to n$ and the second is a map $\ell \to m$.

4a *(4 points). Let g be an element of the finite group G. Let $\sigma\colon G \to G$ be the permutation $x \mapsto gx$. Show that the sign of this permutation is $\left((-1)^{\ell+1}\right)^{n/\ell}$, where ℓ is the order of g and n the order of G.*

To calculate the sign of a permutation, you write the permutation is a product of disjoint cycles and then use the rule that a cycle of length ℓ has sign $(-1)^{\ell+1}$. The cycles here are the orbits under the action of $\langle g \rangle$ on G; $\langle g \rangle$ is the group generated by g. Notice that $\langle g \rangle$ consists of the powers of g; its order is the order ℓ of g. In fact, the orbits all have length ℓ because the orbit of $x \in G$ under the action of $\langle g \rangle$ is the set of elements of G of the form $g^i x$. The number of orbits is then n/ℓ, where n is the order of G. In summary, the sign of the permutation is $\left((-1)^{\ell+1}\right)^{n/\ell}$. This sign is $+1$ unless both n/ℓ and $\ell+1$ are odd. These conditions mean that ℓ must be (1) even and (2) a multiple of the largest power of 2 in n. If n is even, then condition (2) implies condition (1).

4b *(3 points). Suppose that the 2-Sylow subgroups of G are cyclic and that G has even order. Prove that G has a subgroup of index 2.*

Let g be a generator of a 2-Sylow of G. Then the sign that we calculated in part (a) is -1. The existence of an element g with sign -1 means that the sign map $G \to \{\pm 1\}$ is not identically 1. This sign map is the composite of two maps: the homomorphism $G \to \mathrm{Perm}(G)$ that amount to the action of G on itself by left translation, and the sign homomorphism $\mathrm{Perm}(G) \to \{\pm 1\}$ from a permutation group to $\{\pm 1\}$. (It might be helpful to remember that $\mathrm{Perm}(G)$ becomes $\mathbf{S}_n$ if we order the n elements of G.) The desired subgroup of index 2 in G is the kernel of the non-trivial sign homomorphism $G \to \mathrm{Perm}(G)$ that is under discussion. Note that the existence of an index-2 subgroup of G shows that G cannot be a simple group if it has order > 2 and satisfies the 2-Sylow condition of this problem.

5 *(4 points). Calculate the order of the conjugacy class of* $(1\,2)(3\,4)$ *in the symmetric group* $\mathbf{S}_n$ *($n \geq 4$). Find the order of the centralizer of* $(1\,2)(3\,4)$ *in* $\mathbf{S}_n$*.*

By problem 37a in last week's homework, the conjugate of $(1\,2)(3\,4)$ by γ is the product $(\gamma(1)\,\gamma(2))(\gamma(3)\,\gamma(4))$. Since the $\gamma(i)$ constitute an arbitrary quadruple of distinct numbers, the conjugacy class consists of all products $(a\,b)(c\,d)$ with a, b, c and d distinct. The number of such products is $n(n-1)(n-2)(n-3)/8$. You have to divide by 8 because you can flip the entries in each transposition and flip the two transpositions without changing the value of $(a\,b)(c\,d)$. The order of the centralizer is then $8n!/n(n-1)(n-2)(n-3) = 8(n-4)!$, since the order of the group divided by the order of the centralizer is the number of elements in the conjugacy class.